On Some Boundary Value Problems Connected with the Whithey Problem

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1 Statement of the problem

On the interval $[a,b]$, we consider the problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u'(a) = c, \quad u(a) = u(b), \tag{1.2}$$

where $f \in Car([a,b]\times\mathbb{R}^2;\mathbb{R})$ and $c \in \mathbb{R}$. As usual, by a solution to equation (1.1) we understand a function $u \in AC^1([a,b];\mathbb{R})$ satisfying the given equation almost everywhere on $[a,b]$. A solution u to equation (1.1) satisfying conditions (1.2) is said to be a solution to problem (1.1), (1.2).

Definition 1.1. We say that $\alpha \in AC_\ell([a,b];\mathbb{R})$ is a lower function and $\beta \in AC_u([a,b];\mathbb{R})$ is an upper function of equation (1.1) if

$$\alpha''(t) \ge f(t,\alpha(t),\alpha'(t)), \quad \beta''(t) \le f(t,\beta(t),\beta'(t)) \text{ for a.e. } t \in [a,b].^1$$

The lower and upper functions α and β of equation (1.1) are said to be well-ordered if

$$\alpha(t) \le \beta(t) \text{ for } t \in [a,b]. \tag{1.3}$$

We also consider the mixed conditions

$$u'(a) = c_1, \quad u(b) = c_2, \tag{1.4}$$

where $c_1, c_2 \in \mathbb{R}$.

If α and β are well-ordered lower and upper functions of equation (1.1), then the conditions for the function f are known in the existing literature to guarantee the existence of a solution to problem (1.1), (1.4). More precisely, the following theorem holds.

Definition 1.2. A continuous function $\omega\colon \mathbb{R}_+ \to\,]0,+\infty[$, such that

$$\int\limits_0^{+\infty} \frac{1}{\omega(\eta)}\,\mathrm{d}\eta = +\infty, \tag{1.5}$$

is called a Nagumo function.

[1] $AC_\ell([a,b])$ (or $AC_u([a,b])$) denotes the set of absolutely continuous functions $\gamma\colon [a,b]\to\mathbb{R}$ such that γ' admits the representation $\gamma'(t) = \gamma_0(t)+\sigma(t)$ for a.e. $t\in[a,b]$, where $\gamma_0\colon [a,b]\to\mathbb{R}$ is absolutely continuous and the function σ is non-decreasing (or non-increasing) on $[a,b]$ and its derivative is equal to zero almost everywhere on $[a,b]$.

Theorem 1.1 ([2, Theorem 3.1_2]). *Let α and β be well-ordered lower and upper functions of equation* (1.1) *such that*

$$\alpha'(a+) \geq \beta'(a+). \tag{1.6}$$

Let, moreover, f satisfy the condition

$$f(t,x,y)\operatorname{sgn} y \leq \omega(|y|)\big(h(t)+|y|\big) \ \text{ for a.e. } t\in[a,b] \text{ and all } x,y\in\mathbb{R}, \quad \alpha(t)\leq x\leq\beta(t),\ |y|\geq R, \tag{1.7}$$

where $R \geq 0$, $h \in L^1([a,b];\mathbb{R}_+)$, and ω is a Nagumo function. Then, for any $c_1 \in [\beta'(a+),\alpha'(a+)]$ and $c_2 \in [\alpha(b),\beta(b)]$, problem (1.1), (1.4) *has a solution u satisfying the condition*

$$\alpha(t) \leq u(t) \leq \beta(t) \ \text{ for } t\in[a,b]. \tag{1.8}$$

2 Main results

Theorem 2.1. *Let α and β be well-ordered lower and upper functions of equation* (1.1) *such that* (1.6) *holds and*

$$\alpha(a)=\alpha(b), \quad \beta(a)=\beta(b). \tag{2.1}$$

Let, moreover, f satisfy condition (1.7), *where $R\geq 0$, $h\in L^1([a,b];\mathbb{R}_+)$, and ω is a Nagumo function. Then, for any $c\in[\beta'(a),\alpha'(a)]$, problem* (1.1), (1.2) *has a solution u satisfying* (1.8).

