# Averaging in an Optimal Control Problem for a Nonlinear-in-Control Integro-Differential System on the Infinite Interval

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**Abstract**

The paper studies an optimal control problem for a nonlinear-in-control integro-differential system with rapidly oscillating coefficients on the infinite interval. Using the averaging method, the convergence of optimal controls, trajectories, and the quality functional from the original problem to those of the averaged one is established. The results justify the applicability of the averaging approach to nonlinear integro-differential control systems and considerably simplify their analysis while preserving essential dynamical properties.

## 1 Problem statement

Let us consider a nonlinear optimal control problem on the half-axis with a small parameter and rapidly oscillating coefficients.

$$\begin{cases} \dot{x} = X\Big(\dfrac{t}{\varepsilon}, x(t), \displaystyle\int_0^t \varphi(t,s,x(s))\,ds, u(t)\Big), \\ x(0,u(0)) = x_0, \end{cases} \tag{1.1}$$

where the cost function is given by

$$J_\varepsilon[u] = \int_0^\infty e^{-\gamma t} L(t, x(t), u(t))\,dt \longrightarrow \inf, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, $\gamma > 0$ is a fixed constant characterizing the discount, $x$ is a phase vector from $\mathbb{R}^d$, $u(t)$ is an $m$-dimensional control vector taking values in some set $U \subset \mathbb{R}^m$, $t \geq 0$.

Denote

$$\varphi_1(t,x) = \int\limits_0^t \varphi(t,s,x)\,ds.$$

Assume that the following limit exists uniformly with respect to $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$

$$\lim_{\varepsilon\to 0}\int\limits_0^t \Big[X\Big(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau,x), u\Big) - X_0(x,u)\Big]\,d\tau = 0, \tag{1.3}$$

for $t \geq 0$.

To the optimal control problems on the semi-axis (1.1), (1.2) with rapidly oscillating coefficients, we associate a simpler averaged control problem

$$\begin{cases} \dot{\xi} = X_0(\xi, u(t)), \\ \xi(0, u(0)) = x_0, \end{cases} \tag{1.4}$$

with the cost function

$$J[u] = \int\limits_0^\infty e^{-\gamma t} L(t, \xi(t), u(t))\,dt \longrightarrow \inf. \tag{1.5}$$

In what follows $|\cdot|$ denotes the norm of a vector in a finite-dimensional Euclidean space, and $\|\cdot\|$ denotes the matrix norm consistent with the vector norm.

For problem (1.1), (1.2) and its averaged problem (1.4), (1.5), we assume the following conditions.

**Condition 1.1.** *Admissible controls are $m$-dimensional vector functions $u(t)$ which, for almost all $t \geq 0$, take values in some compact set $U \subset \mathbb{R}^m$, and $u(\cdot)$ for each $T > 0$ belongs to a compact set $U_T$ in $L^p(0,T)$ for some $p \geq 1$.*

**Condition 1.2.** *The function $X(t,x,y,u)$ is defined and continuous with respect to all variables in the domain $Q_0 = \{t \geq 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^m\}$, and the following conditions are satisfied:*

(1) *$X(t,x,y,u)$ is bounded in $Q_0$, i.e., there exists a constant $M > 0$ such that*

$$|X(t,x,y,u)| \leq M,$$

*for all $(t,x,y,u) \in Q_0$.*

(2) *$X(t,x,y,u)$ satisfies the Lipschitz condition in $Q_0$ with respect to $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$, with constant $\lambda$:*

$$|X(t,x,y,u) - X(t,x_1,y_1,u_1)| \leq \lambda\big(|x-x_1| + |y-y_1| + |u-u_1|\big),$$

