Structure of Positive Solutions of Generalized Thomas–Fermi Type Differential Equations

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1 Introduction

This is an interim report on our new study on the structure of global positive solutions of generalized Thomas–Fermi type differential equations of the form

$$\big(p(t)\varphi_\alpha(x')\big)' = q(t)\varphi_\beta(x), \quad t \geqq a, \tag{A}$$

where α and β are different positive constants, $p(t)$ and $q(t)$ are positive continuously differentiable functions on $I = [a,\infty)$, $a \geqq 0$, and the symbol φ_γ with a constant $\gamma > 0$ denotes the operator on $C(I)$ defined by

$$\varphi_\gamma(u(t)) = |u(t)|^\gamma \operatorname{sgn} u(t), \quad u \in C(I).$$

We are concerned exclusively with those solutions of (A) which exist on an infinite interval of the form $J = [T,\infty)$, $T \geqq a$. It is well-known that such a solution may have at most one zero on J, that is, $x(t)$ is nonoscillatory on J. Moreover, if $x(t)$ satisfies (A) on J, then so does $-x(t)$. This is why we are allowed to restrict the object of our study to the set of positive solutions of (A).

A few words about the structure of equations of the form (A). Equation (A) is constructed by two elements, the *exponents* $\{\alpha,\beta\}$ and the *coefficients* $\{p(t),q(t)\}$. The exponents determine the nonlinearity of (A). It is customary to say that (A) is *sublinear* if $\alpha > \beta$ and *superlinear* if $\alpha < \beta$. As for the coefficients, we think that the *size* of equation (A) can be measured by the pair of their integrals $\{\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt, \int_a^\infty q(t)\,dt\}$. According to this measure, the class of all equations of the form (A) is divided into the following four disjoint categories of equations:

Category I: $\displaystyle\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty, \quad \int_a^\infty q(t)\,dt < \infty;$

Category II: $\displaystyle\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty, \quad \int_a^\infty q(t)\,dt < \infty;$

Category III: $\displaystyle\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty, \quad \int_a^\infty q(t)\,dt = \infty;$

Category IV: $\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty, \quad \int\limits_a^\infty q(t)\,dt = \infty.$

Given a positive solution $x(t)$ of (A) on $J = [T, \infty)$ we put

$$D_\alpha x(t) = p(t)\varphi_\alpha(x'(t)), \;\; t \geqq T,$$

and call it the *quasi-derivative* of $x(t)$. From (A) it always follows that $D_\alpha x(t)$ is increasing on J. It is possible that $D_\alpha x(t)$ is either positive or negative on J, so that $x(t)$ is either increasing or decreasing on J. Thus, for any positive solution on J there exist the limits

$$x(\infty) = \lim_{t\to\infty} x(t), \quad D_\alpha x(\infty) = \lim_{t\to\infty} D_\alpha x(t),$$

in the extended real number system. The pair $\{x(\infty), D_\alpha x(\infty)\}$, referred to as the *terminal state* of $x(t)$, is a crucial indicator of the asymptotic behavior at infinity of the solution $x(t)$. For instance, the possible patterns of terminal states are used to classify all the solutions of (A) into suitable subclasses according to their asymptotic behaviors as $t \to \infty$.

This report centers around a summary of our analysis of positive solutions of equation (A) of category I. Use is made of the following functions and symbols:

$$P_\alpha(t) = \int\limits_a^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad P_\alpha(t,T) = \int\limits_T^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad P_\alpha = P_\alpha(\infty); \tag{1.1}$$

$$Q(t) = \int\limits_a^t q(s)\,ds, \quad Q(t,T) = \int\limits_T^t q(s)\,ds, \quad Q = Q(\infty); \tag{1.2}$$

$$\pi_\alpha(t) = \int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds, \quad \rho(t) = \int\limits_t^\infty q(s)\,ds. \tag{1.3}$$

2 Positive increasing solutions of equation (A) of category I

We begin by analyzing the behavior of positive increasing solutions of equation (A) of category I. Let $x(t)$ be one such solution on $J = [T, \infty)$. Letting $t \to \infty$, it goes to a finite limit $x(\infty) < \infty$, in which case it is bounded on J, or else it grows to $x(\infty) = \infty$, in which case it is unbounded on J.

