The Estimate of Lyapunov Exponent for Linear Discrete System

A. V. Lipnitskii

Institute of Mathematics, National Academy of Sciences of Belarus
Minsk, Belarus
E-mail: ya.andrei173@yandex.by

Consider a linear discrete system

$$x(n+1)=B_\mu(n)x(n), \ \ x\in\mathbb{R}^2, \ \ n\in\mathbb{N}\cup\{0\}, \tag{1}$$

with a coefficient matrix $B_\mu(n) := \begin{cases}\operatorname{diag}[e^{d_k(\mu)},e^{-d_k(\mu)}], & n=2k-2,\\ U(\mu+\gamma_k(\mu)+b_k), & n=2k-1,\end{cases}$ where $k\in\mathbb{N}$, the real parameter μ, the numbers $b_k\in\mathbb{R}$ and the continuous 2π-periodic functions $d_k(\,\cdot\,),\gamma_k(\,\cdot\,):\mathbb{R}\to\mathbb{R}$. Here, by $U(\varphi)\equiv\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$ we denote the matrix of the counterclockwise rotation by an angle $\varphi\in\mathbb{R}$.

M. Herman in [4] considered system (1) in the case where for some functions $d(\,\cdot\,)$ and $\gamma(\,\cdot\,)$ the following conditions are satisfied:

$$d_k(\mu)=d(\mu+k\omega), \ \ \gamma_k(\mu)=\gamma(\mu+k\omega), \ \ b_k=k\omega, \ \ k\in\mathbb{N}, \ \ \mu\in\mathbb{R}, \ \ \omega\in\mathbb{R}\setminus\mathbb{Q}, \tag{2}$$

$$d(\,\cdot\,)\equiv d\geq 1, \ \ \gamma(\,\cdot\,)\equiv 0. \tag{3}$$

In this case, the Cauchy operator of system (1) $Y_{B_\mu}(k,n)\overset{\text{def}}{=}B_\mu(k-1)\cdots B_\mu(n)$, $k>n\geq 0$, considered as a function of μ, is a linear cocycle (see the definition, for example, in Section 5 of the article [1], or more detailed in [3]).

If conditions (2) hold, by the ergodic theorem, the upper Lyapunov exponent of the system (1), defined by the formula $\lambda^+(B_\mu)\overset{\text{def}}{=}\overline{\lim\limits_{n\to+\infty}}\frac{1}{n}\ln\|Y_{B_\mu}(n,0)\|$, is equal for almost all $\mu\in\mathbb{R}$ to $L(B):=\lim\limits_{n\to+\infty}\frac{1}{2\pi n}\int\limits_0^{2\pi}\ln\|Y_{B_\mu}(n,0)\|\,d\mu$.

In [4, Section 4.1] under conditions (2) and (3), using a sub-harmonicity “trick”, the following estimate was proven

$$L(B)\geq\ln\frac{e^d+e^{-d}}{2}\,. \tag{4}$$

Sorets–Spencer [12] further developed Herman’s complex-analytic method to show positivity of the upper Lyapunov exponent for the linear differential equation

$$\ddot{x}=-(K^2(\cos t+\cos(\omega t+\theta))+E)x, \ \ x\in\mathbb{R}^2, \ \ t\geq 0$$

with any irrational $\omega\in\mathbb{R}$ and almost all $\theta\in\mathbb{R}$ on the set of values $E\geq 0$, whose relative Lebesgue measure tends to unity for large K.

L.-S. Young in [13, Corollary 1] considered the case when $\mu+\gamma(\mu)$ is a diffeomorphism of the unit circle, i.e., the condition

$$\gamma'(\mu)>-1, \ \ \mu\in\mathbb{R} \tag{5}$$

is true. Provided that ω satisfies some Diophantine condition, which holds almost everywhere, she proved the approximation $L(B) \approx d$ for the average Lyapunov exponent of system (1), (2), (5) with sufficiently big values of $d(\,\cdot\,) \equiv d > 0$.