Theorem 2.2. *Let α and β be well-ordered lower and upper functions of equation* (1.1) *such that* (1.6) *and* (2.1) *hold and let f satisfy the condition*

$$f(t,x,y)\operatorname{sgn} y \leq q(t,|y|)+\sum_{k=1}^{n} h_k(t)|y|^{1+\frac{1}{\mu_k}} \ \text{ for a.e. } t\in[a,b] \text{ and all } x,y\in\mathbb{R}, \quad \alpha(t)\leq x\leq\beta(t),\ |y|\geq R,$$

where $R\geq 0$, $q\in Car_{sl}([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$[2]*, and for $k=1,\dots,n$, $h_k\in L^{\nu_k}([a,b];\mathbb{R}_+)$ and $\mu_k,\nu_k\in[1,+\infty]$ are conjugate numbers (we put $\frac{1}{\mu_k}=0$ if $\mu_k=+\infty$). Then, for any $c\in[\beta'(a+),\alpha'(a+)]$, problem* (1.1), (1.2) *has a solution u satisfying* (1.8).

Definition 2.1. Let $\alpha\colon[a,b]\to\mathbb{R}$, $\beta\colon[a,b]\to\mathbb{R}$ be continuous functions such that (1.3) holds. We say that u^* (or u_*) is a maximal solution (or a minimal solution) to problem (1.1), (1.2) (or (1.1), (1.4)) in the segment $[\alpha,\beta]$, if it is a solution to this problem satisfying $\alpha(t)\leq u^*(t)\leq\beta(t)$ (or $\alpha(t)\leq u_*(t)\leq\beta(t)$) for $t\in[a,b]$ and for any solution u to problem (1.1), (1.2) (or (1.1), (1.4)) with the property (1.8), the inequality $u(t)\leq u^*(t)$ (or $u(t)\geq u_*(t)$) holds for $t\in[a,b]$.

Theorem 2.3. *Let the hypotheses of Theorem 2.1 (or Theorem 2.2) be satisfied. Then, for any $c\in[\beta'(a+),\alpha'(a+)]$, problem* (1.1), (1.2) *has maximal and minimal solutions in the segment $[\alpha,\beta]$.*

Theorem 2.4. *Let the hypotheses of Theorem 1.1 be satisfied. Then, for any $c_1\in[\beta'(a+),\alpha'(a+)]$ and $c_2\in[\alpha(b),\beta(b)]$, problem* (1.1), (1.4) *has maximal and minimal solutions in the segment $[\alpha,\beta]$.*

3 Applications of the main results

In this part, we provide the consequences of the main results for a pendulum inside a closed box which moves in a plane. More precisely, consider free damped pendulum consisting of a small solid of the weight m attached to the massless rod of the length ℓ, see Fig. 1.

[2] $Car_{sl}([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ is the set of functions $q\in Car([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ such that for almost every $t\in[a,b]$, the function $q(t,\cdot)$ is non-decreasing on $\mathbb{R}_+$ and $\lim\limits_{\varrho\to+\infty}\frac{1}{\varrho}\int_a^b q(s,\varrho)\,\mathrm{d}s=0$.

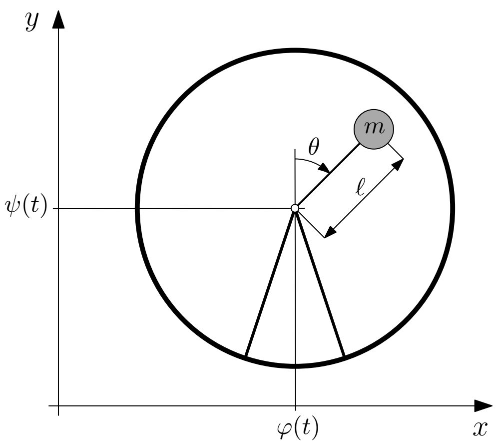

Figure 1.

