Non-Force Approach to Lyapunov Exponents Control

E. K. Makarov, D. A. Korshukov
Institute of Mathematics of the National Academy of Science of Belarus
Minsk, Belarus
E-mail: jcm@im.bas-net.by

1 Introduction

The problem on Lyapunov exponents control has been formulated by E. L. Tonkov and L. E. Zabello during a heated discussion at the IV Conference of Belarusian Mathematicians in 1975 (see [11] and [7, p. 5, 13]). We can state a simple variant of this problem as follows.

For a control system

$$\dot{x} = A(t)x + B(t)u, \;\; x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m, \;\; t \geq 0, \tag{1.1}$$

it is required to find a linear feedback

$$u = U(t)x, \;\; U \in \mathbb{R}^{m\times n}, \tag{1.2}$$

such that the closed-loop system

$$\dot{x} = (A(t) + B(t)U(t))x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1.3}$$

have a prescribed spectrum of Lyapunov exponents.

First substantial results on Lyapunov exponents control was presented in [9]. For basic information on the issue see [7]. The current state of this theory is characterized by the predominance of so-called forceful (or coercive) control methods, which rely primarily on the resources of the control subsystem to change the state of the entire controlled system. For example, theorems on local proportional assignability of Lyapunov spectrum [7, p. 186], [5], [6] provide Lipshitz estimate for control matrix U with respect to exponents shift produced by U. An alternative to this approach is the concept of non-force control in the general theory of controlled systems proposed by S. V. Yemelyanov and S. K. Korovin in [10]. This concept involves using the resources of the controlled subsystem to the same purposes. It was noted in [10, p. 8, 9] that the stabilization methods "are primarily oriented toward force-based solutions to the problem, whereas nature demonstrates remarkable examples of solving stabilization problems with very limited means and under very constrained circumstances". To construct technical systems with similar properties it is necessary to develop "fundamentally new, 'non-force' mechanisms for suppressing uncertainty factors . These include, in particular, techniques based on the use of positive feedback and unstable motions, allowing the system to self-accelerate until conditions are created for suppressing interference and uncertainty factors".

Our aim is to propose a way to transfer the above concept to problems of Lyapunov exponents control. Since no remarkable results are available in general settings, we consider a model situation taking $B \equiv I$.

2 Results for a model situation

Consider a linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \ge 0, \tag{2.1}$$

with piecewise continuous and bounded coefficient matrix A such that $\|A(t)\| \le M < +\infty$ for all $t \ge 0$. We denote the Cauchy matrix of (2.1) by X_A and the highest Lyapunov exponent of (2.1) by $\lambda_n(A)$. Together with system (2.1) consider a perturbed system

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \quad t \ge 0, \tag{2.2}$$

with piecewise continuous and bounded perturbation matrix Q. Denote the higher exponent of (2.2) by $\lambda_n(A+Q)$.

Definition 2.1 (see [2, p. 162], [3, p. 48] or [4, p. 42])**.** The number

$$\Omega'(A) = \lim_{\varepsilon \to 0} \sup_{\|Q\| \le \varepsilon} \lambda_n(A+Q)$$

is said to be the exact upper mobility limit for the higher exponent of system (2.1).

Definition 2.2 (see [2, p. 116], [3, p. 48] or [4, p. 43])**.** The upper central exponent of system (2.1) is the number

$$\Omega(A) = \lim_{T \to +\infty} \overline{\lim_{m \to \infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT-T)\|.$$

V. M. Millionschikov in [8] has proved that the upper central exponent is attainable, i.e. the equality

$$\Omega'(A) = \Omega(A)$$

holds.

The attainability of the upper central exponent is proved in [8] under the assumption that there are no restrictions on the set of values taken by the perturbation Q at an arbitrary point on the positive semiaxis. This condition can be relaxed for classes of perturbations that vanish on some given set.

For a control system (1.1) with linear feedback (1.2) we have a natural constraint on the values of the disturbances

$$Q(t)x(t) \in \operatorname{Im} B(t).$$

Hence, from the point of view of the theory of Lyapunov exponents control, it is of interest to consider classes of disturbances with restrictions on values taken by the disturbance Q on the positive semi-axis.

To construct a simple sample of such class, we take an arbitrary set $\mathcal{Z} \subset \mathbb{R}$ with a piecewise continuous characteristic function and consider the set $\mathfrak{B}(\mathcal{Z})$ of all piecewise continuous bounded perturbations Q that identically vanish on $\mathcal{Z}$.

Definition 2.3. Let $\mathfrak{M}$ be any set of bounded piecewise continuous perturbations Q. We say that the upper central exponent is *attainable within the class of* $\mathfrak{M}$*-small perturbations* (i.e. small perturbations from $\mathfrak{M}$) if the equality

$$\Omega(A) = \Omega'(A, \mathfrak{M}) \doteq \lim_{\varepsilon \to 0} \sup_{\|Q\| \le \varepsilon} \big\{ \lambda_n(A+Q) : \ Q \in \mathfrak{M} \big\}$$

holds.