A. Avila and R. Krikorian in [2] studied so-called "monotonic" cocycles, i.e. systems (1), (2), that the polar angle of any their solution is a monotonically increasing function of the parameter μ. This property can be represented by the inequality (here $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product of vectors in $\mathbb{R}^2$)

$$\Big(B_\mu(2k)B_\mu(2k-1)y, U\Big(\frac{\pi}{2}\Big)\big(B_\mu(2k)B_\mu(2k-1)y\big)'_\mu\Big) \geq \varepsilon > 0, \;\; 0 \neq y \in \mathbb{R}^2, \;\; k \in \mathbb{N}. \tag{6}$$

Assuming the inclusion $d(\,\cdot\,), \gamma(\,\cdot\,) \in C^\infty$, they showed the smoothness of its upper Lyapunov exponent considered as a function of μ. Clearly, each of conditions (3) or (5) implies inequality (6). Therefore, the major Lyapunov exponent remains positive in some C^∞-neighborhood of system (1)–(3).

Besides of the discrete system (1), we consider also the linear differential system

$$\dot{x} = A_\mu(t)x, \;\; x \in \mathbb{R}^2, \;\; t \geq 0 \tag{7}$$

with matrices $B_\mu(t) := \begin{cases} d_k(\mu)\,\mathrm{diag}\,[1,-1], & 2k-2 \leq t < 2k-1, \\ (\mu + \gamma_k(\mu) + b_k)J, & 2k-1 \leq t < 2k, \end{cases}$ where $k \in \mathbb{N}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and μ is a real parameter; the numbers b_k and the functions $d_k(\,\cdot\,)$, $\gamma_k(\,\cdot\,)$ are assumed to be as those in system (1).

The Cauchy operator $X_{A_\mu}(t,s)$, $t, s \geq 0$, of system (7) for any $n \in \mathbb{N}$ satisfies the equality $X_{A_\mu}(n, n-1) = B_\mu(n-1)$. Thus systems (1) and (7) are asymptotically equivalent.

The case

$$d_k(\mu) \equiv d_k \geq d > 0, \;\; \gamma_k(\,\cdot\,) \equiv 0, \;\; k \in \mathbb{N} \tag{8}$$

was considered in [7, 8].

In [7], under the constraint $d_k \equiv d > 4\ln 2$, it is shown that the upper Lyapunov exponent of system (7) is positive in a set of parameter μ values of positive Lebesgue measure. The result of [8] implies the absence of any upper bound of the norm for system (7) solutions uniform with respect to $\mu \in \mathbb{R}$ and $t \geq 0$.

We denote by $\rho(H)$ the spectral radius of the matrix H, i.e., the maximum of the absolute values of its eigenvalues.

Due to Theorem 2 in [1], for any 2×2-matrices $H_1, \ldots, H_n$, $n \in \mathbb{N}$, with unit determinant the equality holds

$$\frac{1}{2\pi}\int_0^{2\pi} \ln \rho(H_n U(\theta) \cdots H_1 U(\theta))\, d\theta = \sum_{k=1}^{n} \ln\Big(\frac{\|H_k\| + \|H_k\|^{-1}}{2}\Big). \tag{9}$$

Another proof of this formula is obtained in [3] using the geometry of the action of $SL(2,\mathbb{R})$ on the projective line.

From equality (9) we have

Lemma 1. *Under conditions* (8) *we have the estimate*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln \|X_{B_\mu}(2n, 0)\|\, d\mu \geq \sum_{k=1}^{n} \ln\Big(\frac{d_k + d_k^{-1}}{2}\Big).$$

The next lemma is a general form of the statement previously used in [7].

Lemma 2. *Suppose for some constants $C, h > 0$ and an infinitely increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}$ it holds that*

$$\|X_{B_\mu}(t_k, 0)\| \leq e^{C\,t_k}, \quad \frac{1}{t_k}\int\limits_0^{2\pi} \ln\|X_{B_\mu}(t_k, 0)\|\,d\mu \geq h, \quad k \in \mathbb{N}.$$

Then the major Lyapunov exponent of system (7_μ) *is positive in the set of parameter* μ *values of positive Lebesgue measure.*

Lemmas 1 and 2 allow us to generalize Herman's estimate (4) to the general case of systems (1), (8), not necessarily quasi-periodic. Specifically, we have the following statement.