*for all $(t,x,y,u), (t,x_1,y_1,u_1) \in Q_0$.*

**Condition 1.3.** *The function $\varphi(t,s,x)$ is defined and continuous in the domain $Q_1 = \{t \geq 0,\ s \geq 0,\ x \in \mathbb{R}^d\}$ and satisfies linear growth and a Lipschitz condition with respect to $x$. That is, there exists a constant $L_\varphi > 0$ such that*

$$|\varphi(t,s,x) - \varphi(t,s,x_1)| \leq L_\varphi |x - x_1|,$$
$$|\varphi(t,s,x)| \leq L_\varphi(1 + |x|).$$

**Condition 1.4.** *The limit in* (1.3) *exists uniformly with respect to* $x \in \mathbb{R}^d$ *and* $u \in U$.

**Condition 1.5.** *The scalar function* $L(t,x,u)$ *is defined and continuous with respect to all its arguments in the domain* $Q_2 = \{t \geq 0,\ x \in \mathbb{R}^d,\ u \in U\}$, *and satisfies a linear growth condition with respect to* $x$ *and* $u$ *in* $Q_2$ *with constant* $M > 0$; *specifically,*

$$|L(t,x,u)| \leq M\big(1 + |x| + |u|\big).$$

## 2 Main results

First, we note that from Conditions 1.1, 1.2 and Theorem 3.1 from [4] it follows that for each admissible control $u(t)$, the solution $x(t,u)$ of the Cauchy problem (1.1) exists, is unique on $[0,\infty)$, and is an absolutely continuous function. Moreover, the solution of problem (1.1) satisfies the estimate

$$|x(t)| \leq |x_0| + Mt. \tag{2.1}$$

Hence, for the cost function (1.2), taking into account (2.1) and Condition 1.5, we obtain

$$|J_\varepsilon[u]| \leq \int_0^\infty e^{-\gamma t} M\big(1 + |x(t)| + |u(t)|\big)\,dt \leq \int_0^\infty e^{-\gamma t} M(1 + |x_0| + Mt)\,dt + C\int_0^\infty e^{-\gamma t}|u|\,dt < \infty,$$

for some constant $C > 0$.

Thus, the cost function (1.2) is well-defined for all admissible controls.

From Condition 1.4 it follows that similar conclusions hold for the averaged problem.

The following lemma will be needed later, it guarantees the convergence of the solutions of the exact system (1.1) to the corresponding solutions of the averaged system (1.4). Let $T > 0$ be fixed.

**Lemma 2.1.** *Assume that Conditions 1.2–1.4 hold. Then, if* $u_\varepsilon \to u_0$ *as* $\varepsilon \to 0$ *in the norm of the space* $L^p(0,T)$, *the solution of the Cauchy problem* (1.1) *satisfies*

$$x_\varepsilon(t) \rightrightarrows \xi_0(t),\ \ \varepsilon \to 0,$$

*on* $[0,T]$, *where* $\xi_0(t)$ *is the solution of the Cauchy problem corresponding to* $u = u_0$.

*Proof.* In addition to systems (1.1) and (1.4), consider the auxiliary system

$$\dot z_\varepsilon = X\bigg(\frac{t}{\varepsilon}, z_\varepsilon(t), \int_0^t \varphi(t,s,z_\varepsilon(s))\,ds, u_0\bigg),\ \ z_\varepsilon(0,u_0(0)) = x_0.$$

Then

$$\begin{aligned}
&|z_\varepsilon(t) - x_\varepsilon(t)| \\
&\quad \leq \int_0^t \bigg| X\bigg(\frac{s}{\varepsilon}, z_\varepsilon(s), \int_0^s \varphi(s,\tau,z_\varepsilon(\tau))\,d\tau, u_0(s)\bigg) - X\bigg(\frac{s}{\varepsilon}, x_\varepsilon(s), \int_0^s \varphi(s,\tau,x_\varepsilon(\tau))\,d\tau, u_\varepsilon(s)\bigg)\bigg|\,ds \\
&\quad \leq \lambda \int_0^t \bigg(|z_\varepsilon(s) - x_\varepsilon(s)| + L_\varphi \int_0^s |z_\varepsilon(\tau) - x_\varepsilon(\tau)|\,d\tau\bigg)\,ds + \lambda \int_0^T |u_\varepsilon(s) - u_0|\,ds \\
&\quad = \lambda \int_0^t |z_\varepsilon(s) - x_\varepsilon(s)|\,ds + \lambda L_\varphi \int_0^t\int_0^s |z_\varepsilon(\tau) - x_\varepsilon(\tau)|\,d\tau\,ds + \lambda \int_0^T |u_\varepsilon(s) - u_0|\,ds.
\end{aligned}$$