Integrating (A) twice over $[T, t]$, we are led to the integral equation

$$x(t) = c + \int\limits_T^t p(s)^{-\frac{1}{\alpha}} \left(d + \int\limits_T^s q(r)x(r)^\beta\,dr \right)^{\frac{1}{\alpha}} ds, \tag{2.1}$$

for $t \geqq T$, where $c = x(T) > 0$ and $d = D_\alpha x(T) \geqq 0$. The pair of constants $\{c, d\}$ is called the *initial state* of $x(t)$.

By estimating the right-hand side of (2.1) from above, we obtain the following inequality

$$x(t) \leqq \big(c^{1-\frac{\beta}{\alpha}} + P_\alpha(c^{-\beta}d + Q)^{\frac{1}{\alpha}}\big)x(t)^{\frac{\beta}{\alpha}}, \;\; t \geqq T,$$

which implies that if $\alpha > \beta$, then $x(t)$ satisfies

$$x(t) \leqq \big(c^{1-\frac{\beta}{\alpha}} + P_\alpha(c^{-\beta}d + Q)^{\frac{1}{\alpha}}\big)^{\frac{\alpha}{\alpha-\beta}}, \;\; t \geqq T, \tag{2.2}$$

and that if $\alpha < \beta$, then $x(t)$ satisfies

$$x(t) \geqq \left(c^{1-\frac{\beta}{\alpha}} + P_\alpha(c^{-\beta} d + Q)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}}, \quad t \geqq T. \tag{2.3}$$

We may denote by $M(c,d)$ the constants (formlly the same) on the right-hand sides of (2.2) and (2.3), since other four $\{\alpha, \beta, P_\alpha, Q\}$ are the structural constants of equation (A) itself.

Inequality (2.2) can be paraphrased as follows.

Theorem 2.1. *Assume that equation* (A) *is sublinear and of category* I*. Then, every positive increasing solution of* (A) *on* $[T,\infty)$*, if exists, is bounded there and tends to a limit not exceeding* $M(c,d)$ *as* $t \to \infty$*, where* $c = x(T) > 0$ *and* $d = D_\alpha x(T) \geqq 0$*.*

A question naturally arises as to the existence of bounded positive increasing solutions of (A). This question can be affirmatively answered for both sublinear and superlinear equations (A) of category I.

Theorem 2.2. *Assume that equation* (A) *of category* I *is either sublinear or superlinear. Let there be given a pair of constants* $\{c > 0, d \geqq 0\}$*. Then, there is an interval* $[T,\infty)$*,* $T \geqq a$*, on which equation* (A) *of category* I *possesses a bounded positive solution* $x(t)$ *satisfying the initial condition* $x(T) = c$ *and* $D_\alpha x(T) = d$*.*

To prove this theorem we proceed as follows. Choose $T > a$ so large that

$$(2d)^{\frac{\beta}{\alpha}} \int_T^\infty p(t)^{-\frac{1}{\alpha}}\, dt \leqq c, \quad (2c)^\beta \int_T^\infty q(t)\, dt \leqq d,$$

and define $\mathcal{X} = \{x \in C[T,\infty) :\ c \leqq x(t) \leqq 2c,\ t \geqq T\}$. Finally define the operator F by

$$Fx(t) = c + \int_T^t p(s)^{-\frac{1}{\alpha}} \left(d + \int_T^s q(r) x(r)^\beta\, dr\right)^{\frac{1}{\alpha}} ds,$$

and let it act on the set $\mathcal{X}$. Then, show that F is a continuous self-map of $\mathcal{X}$ whose image set $F(\mathcal{X})$ is locally uniformly bounded and locally equicontinuous on $[T,\infty)$. Thus by the Schauder–Tikhonov fixed point theorem F has a fixed point $x \in \mathcal{X}$, which is nothing else but a bounded positive increasing solution of (A).

