On a General Linear Boundary Value Problem for Fractional Functional Differential Equations with Aftereffect

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1 Introduction

The efficiency of the main results of the abstract functional differential equation (AFDE) theory in the study of boundary value problems, control problems, and stability issues is demonstrated in [1–3] by examples of wide classes of functional differential equations with an integer-order derivative, including delay differential equations, integro-differential equations, impulse systems and others. It turned out that some classes of equations with a fractional-order derivative can also be considered as a concrete realization of AFDE.

We consider here a system of functional differential equations with fractional derivative (FFDE) and extend the idea of [6] as applied to the question on solvability of a general boundary value problem (BVP). First we describe in detail a class of fractional functional differential equations and appropriate spaces where those are considered. We concerned with the representation of general solution to the system and basic relationships for the Cauchy operator and the fundamental matrix. Next the setting of the general linear BVP is given and discussed, and conditions for the solvability of BVP are obtained in two cases: i) when it is uniquely and everywhere solvable, and ii) when it is not everywhere solvable. As for linear BVP's for AFDE in general, the principal results by L. F. Rakhmatullina are given in detail in [1–3]. Here we propose a somewhat different approach without recourse to the adjoint BVP and an extension of the original BVP.

2 Preliminaries

First, let us introduce the Banach spaces where operators and equations are considered.

Fix a segment $[0,T]\subset\mathbb{R}$. Let $L_\infty^n = L_\infty^n[0,T]$ be the space of measurable and bounded in essence functions $z:[0,T]\to\mathbb{R}^n$ with the norm $\|z\|_{L_\infty} = \operatorname{vraisup}(|z(t)|,\ t\in[0,T])$ (here and below $|\cdot|$ stands for a norm in $\mathbb{R}^n$.

By $L_2^r = L_2^r[0,T]$ we denote the Hilbert space of square summable functions $v:[0,T]\to R^r$ with the inner product $(u,v)=\int\limits_0^T u'(t)v(t)\,dt$ ($\cdot'$ is the symbol of transposition).

Below we will use the following two known operators: the Caputo derivative $\mathcal{D}^\alpha$ of the order $\alpha\in(0,1)$ (see, for instance, [4]) defined by the equality

$$(\mathcal{D}^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{\dot{x}(s)}{(t-s)^\alpha}\,ds$$

($\Gamma(\cdot)$ is the Euler gamma-function), and the fractional integration operator J^α defined by the equality

$$(J^\alpha z)(t) = \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,z(s)\,ds$$

(see, for instance, [4]).

Introduce the space $AC^\alpha_\infty = AC^\alpha_\infty[0,T]$ as the space of functions $y:[0,T]\to\mathbb{R}^n$ representable in the form $y(t)=y(0)+(\mathcal{J}^\alpha z)(t)$, where $z\in L^n_\infty$ with the norm $\|y\|_{AC^\alpha_\infty}=\|\mathcal{D}^\alpha y\|_{L^n_\infty}+|y(0)|_{\mathbb{R}^n}$.

Note that the operators $\mathcal{D}^\alpha : AC^\alpha_\infty\to L^n_\infty$ and $J^\alpha : L^n_\infty\to AC^\alpha_\infty$ are bounded, and $\mathcal{D}^\alpha J^\alpha = I$, I is the identity operator.

Consider the linear fractional functional differential system

$$\mathcal{D}^\alpha x=\mathcal{T}x+f, \tag{1}$$

where $\mathcal{T} : AC^\alpha_\infty[0,T]\to L^n_\infty[0,T]$ is linear bounded Volterra operator with the property: there exists $p>0$ such that the inequality

$$|(\mathcal{T}x)(t)|\le p\max_{s\in[0,t]}|x(s)|,\ \ t\in[0,T] \tag{2}$$

holds for any $x\in AC^\alpha_\infty[0,T]$. Let us denote

$$(Kz)(t)=(\mathcal{T}J^\alpha z)(t).$$

Throughout the following, we assume that $K : L^n_\infty[0,T]\to L^n_\infty[0,T]$ is a regular integral Volterra operator:

$$(Kz)(t)=\int\limits_0^t K(t,s)z(s)\,ds.$$

This condition is fulfilled for wide classes of operators $\mathcal{T}$ including the operators of inner superposition with delay [2] and Volterra integral ones. In such cases, the representation of the kernel $K(t,s)$ can be obtained in an explicit form.