By the generalized Gronwall–Bellman lemma,

$$|z_\varepsilon(t) - x_\varepsilon(t)| \le \lambda T^{\frac{1}{q}} \|u_\varepsilon - u_0\|_{L^p} \exp(\lambda T(1 + L_\varphi T)).$$

Furthermore, by Lemma 3.1 [3], it follows that $z_\varepsilon(t)$ converges to $\xi_0(t)$ as $\varepsilon \to 0$ uniformly on $[0, T]$.

Note that replacing the class $L^2(0,T)$ with $L^p(0,T)$ does not affect on the proof, since any function in $L^p(0,T)$, $p \ge 1$, can be approximated arbitrarily closely by piecewise constant functions. □

*Remark.* It follows from this theorem and the uniqueness of the Cauchy problem solution that if $u_\varepsilon \to u_0(t)$ as $\varepsilon \to 0$ in the norm $L^p(0,T)$ for each $T > 0$, then $x_\varepsilon(t)$ converges to $\xi_0(t)$ uniformly on every interval $[0,T]$. Hence, $x_\varepsilon(t) \to \xi_0(t)$ as $\varepsilon \to 0$ for any $t \ge 0$. Therefore, in this case we obtain pointwise convergence of the solutions of the original problem to the corresponding solutions of the averaged one.

The next theorem establishes a connection between optimal controls, optimal trajectories, and cost functionals of the exact problem (1.1), (1.2) and the averaged problem (1.4), (1.5).

Denote $J_\varepsilon^* = \inf_u J_\varepsilon[u]$, $J_0^* = \inf_u J_0[u]$, where the infimum is taken over all admissible controls.

**Theorem 2.1.** *Suppose that Conditions 1.1–1.5 hold, and there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ problems* (1.1), (1.2) *and* (1.4), (1.5) *have solutions $(x_\varepsilon^*(t), u_\varepsilon^*(t))$, $(\xi^*(t), u^*(t))$, respectively. Then the following variational relations hold:*

(1) *$J_\varepsilon^* \to J_0^*$ as $\varepsilon \to 0$;*

(2) *for each $\eta > 0$ there exists $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$:*

$$\left| J_\varepsilon^* - J_\varepsilon[u^*] \right| < \eta,$$

*i.e., the optimal control of the averaged problem is nearly optimal for the original one;*

(3) *there exists a sequence $\varepsilon_n \to 0$, $n \to \infty$ such that*

$$x_{\varepsilon_n}^*(t) \to \xi^*(t), \tag{2.2}$$

*uniformly on every interval $[0,T]$, $T > 0$, and*

$$u_{\varepsilon_n}^*(t) \to u^*(t), \tag{2.3}$$

*almost everywhere on $[0, \infty)$, and $u_{\varepsilon_n}^*(\,\cdot\,)$ converges to $u^*(\,\cdot\,)$ in the norm $L^p(0,T)$ for each $T > 0$.*

*If the averaged problem* (1.4), (1.5) *has a unique solution, then the convergences in* (2.2), (2.3) *hold for all $\varepsilon > 0$.*

We approximate the solutions of the optimal control problem (1.1) with the cost functional

$$I_\varepsilon[u] = \int_0^\infty e^{-\gamma t} \big( A(t, x(t)) + B(t, u(t)) \big)\, dt \longrightarrow \inf, \tag{2.4}$$

on the half-line by the solutions of the averaged control problems on the interval $[0,T]$ as $T \to \infty$.