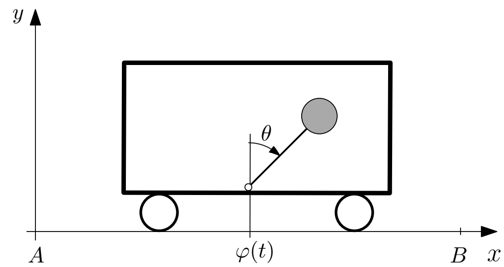

Figure 2.

The pendulum swings inside a closed box filled with a fluid; the given functions φ, ψ determine box's position in time. It is a system with one degree of freedom, described by the generalized coordinate θ, whose equation of motion is of the form

$$\ddot{\theta}+\frac{\mu}{m}\,\dot{\theta}+\frac{\ddot{\varphi}(t)}{\ell}\,\cos\theta-\Big(\frac{\mathrm{g}}{\ell}+\frac{\ddot{\psi}(t)}{\ell}\Big)\sin\theta=0. \tag{3.1}$$

It is worth mentioning here that if we consider another model of fluid friction, namely, $\vec{F}_{\rm fric}=-\mu|\vec{v}_{\rm rel}|\vec{v}_{\rm rel}$, then the term $\frac{\mu\ell}{m}\,|\dot{\theta}|\dot{\theta}$ is presented in equation (3.1) instead of $\frac{\mu}{m}\,\dot{\theta}$.

Courant and Robbins in their book "What is Mathematics" (see [1]) formulated a problem stated up by H. Whitney. The problem is as follows. "Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s=\varphi(t)$ is given, where s is the distance of the train from station A, and t is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce. Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B?" It is clear that, assuming a point mass m attached to the massless rod instead of the rod itself (see Fig. 2), a motion of the rod is determined by a solution to the equation

$$\ddot{\theta}+\frac{\ddot{\varphi}(t)}{\ell}\,\cos\theta-\frac{\mathrm{g}}{\ell}\,\sin\theta=0$$

on the interval $[0,T]$ satisfying the condition

$$-\frac{\pi}{2}<\theta(t)<\frac{\pi}{2}\ \text{ for }\ t\in[0,T].$$

Inspired by the above problems, we now apply the main results and provide the effective conditions for the existence as well as uniqueness of a solution u to the equation

$$u''=p(t)\sin u+g(t)\cos u+h(t)|u'|^{\lambda}\operatorname{sgn}u'(t), \tag{3.2}$$

satisfying the boundary conditions of types (1.2) and (1.4), and the inequalities

$$-\frac{\pi}{2} < u(t) < \frac{\pi}{2} \quad \text{for} \quad t \in \,]a,b[\,.$$

We assume in what follows that $p, g, h \in L^1([a,b];\mathbb{R})$ and $\lambda \geq 1$.

The following proposition concerning the boundary conditions

$$u'(a) = 0, \quad u(a) = u(b) \tag{3.3}$$

follow from Theorems 2.2 and 2.3.

Proposition 3.1. *Let $\lambda \geq 1$ and $p, g, h \in L^1([a,b];\mathbb{R})$ be such that*

$$p(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b], \quad p(t) \not\equiv 0 \ \text{ on } [a,b], \tag{3.4}$$

and

$$\left.\begin{array}{ll} [h]_+ \in L^{\frac{1}{2-\lambda}}([a,b];\mathbb{R}_+), & \text{if } \ \lambda \in \,]1,2[, \\ [h]_+ \in L^{\infty}([a,b];\mathbb{R}_+), & \text{if } \ \lambda = 2, \\ [h(t)]_+ \equiv 0 \ \text{ on } [a,b], & \text{if } \ \lambda > 2. \end{array}\right\} \tag{3.5}$$