Inequality (2.3) suggests that a superlinear equation of category I possibly has an unbounded positive increasing solution. That this is indeed the case confirmed by the following example.

Example. Let $\alpha < \beta$ and consider the differential equation

$$\left(e^{\alpha t} \varphi_\alpha(x')\right)' = e^{-\beta t} \varphi_\beta(x), \tag{2.4}$$

which is of category I on $[0,\infty)$. This equation has an unbounded positive increasing solution

$$x_0(t) = k \exp\left(\frac{\alpha+\beta}{\beta-\alpha} t\right), \quad \text{where} \quad k = \left[\frac{2\alpha\beta}{\beta-\alpha} \left(\frac{\alpha+\beta}{\beta-\alpha}\right)^\alpha\right]^{\frac{1}{\beta-\alpha}}. \tag{2.5}$$

For now we are unable to logically justify the existence of an unbounded positive solution (2.5) for the special superlinear equation (2.4). A much more difficult question is: Do all superlinear equations (A) have unbounded positive increasing solutions?

Returning to the existence of bounded positive increasing solutions again, we often have to solve the initial value problem for equation (A) in the sense that the interval $[T,\infty)$ of existence of solutions is prescribed in advance.

Theorem 2.3. *Consider an interval* $[T,\infty)$ *and let it be fixed. Let* $c>0$ *and* $d \geqq 0$ *be arbitrary constants. Then, any sublinear equation* (A) *of category* I *has a bounded positive solution on* $[T,\infty)$ *satisfying the initial condition* $x(T)=c$ *and* $D_\alpha x(T)=d$.

Proof. Using the constant $M(c,d)$ appearing in (2.2), we define the set $\mathcal{Y}$ of continuous functions on $[T,\infty)$ by

$$\mathcal{Y}=\big\{x\in C[T,\infty):\ c\leqq x(t)\leqq M(c,d),\ t\geqq T\big\},$$

and we let F operate on c and $\mathcal{Y}$. Then, we see that $c\leqq Fc(t)$ and $FM(c,d)(t)\leqq M(c,d)$ for $t\geqq T$, which means that F is a self-map of $\mathcal{Y}$. Moreover, it can be cheked routinely that the hypotheses of the Schauder–Tikhonov theorem are fulfilled by F. Therefore, F has a fixed element $x\in\mathcal{Y}$, which means that $x(t)$ is a bounded positive solution of (A) satisfying the prescribed initial condition. This outlines the proof of Theorem 2.3. □

It seems difficult to prove or disprove the superlinear version of Theorem 2.3.

3 Positive decreasing solutions of equation (A) of category I

We turn our attention to positive decreasing solutions of equation (A) of category I. A positive solution $x(t)$ of (A) is decreasing on $J=[T,\infty)$ if and only if its quasi-derivative $D_\alpha x(t)$ is negative there, i.e.,

$$D_\alpha x(t)=-p(t)(-x'(t))^\alpha<0\ \text{ for }\ t\geqq T.$$

Since $D_\alpha x(t)$ is increasing on J, it goes to a nonpositive limit $D_\alpha x(\infty)$ as $t\to\infty$, that is, either $D_\alpha x(\infty)<0$ or $D_\alpha x(\infty)=0$. In either of these cases $x(t)$ is decreasing and goes to a limit, positive or zero, as $t\to\infty$. Thus we see that any positive decreasing solution of (A) has one of the following four patterns for its terminal states:

(i) $\big\{x(\infty)>0,\ -\infty<D_\alpha x(\infty)<0\big\}$,

(ii) $\big\{x(\infty)=0,\ -\infty<D_\alpha x(\infty)<0\big\}$,

(iii) $\big\{x(\infty)>0,\ D_\alpha x(\infty)=0\big\}$,

(iv) $\big\{x(\infty)=0,\ D_\alpha x(\infty)=0\big\}$.