Theorem 2.1. *If there exist numbers $b > \delta > 0$ and a sequence $t_k > 0$, $k \in \mathbb{N}$, such that $\delta < t_{k+1} - t_k < b$ and $\mathcal{Z} \cap [t_k, t_k + \delta] = \varnothing$ for all $k \in \mathbb{N}$, then the highest central exponent is attainable in the class of $\mathfrak{B}(\mathcal{Z})$-small perturbations for any system* (2.1)*.*

Sketch of the proof. First we prove the equality

$$\Omega(A) = \lim_{j \to +\infty} \overline{\lim_{m \to \infty}}\, t^{-1}_{2^j(m+1)} \sum_{k=1}^{m} \ln \|X_A(t_{2^j(k+1)}, t_{2^j k})\|.$$

Then we prove the attainability of $\Omega(A)$ applying Millionshchikov rotation method at points $2^j(k+1,$ $k \in \mathbb{N}$, for successive $j \in \mathbb{N}$. □

Theorem 2.2. *If there exists an increasing sequence of positive numbers t_k, $k \in \mathbb{N}$, tending to $+\infty$ and a number $\vartheta > 1$ such that $\mathcal{Z} \supset [t_k, \vartheta t_k]$ for all $k \in \mathbb{N}$, then there exists a system* (2.1) *such that $\Omega'(A, \mathfrak{B}(\mathcal{Z})) \neq \Omega(A)$.*

Sketch of the proof. To prove the theorem we construct a linear system (2.1) satisfying the following conditions:

(i) coefficient matrix $A(t)$ equals zero for all $t \notin \mathcal{Z}$;

(ii) $0 < \lambda_n(A) < \Omega(A)$.

Such a system can be obtained by a slight rearrangement of example from [1]. It can be easily seen that $\Omega(A, \mathfrak{B}(\mathcal{Z})) = \lambda_n(A)$, i.e. the upper central exponent of this system is not attainable in the class of $\mathfrak{B}(\mathcal{Z})$-small perturbations. □

3 Discussion and conclusions

It should be noted that

(i) we have a gap between Theorems 2.1 and 2.2, i.e. we do not obtain a necessary and sufficient conditions for attainability of $\Omega(A)$ in the class of $\mathfrak{B}(\mathcal{Z})$-small perturbations;

(ii) we do not obtain intermediate values of $\lambda_n(A+Q)$, i.e. values from $]\lambda_n(A), \Omega(A)[$;

(iii) we do not obtain values of $\lambda_n(A+Q)$ less than $\lambda_n(A)$.

These are problems to solve even in our model situation and some of them seem to be very difficult.

References

[1] E. A. Barabanov, Properties of the highest σ-exponent. (Russian) *Differentsial'nye Uravneniya* **18** (1982), no. 5, 739–744.

[2] B. F. Bylov, R. È. Vinograd, D. M. Grobman and V. V. Nemyckiĭ, *Theory of Ljapunov Exponents and its Application to Problems of Stability.* (Russian) Izdat. "Nauka", Moscow, 1966.

[3] N. A. Izobov, *Introduction to the Theory of Lyapunov Exponents.* (Russian) BGU, Minsk, 2006.

[4] N. A. Izobov, *Lyapunov Exponents and Stability.* Stability, Oscillations and Optimization of Systems, 6. Cambridge Scientific Publishers, Cambridge, 2012.

[5] E. K. Makarov and S. N. Popova, Local controllability of characteristic Lyapunov exponents of systems with simple exponents. (Russian) *Differ. Uravn.* **33** (1997), no. 4, 495–499; translation in *Differential Equations* **33** (1997), no. 4, 496–500.

[6] E. K. Makarov and S. N. Popova, On sufficient conditions for the local proportional controllability of the Lyapunov exponents of linear systems. (Russian) *Differ. Uravn.* **39** (2003), no. 2, 217–226; translation in *Differ. Equ.* **39** (2003), no. 2, 234–245.

[7] E. K. Makarov and S. N. Popova, *Controllability of Asymptotic Invariants of Nonstationary Linear Systems.* (Russian) Navuka, Minsk, Belarus, 2012.

[8] V. M. Millionshchikov, A proof of accessibility of the central exponents of linear systems. (Russian) *Sibirsk. Mat. Zh.* **10** (1969), 99–104.

[9] S. N. Popova, On the issue of Lyapunov exponents control. (Russian) *Vestn. Udmurt. Univ., Mat. Mekh. Komp'yut. Nauki*, 1992, no. 1, 23–39.

[10] S. V. Yemelyanov and S. K. Korovin, *New Types of Feedback: Control under Uncertainty.* (Russian) Nauka, Moscow, 1997.

[11] L. E. Zabello, On controllability of Lyapunov exponents. *Problems of Development of Applied Mathematical Research, Abstracts of IV Republican Conference of Mathematicians of Belarus (Minsk, June, 3–4, 1975)*, p. 38, BSU, Minsk, Vol. 2, 1975.