Theorem 1. *Let conditions* (8) *be satisfied and the sequence* $\{d_k\}_{k=1}^{\infty}$ *be bounded. Then the upper Lyapunov exponent of system* (1_μ) *is positive for all* μ *in some set of positive Lebesgue measure.*

For any natural number n, we denote by $\nu_2(n)$ the maximum power of 2 that divides n. Thus, the equality $n = 2^{\nu_2(n)}p(n)$ holds, where $p(n)$ is odd.

The main result of this report is the next

Theorem 2. *Let* $\{\alpha_m\}_{m=1}^{\infty}$ *be an arbitrary sequence of real numbers, the sequence* $\{b_k\}_{k=1}^{\infty}$ *be defined by the equality*

$$b_k = \alpha_{1+\nu_2(k)}, \quad k \in \mathbb{N}, \tag{10}$$

and the functions d_k *satisfy the conditions* $d_k(\,\cdot\,) \equiv d(\,\cdot\,) > 0$ *for all* $k \in \mathbb{N}$*; besides of that assume the functions* $d(\,\cdot\,)$ *and* $\gamma(\,\cdot\,)$ *to be differentiable on* $\mathbb{R}$, π*-periodic and such that the following inequalities hold*

$$\gamma'(\mu) > 2|d'(\mu)| - 2^{-1}, \quad \mu \in \mathbb{R}, \tag{11}$$

$$\int\limits_0^{\pi} d(\mu)\,d\mu > 2^{13}. \tag{12}$$

Then the largest Lyapunov exponent $\lambda_{\max}(A_\mu)$ *of system* (7) *is positive for all* μ *from some set of positive Lebesgue measure.*

If the matrix $A_\mu(\,\cdot\,)$ is defined by conditions (10), Lemma 2 from [6] asserts that the following equality holds:

$$X_{A_\mu}(2^{k+1}, 0) = U(\alpha_{k+1} - \alpha_k)X^2_{A_\mu}(2^k, 0), \quad k \in \mathbb{N}. \tag{13}$$

Systems, whose coefficients satisfy (10), allow one to construct one-parameter families with various asymptotic properties. For example, as shown in [6] after formula (32), if the sequence $\{\alpha_n\}_{n=1}^{\infty}$ converges, then the matrix $A_\mu(\,\cdot\,)$ is the uniform limit as $t \to \infty$ of periodic matrices sequence. V. M. Millionshchikov used these systems in [10,11] (see also [5]) to prove the existence of irregular under Lyapunov linear differential systems with limit-periodic and quasi-periodic coefficients.

In [9] we proved the existence of the parameter value $\mu \in \mathbb{R}$, such that the corresponding system (7) is unstable, in more general case when $d(\,\cdot\,)$ is supposed to be an arbitrary continuous function, $\gamma(\,\cdot\,) \equiv 0$ and conditions (10) hold.

In the rest of our report we will assume that the conditions of Theorem 2 are satisfied.

Now let us explain the method of its proof.

We suppose $\gamma(\mu) \equiv 0$ for simplicity.

For any fixed $0 \neq z \in \mathbb{R}^2$, let $x_\mu(t)$ denote the solution of system (7) that satisfies the initial condition $x_\mu(0) = z$, and let $\varphi_\mu(t)$ be its polar angle, which is a continuous function of two variables t and μ (its values are not limited to any interval of length 2π).

On even time intervals of the form $(2k-1, 2k)$, $k \in \mathbb{N}$, any solution of system (7) rotates by an angle proportional to μ.

Therefore, for any μ_0, the difference between the polar angles of solutions with $\mu = \mu_0$ and $\mu = \mu_0 + \pi$ increases on π.

On odd intervals of the form $(2k, 2k+1)$, by the diagonality of the coefficient matrix, any solution does not leave a certain coordinate plane.

Thus the difference between φ_μ and $\varphi_{\mu+\pi}$ remains the same, since the coefficients of system (7) are π-periodic with respect to μ, and therefore this difference is a multiple of π.

By induction, we obtain the equality

$$\varphi_{\mu+\pi}(2n) - \varphi_\mu(2n) = n\pi, \quad n \in \mathbb{N}. \tag{14}$$

Let us take two solutions of system (7) for the parameter values $\mu = \mu_0$ and $\mu = \mu_0 + \Delta\mu$, and such that their norm at the moment $t = 2k$ are equal to unity.

For small $\Delta\mu$, they are close to each other, and the distance between them at $t = 2k+1$ will be not much more bigger than the coefficients difference of the corresponding systems (7) in the interval $(2k, 2k+1)$.