Under above condition (2), the operator K has the resolvent operator $R : (I-K)^{-1}=I+R$ (see [7]), and R is an integral Volterra operator too:

$$(Rf)(t)=\int\limits_0^t R(t,s)f(s)\,ds$$

with the resolvent kernel $R(t,s)$ [8, Theorem 2.2, p. 119]).

All our constructions below are based on the representation of solutions to (1) with the initial condition $x(0)=a$ [7]:

$$x(t)=X(t)a+\int\limits_0^t C(t,s)f(s)\,ds,\ \ t\in[0,T],$$

where the Cauchy matrix $C(t,s)$ is defined by the equality

$$C(t,s)=\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}E+\int\limits_s^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}R(\tau,s)\,d\tau,$$

$X(t)$ is the fundamental matrix of the homogeneous equation (1) that is defined by the equality

$$X(t)=E+\int\limits_0^t C(t,s)(\mathcal{T}E)(s)\,ds.$$

Remark. Equations (1) and $\dot{x}=\mathcal{T}x+f$ are two different representatives of the AFDE [3] $\delta x=\mathcal{T}x+f$. Therewith the theory of the latter uses the representation $x=V\,\frac{d}{dt}\,x+x(0)$, where $(Vz)(t)=\int\limits_0^t z(s)ds$, while the theory of (1) is based on the representation $x=J^\alpha\mathcal{D}^\alpha x+x(0)$. The space AC^α_∞ is isomorphic to the direct product $L^n_\infty\times\mathbb{R}^n$ with $x=J^\alpha z+\beta$.

3 The boundary value problem

We consider the boundary value problem for the system (1) with the following boundary conditions:

$$\ell x \equiv \sum_{i=1}^{m} A_i x(t_i) + \int_0^T B(\tau) x(\tau)\, d\tau = \beta \in \mathbb{R}^N, \tag{3}$$

where t_i, $i = 1, \ldots, m$ are fixed points from $[0, T]$, A_i are constant $(N \times n)$-matrices, $B(\cdot)$ is $(N \times n)$-matrix with summable elements, β is a prescribed constant vector.

The general solution of (1) has the form

$$x(t) = X(t)a + \int_0^t C(t,s) f(s)\, ds, \tag{4}$$

with arbitrary $a \in \mathbb{R}^n$.

From (4) it follows that if $N = n$, then BVP (1), (3) is uniquely solvable for any f, β if and only if the matrix

$$\ell X = (\ell X^1, \ldots, \ell X^n),$$

where X^j is the j-th column of X, is nonsingular, i.e. $\det \ell X \neq 0$.

In the case under consideration we have the representation of ℓX in the following explicit form:

$$\ell X = \sum_{i=1}^{m} A_i X(t_i) + \int_0^T B(\tau)\, X(\tau)\, d\tau. \tag{5}$$

We assume in the sequel that $N > n$ and the system $\ell^i : AC_\infty^\alpha \to \mathbb{R}$, $i = 1, \ldots, N$ can be splitted into two subsystems $\ell_1 : AC_\infty^\alpha \to \mathbb{R}^n$ and $\ell_2 : AC_\infty^\alpha \to \mathbb{R}^{N-n}$ such that the BVP

$$\mathcal{D}^\alpha x = \mathcal{T} x + f, \quad \ell_1 x = \beta_1$$

is uniquely solvable. Without loss of generality we will assume that ℓ_1 is formed by first n components of ℓ and the elements of β_1 in (4) are the corresponding components of β. Thus ℓ_2 will stand for the final $(N - n)$ components of ℓ, and elements of $\beta_2 \in R^{N-n}$ are defined as the final $(N - n)$ components of β.