This fact makes it possible to divide the set of all positive decreasing solutions (A) into four disjoint subsets, each of which consists of solutions having the same pattern of terminal states. Each of these four subclasses (i)–(iv) is characterized by one of the corresponding integral equations (3.1)–(3.3) arranged below:

$$x(t)=\gamma+\int_t^\infty p(s)^{-\frac{1}{\alpha}}\left(\delta+\int_s^\infty q(r)x(r)^\beta\,dr\right)^{\frac{1}{\alpha}}ds,\quad t\geqq T, \tag{3.1}$$

where both γ and δ are positive constants;

$$x(t)=\int_t^\infty p(s)^{-\frac{1}{\alpha}}\left(\delta+\int_s^\infty q(r)x(r)^\beta\,dr\right)^{\frac{1}{\alpha}}ds,\quad t\geqq T, \tag{3.2}$$

where δ is a positive constant;

$$x(t)=\omega+\int_t^\infty p(s)^{-\frac{1}{\alpha}}\left(\int_s^\infty q(r)x(r)^\beta\,dr\right)^{\frac{1}{\alpha}}ds,\quad t\geqq T, \tag{3.3}$$

where ω is a constant;

$$x(t)=\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\bigg(\int\limits_s^\infty q(r)x(r)^\beta\,dr\bigg)^{\frac{1}{\alpha}}\,ds,\ \ t\geqq T. \tag{3.4}$$

These four integral equations can be solved without difficulty. To solve (3.1) for any positive $\gamma>0$ and $\delta>0$, we first choose $T>a$ so that

$$(2\delta)^{\frac{1}{\alpha}}\int\limits_T^\infty p(t)^{-\frac{1}{\alpha}}\,dt\leqq\gamma\ \ \text{and}\ \ (2\gamma)^\beta\int\limits_T^\infty q(t)\,dt\leqq\delta,$$

and define the set $\mathcal{U}=\{x\in C[T,\infty):\ \gamma\leqq x(t)\leqq 2\gamma,\ t\geqq T\}$. Then, we let the following integral operator G given by

$$Gx(t)=\gamma+\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\bigg(\delta+\int\limits_s^\infty q(r)x(r)^\beta\,dr\bigg)^{\frac{1}{\alpha}}\,ds,\ \ t\geqq T, \tag{3.5}$$

operate on $\mathcal{U}$, and confirm that all the hypotheses of the Schauder–Tikhonov theorem are fulfilled by G.

To solve (3.2) for any $\delta>0$ we first choose $T>a$ so that $\pi_\alpha(T)^\beta\rho(T)\leqq 2^{-\frac{\beta}{\alpha}}\delta^{1-\frac{\beta}{\alpha}}$, and define the set $\mathcal{U}_0=\{x\in C[T,\infty):\delta^{\frac{1}{\alpha}}\pi_\alpha(t)\leqq x(t)\leqq(2\delta)^{\frac{1}{\alpha}}\pi_\alpha(t),t\geqq T\}$. Then, let the integral operator G_0 given by

$$G_0x(t)=\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\bigg(\delta+\int\limits_s^\infty q(r)x(r)^\beta\,dr\bigg)^{\frac{1}{\alpha}}\,ds,\ \ t\geqq T,$$

operate on $\mathcal{U}_0$, and confirm that G_0 satisfies all hypotheses of the Schauder–Tikhonov theorem.

To solve (3.3) for any $\omega>0$, first choose $T>a$ so that $\pi_\alpha(T)^\alpha\rho(T)\leqq 2^{-\beta}\omega^{\alpha-\beta}$, and define the set $\mathcal{V}=\{x\in C[T,\infty):\ \omega\leqq x(t)\leqq 2\omega,\ t\geqq T\}$. Then, consider the integral operator H defined by

$$Hx(t)=\omega+\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\bigg(\int\limits_s^\infty q(r)x(r)^\beta\,dr\bigg)^{\frac{1}{\alpha}}\,ds,\ \ t\geqq T,$$

let H act on $\mathcal{V}$, and show that the Schauder–Tikhonov theorem can be applied to H.