The latter does not exceed approximately $|d'_\mu|\,\Delta\mu$.

Hence, as a part the difference between the polar angles of these solutions at this moment is not greater than $\approx |d'_\mu|\,\Delta\mu$.

Thus, because of $\varphi_{\mu+\Delta\mu} - \varphi_\mu$ increases by $\Delta\mu$ over time interval $(2k-1, 2k)$, if $|d'_\mu| \ll 1$, then the monotonic growth of the polar angle with respect to μ, assumed at the moment $t = 2k-1$, will also hold at $t = 2k+1$.

Therefore, by induction, we obtain the inequality stated in Lemma 1

$$(\varphi_\mu(k))'_\mu > 0, \quad k \in \mathbb{N}. \tag{15}$$

Lemma 2 establishes the symmetry in singular value decomposition of the Cauchy operator $X_A(\theta, \tau)$ of system (7) for time interval from 0 to $2^k - 1$, namely, the equality of rotation angles in right and left sides. So, it can be represented as (here $D_k = \operatorname{diag}[r_k, r_k^{-1}]$ for some $0 \neq r_k \in \mathbb{R}$)

$$X_k := X_A(2^k - 1, 0) = U(\psi_k) D_k U(\psi_k), \quad \psi_k \in \mathbb{R}. \tag{16}$$

Similarly to formula (13), we prove the equality

$$X_{k+1} = X_k U(\mu + b_{2^k - 1}) X_k. \tag{17}$$

If relation (16) is valid at k-th step, then the mentioned symmetry together with (17) implies the same symmetry at $k+1$-th step, which, by induction, gives representation (16) for all k.

Denote $y = U(-\psi_{k,\mu})e_1$, $e_1 = (1,0)^t$. The vectors $D_{k,\mu}U(\psi_{k,\mu})y$ and e_1 are obviously collinear.

Hence, in the case $(\psi_{k,\mu})'_\mu < 0$, if ν is little bigger than μ, by representation (16), it is easy to get the inclusion of the vector $X_{k,\nu}y$ to the right-hand neighborhood of $U(\psi_{k,\mu})e_1$.

While (15) implies that the vector $X_{k,\nu}y$ belongs to the left-hand neighborhood of the vector $X_{k,\mu}y \overset{(16)}{=} U(\psi_{k,\mu})e_1$.

So, we have the contradiction that proves the inequality

$$(\psi_{k,\mu})'_\mu > 0. \tag{18}$$

Further, if $\psi_{k,\mu}$ is a multiple of $\frac{\pi}{2}$, then the matrix X_k is diagonal. Thus the polar angle φ of the vector $X_k e_1$ satisfies for some integer m the equality

$$\varphi(X_k e_1) = m\pi. \tag{19}$$

Because of (18) the value of ψ_k strongly increases with respect to μ, therefore the segment $[0,\pi]$ is divided into intervals, at whose boundary points the angles $\psi_{k,\mu}$ are some multiplies of $\frac{\pi}{2}$.

It follows from (19) that their quantity does not exceed the number of vector $X_{k,\mu}e_1$ intersections with the Ox_1 axis.

Equality (14) means that the latter is not bigger than 2^{k-1}.

Thus we have asserted the estimate

$$\psi_{k,\pi} - \psi_{k,0} \le 2^{k-2}. \tag{20}$$

Denote $\theta_\mu = 2\psi_k + b_{2^k-1} + \mu$. By use of (17) we have

$$\|X_{k+1}\| \overset{(16),(17)}{=} \|D_k U(\theta_\mu) D_k\| = \begin{pmatrix} r_k^2 \cos\theta_\mu & -\sin\theta_\mu \\ \sin\theta_\mu & r_k^{-2}\cos\theta_\mu \end{pmatrix} \approx r_k^2 |\cos\theta_\mu|. \tag{21}$$

(18) and (20) gives the continuity and monotonic growth of θ_μ with respect to μ, together with the inequality $\theta_{\mu+\pi} - \theta_\mu \le n\pi$, where $n := 2^{k-1} + 1$.

Therefore, there exist no more than n intervals I_j, where $|\cos\theta_\mu| < \frac{1}{n}$.