The representation (4) allows us to extend the idea of [6] onto the described class of fractional functional differential equations (1).

To describe a set of f for which BVP (1), (3) is solvable, we introduce an auxiliary linear bounded Volterra operator $F : L_2^r \to L_\infty^n$, $(Fu)(t) = \int_0^t F(t,s) u(s)\, ds$ and will study the solvability for $f \in F L_2^r$.

Having in mind (4), we obtain for $\ell_1 x$ the representation in operator form:

$$\ell_1 x = \ell_1 X a + \ell_1 C f = \beta_1,$$

where

$$a = (\ell_1 X)^{-1} \beta_1 - (\ell_1 X)^{-1} \ell_1 C f.$$

Hence,

$$x = X(\ell_1 X)^{-1} \beta_1 - X(\ell_1 X)^{-1} \ell_1 C f + C f$$

or

$$x = X(\ell_1 X)^{-1}\beta_1 + Gf, \tag{6}$$

where $G : L_\infty^n \to AC_\infty^\alpha$ is the Green operator to the problem (5). The expression for the kernel $G(t,s)$ of this integral operator can be derived from its definition.

Let us apply the vector-functional ℓ_2 to both sides of (6):

$$\ell_2 x = \ell_2 X(\ell_1 X)^{-1}\beta_1 + \ell_2 Gf.$$

Thus the question under consideration is the question on a set of f such that the equation

$$\ell_2 Gf = \beta_2 - \ell_2 X(\ell_1 X)^{-1}\beta_1 \tag{7}$$

is solvable with $f \in FL_2^r$, i.e.,

$$f(t) = \int_0^t F(t,s)u(s)\,ds, \quad u \in L_2^r. \tag{8}$$

Define the $(N-n)\times n$-matrix Γ with summable elements by the representation

$$\ell_2 Gf = \int_0^T \Gamma(s)f(s)\,ds \tag{9}$$

for all $f \in L_\infty^n$. An explicit form of $\Gamma(\cdot)$ can be easily derived by elementary transformations taking into account the representation of ℓ_2. After substitution of (8) into (9) we obtain

$$\ell_2 Gf = \int_0^T \Gamma(t)\int_0^t F(t,s)u(s)\,ds\,dt = \int_0^T\int_s^T \Gamma(t)F(t,s)\,dt\,u(s)\,ds$$

or

$$\ell_2 Gf = \int_0^T\int_s^T \Gamma(t)F(t,s)\,dt\,u(s)\,ds = \int_0^T M(s)u(s)\,ds,$$

where

$$M(s) = \int_s^T \Gamma(t)F(t,s)\,dt. \tag{10}$$

To solve (7) with respect to f of the form (8), we propose to find u in the form

$$u(s) = M'(s)d + h(s)$$

with constant d and $h \in L_2^r$ being orthogonal to the columns of M'.

Theorem 1. *Let the matrix* $W = \int_0^T M(s)M'(s)\,ds$, *where* M *is defined by* (10), *be nonsingular. Then BVP* (1), (3) *is solvable for all* f *of the form*

$$f(t) = f_0(t) + \varphi(t),$$

where

$$f_0(t)=\int_0^t F(t,s)M'(s)\,ds\,\big[W^{-1}\beta_2-W^{-1}(\ell_2X)(\ell_1X)^{-1}\beta_1\big],\ \ \varphi(t)=\int_0^t F(t,s)h(s)\,ds,$$

$h(\,\cdot\,)\in L_2^r$ *is an arbitrary function orthogonal to each column of* $M'(\,\cdot\,)$: $\int_0^T M(s)h(s)\,ds=0.$

In conclusion, we note that the choice of the operator F allows one to adjust the smoothness of the functions for which the problem (1), (3) turns out to be solvable.

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