Then, problem (3.2), (3.3) has a maximal solution u^* and a minimal solution u_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$-\frac{\pi}{2} < u_*(t) \leq u^*(t) < \frac{\pi}{2} \quad \text{for} \quad t \in [a,b].$$

For the boundary conditions

$$u'(a) = 0, \quad u(b) = c, \tag{3.6}$$

Theorems 1.1 and 2.4 yield

Proposition 3.2. *Let $\lambda \geq 1$ and $p, g, h \in L^1([a,b];\mathbb{R})$ be such that (3.4) and (3.5) are fulfilled. Then, for any $c \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, problem (3.2), (3.6) has a maximal solution u^* and a minimal solution u_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and*

$$-\frac{\pi}{2} < u_*(t) \leq u^*(t) < \frac{\pi}{2} \quad \text{for} \quad t \in [a,b[\,.$$

3.1 On the Whitney problem

As a possible use of Propositions 3.1 and 3.2, we provide two modifications of the Whitney problem.

Question 3.1. Is there any initial position θ_0 of the rod such that if the motion of the rod starts at the station A (i.e., at the time $t = 0$) from the position θ_0 with the zero angular velocity, then the rod is at the station B (i.e., at the time $t = T$) again at the position θ_0 without the rod touching the floor during the motion?

Corollary 3.1. *The problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\cos\theta - \frac{\mathrm{g}}{\ell}\sin\theta = 0; \quad \dot{\theta}(0) = 0, \ \ \theta(0) = \theta(T) \tag{3.7}$$

has a maximal solution θ^* and a minimal solution θ_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$-\frac{\pi}{2} < \theta_*(t) \leq \theta^*(t) < \frac{\pi}{2} \quad \text{for} \quad t \in [0,T].$$

Answer 3.1. Yes; $\theta_0 = \theta(0)$, where θ is a solution to problem (3.7).

Question 3.2. Let the terminal position θ_T of the rod be given. Is there any initial position θ_0 of the rod such that if the motion of the rod starts at the station A (i.e., at the time $t = 0$) from the position θ_0 with the zero angular velocity, then the rod is at the station B (i.e., at the time $t = T$) at the position θ_T without the rod touching the floor during the motion?

Corollary 3.2. *For any $\theta_T \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell} \cos\theta - \frac{g}{\ell} \sin\theta = 0; \quad \dot{\theta}(0) = 0, \ \ \theta(T) = \theta_T \tag{3.8}$$

has a maximal solution θ^* and a minimal solution θ_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$-\frac{\pi}{2} < \theta_*(t) \le \theta^*(t) < \frac{\pi}{2} \ \ \textit{for} \ \ t \in [0, T[\,.$$

Answer 3.2. Yes; $\theta_0 = \theta(0)$, where θ is a solution to problem (3.8).

Corollary 3.3. *Let $\theta_{\max} := \theta^*(0)$, where θ^* is a maximal solution to the problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell} \cos\theta - \frac{g}{\ell} \sin\theta = 0; \quad \dot{\theta}(0) = 0, \ \ \theta(T) = \frac{\pi}{2}\,.$$

Then, for any $\theta_0 \in \,]\theta_{\max}, \frac{\pi}{2}[\,$, the initial value problem

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell} \cos\theta - \frac{g}{\ell} \sin\theta = 0; \quad \theta(0) = \theta_0, \ \ \dot{\theta}(0) = 0$$

has a unique solution θ and this solution satisfies

$$\theta(t) = \frac{\pi}{2} \ \ \textit{for some} \ \ t \in \,]0, T[\,.$$

It follows from Corollary 3.3 that if the motion of the rod starts at the station A (i.e., at the time $t = 0$) from the position $\theta_0 \in \,]\theta_{\max}, \frac{\pi}{2}[$ with the zero angular velocity, then the rod falls to the floor before the train reaches the station B.

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