By (18), $\theta'_\mu \ge 1$. Hence, the length of each I_j does not substantially exceed $\frac{1}{n}$. Moreover, by the same reason we have approximately $|\cos\theta_\mu| \approx \frac{1}{n}$ for prevailing set of points in I_j.

Thus, in integral average, a similar inequality will hold on the entire interval $[0,\pi]$:

$$|\cos\theta_\mu| \ge \varkappa_n, \quad \varkappa_n \approx \frac{1}{n}\,. \tag{22}$$

It gives us the estimates

$$L_{k+1} := \int_0^\pi \ln\|X_{k+1}\|\,d\mu \overset{(21)}{\ge} 2\int_0^\pi \ln r_{k,\mu}\,d\mu + \int_0^\pi \ln|\cos\theta_\mu|\,d\mu \overset{(16),\,(22)}{\ge} 2L_k - \ln\varkappa_n \approx 2L_k - k. \tag{23}$$

Increasing r_1, one can choose the norm L_1 so big that, by (23), L_{k+1} does not significantly less than $2L_k$. This, by induction, leads to the positivity of the integral of system (7) upper Lyapunov exponent.

References

[1] A. Avila and J. Bochi, A formula with some applications to the theory of Lyapunov exponents. *Israel J. Math.* **131** (2002), 125–137.

[2] A. Avila and R. Krikorian, Monotonic cocycles. *Invent. Math.* **202** (2015), no. 1, 271–331.

[3] A. T. Baraviera, J. Lopes Dias and P. Duarte, On the Herman–Avila–Bochi formula for Lyapunov exponents of $SL(2,\mathbb{R})$-cocycles. *Nonlinearity* **24** (2011), no. 9, 2465–2476.

[4] M.-R. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2. (French) [A method for determining the lower bounds of Lyapunov exponents and some examples showing the local character of an Arnol'd-Moser theorem on the two-dimensional torus] *Comment. Math. Helv.* **58** (1983), no. 3, 453–502.

[5] A. V. Lipnitskiĭ, On V. M. Millionshchikov's solution of the Erugin problem. (Russian) *Differ. Uravn.* **36** (2000), no. 12, 1615–1620; translation in *Differ. Equ.* **36** (2000), no. 12, 1770–1776.

[6] A. V. Lipnitskiĭ, On the singular and higher characteristic exponents of an almost periodic linear differential system that depends affinely on a parameter. (Russian) *Differ. Uravn.* **42** (2006), no. 3, 347–355; translation in *Differ. Equ.* **42** (2006), no. 3, 369–379.

[7] A. V. Lipnitskiĭ, On the positivity of the upper Lyapunov exponent in one-parameter families of linear differential systems. (Russian) *Differ. Uravn.* **45** (2009), no. 8, 1095–1101; translation in *Differ. Equ.* **45** (2009), no. 8, 1116–1123.

[8] A. V. Lipnitskii, Lower bounds for the norm of solutions of linear differential systems with a linear parameter. (Russian) *Differ. Uravn.* **50** (2014), no. 3, 412–416; translation in *50* (2014), no. 3, 410–414.

[9] A. V. Lipnitskii, On the instability of Millionshchikov linear differential systems with continuous dependence on a real parameter. (Russian) *Differ. Uravn.* **58** (2022), no. 4, 470–476; translation in *Differ. Equ.* **58** (2022), no. 4, 468–474.

[10] V. M. Millionshchikov, Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients. (Russian) *Differencial'nye Uravnenija* **4** (1968), 391–396; translation in *Differ. Equations* **4** (1968), no. 3, pp. 203–205 (1972).

[11] V. M. Millionshchikov, A proof of the existence of nonregular systems of linear differential equations with quasiperiodic coefficients. (Russian) *Differencial'nye Uravnenija* **5** (1969), 1979–1983; translation in *Differ. Equations* **5** (1969), no. 11, 1475–1478 (1972).

[12] E. Sorets and Th. Spencer, Positive Lyapunov exponents for Schrödinger operators with quasiperiodic potentials. *Comm. Math. Phys.* **142** (1991), no. 3, 543–566.

[13] L.-S. Young, Lyapunov exponents for some quasi-periodic cocycles. *Ergodic Theory Dynam. Systems* **17** (1997), no. 2, 483–504.