It should be noted that the three results mentioned above are true for both sublinear and superlear cases of (A). However, the situation around the remaining equation (3.4) is different from the others. It is solvable only in the sublinear case. To see this we derive from (3.5) the inequalities

$$x(t)\leqq\pi_\alpha(t)\rho(t)^{\frac{1}{\alpha}}x(t)^{\frac{\beta}{\alpha}}\implies x(t)^{1-\frac{\beta}{\alpha}}\leqq\pi_\alpha(t)\rho(t)^{\frac{1}{\alpha}},\ \ t\geqq T. \tag{3.6}$$

Suppose that $\alpha<\beta$. Letting $t\to\infty$ in the second inequality of (3.6), we are led to $\infty\leqq 0$, a contradiction. This means that any superlinear equation (A) of category I can never have a positive decreasing solution decaying to 0 as $t\to\infty$. On the other hand, if $\alpha>\beta$, then (3.6) implies that any positive decreasing solution $x(t)$ tending to 0 as $t\to\infty$ satisfy the inequality

$$x(t)\leqq\big(\pi_\alpha(t)^\alpha\rho(t)\big)^{\frac{1}{\alpha-\beta}},\ \ t\geqq T. \tag{3.7}$$

The function on the right-hand side of (3.7) is denoted by $\sigma(t)$. Let us solve (3.4) with the help of $\sigma(t)$ as follows. Define the integral operator H_0 by

$$H_0x(t)=\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\bigg(\int\limits_s^\infty q(r)x(r)^\beta\,dr\bigg)^{\frac{1}{\alpha}}\,ds,\ \ t\geqq T,$$

where $T > a$ is given in advance, let it act on the set $\mathcal{V}_0$ defined by

$$\mathcal{V}_0 = \{x \in C[T,\infty) : \ \lambda\sigma(t) \leqq x(t) \leqq \sigma(t), \ t \geqq T\},$$

where λ is a positive constant less than 1, and finally show that H_0 has the properties that

$$H_0\sigma(t) \leqq \sigma(t) \ \text{ and } \ H_0\lambda\sigma(t) \geqq \lambda\sigma(t), \quad t \geqq T, \tag{3.8}$$

provided λ is sufficiently small. Clearly (3.8) implies that H_0 is a self-map of $\mathcal{V}_0$. Moreover, the continuity of H_0 and the relative compactness of $H_0(\mathcal{V}_0)$ can be verified with relative ease. It follows therefore that H_0 has a fixed element $x_0 \in \mathcal{V}_0$ which gives rise to a positive decreasing solution of (A) on the interval $[T,\infty)$ prescribed in advance.

All of the discussions on positive decreasing solutions are summarized in the following two theorems.

Theorem 3.1. *Any equation* (A) *of category* I*, either sublinear or superlinear, always has positive decreasing solutions with the following three patterns of terminal states:*

(i) $x(\infty) > 0, \ -\infty < D_\alpha x(\infty) < 0$*;*

(ii) $x(\infty) = 0, \ -\infty < D_\alpha x(\infty) < 0$*;*

(iii) $x(\infty) > 0, \ D_\alpha x(\infty) = 0$*.*

Theorem 3.2.

(i) *No superlinear equation* (A) *of category* I *has a positive decreasing solution* $x(t)$ *with the terminal state* $\{x(\infty) > 0, D_\alpha x(\infty) = 0\}$*.*

(ii) *Every sublinear equation* (A) *of category* I *always has a positive decreasing solution* $x(t)$ *with the terminal state* $\{x(\infty) = 0, D_\alpha(\infty) = 0\}$ *on any interval of the form* $[T,\infty)$*.*