Let $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ be solution of problem (1.1), (2.4) on the half-axis. Fix $T > 0$ and consider problem (1.4) with the cost function

$$I_{0T}[u] = \int_0^T e^{-\gamma t}\big(A(t, \xi(t)) + B(t, u(t))\big)\, dt \longrightarrow \inf, \tag{2.5}$$

on the interval $[0, T]$. By Theorem 2.1 [3], this problem has an optimal solution $(\xi^*_T, u^*_T)$ on $[0, T]$.

Let the cost function (2.5) satisfy the following condition.

**Condition 2.1.** *The scalar functions $A(t, x)$, $B(t, u)$, and $\frac{\partial B}{\partial u}(t, u)$ are defined for $t \geq 0$, $x \in \mathbb{R}^d$, $u \in V$, and continuous with respect to all their arguments. Moreover,*

(1) *$A(t, x) \geq 0$ and satisfies a linear growth condition in $x \in \mathbb{R}^d$ with a constant $M$, i.e., $|A(t, x)| \leq M(1 + |x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$;*

(2) *there exist constants $a > 0, a_1 > 0$ such that $a_1|u|^2 \geq B(t, u) \geq a|u|^2$ for all $t \geq 0$, $B(t, u)$ is convex with respect to $u \in V$, and there exists $a_2 > 0$ such that*

$$\Big|\frac{\partial B}{\partial u}(t, u)\Big| \leq a_2|u|.$$

For problem (1.4), define an admissible control on $[0, \infty)$ by

$$\overline{u}_{T,\infty}(t) = \begin{cases} u^*_T(t), & t \in [0, T], \\ 0, & t > T, \end{cases}$$

and let $\xi_{T,\infty}$ denote the corresponding admissible trajectory.

Let $I^*_{0T} = \inf\limits_u I_{0T}[u]$.

**Theorem 2.2.** *Let Conditions 1.1–1.4 and 2.1 be satisfied. Then*

(1)
$$|I^*_\varepsilon - I^*_{0T}| \to 0 \ \text{ as } \ \varepsilon \to 0, \ \ T \to \infty;$$

(2) *there exist sequences $T_n \to \infty$, $\varepsilon_n \to 0$ such that for any $t > 0$ we have*

$$|x^*_{\varepsilon_n}(t) - \xi_{T_n,\infty}(t)| \to 0, \ \ T_n \to \infty, \ \ \varepsilon_n \to 0; \tag{2.6}$$

*If the averaged problem (1.4), (2.5) has a unique solution, then the convergence in (2.6) holds for all $T \to \infty$ and $\varepsilon \to 0$.*

*Proof.* We show that $\forall\,\eta > 0$ there exist $T_0$ and $\varepsilon_0 > 0$ such that $\forall\,T > T_0$, $\forall \varepsilon < \varepsilon_0$:

$$|I^*_{0T} - I^*_\varepsilon| < \eta.$$

Indeed, we have

$$|I^*_{0T} - I^*_\varepsilon| \leq |I^*_{0T} - I^*_0| + |I^*_0 - I^*_\varepsilon|.$$

The first term does not depend on $\varepsilon$ and tends to 0 as $T \to \infty$ by Theorem 2.5 [2]. The convergence of the second term to zero as $\varepsilon \to 0$ follows from item 1) of Theorem 2.1.

(2) The proof of this item follows from the inequality

$$|x^*_{\varepsilon_n}(t) - \xi_{T_n,\infty}(t)| \leq |x^*_{\varepsilon_n}(t) - \xi^*(t)| + |\xi^*(t) - \xi_{T_n,\infty}(t)|.$$

The convergence of the first term to zero as $\varepsilon_n \to 0$, for each fixed $t > 0$, follows from Theorem 2.1. The convergence $|\xi^*(t) - \xi_{T_n,\infty}(t)| \to 0$ as $T_n \to \infty$ follows from item (iv) of Theorem 2.5 [2]. □

## References


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