

## On Well-Posed Nonlocal Boundary Value Problems for Second Order Linear Singular Differential Equations

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In our report, we consider the second order linear differential equation

$$u'' = p_1(t)u + p_2(t)u' + p_0(t) \quad (1)$$

with the boundary conditions

$$\int_a^b u(s) d\sigma_i(s) = c_i \quad (i = 1, 2). \quad (2)$$

Below everywhere it is assumed that  $p_i \in L_{loc}(]a, b[)$  ( $i = 0, 1, 2$ ), i.e.  $p_0, p_1$  and  $p_2$  are Lebesgue integrable functions on every closed interval contained in  $]a, b[$ , while  $\sigma_i : [a, b] \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) are functions of finite variations. Moreover, it is assumed that  $u(a)$  and  $u(b)$  are the right and the left limits of the function  $u$  at the points  $a$  and  $b$ , respectively.

Equation (1) is said to be **singular** if one of its coefficients has a nonintegrable singularity either at  $a$  or  $b$ , i.e. when

$$\int_a^b (|p_0(t)| + |p_1(t)| + |p_2(t)|) dt = +\infty.$$

If

$$\sigma_1(t) = \begin{cases} 0 & \text{for } t = a, \\ 1 & \text{for } a < t \leq b, \end{cases} \quad \sigma_2(t) = \begin{cases} 0 & \text{for } a \leq t < b, \\ 1 & \text{for } t = b, \end{cases}$$

then conditions (2) coincide with the Dirichlet conditions

$$u(a) = c_1, \quad u(b) = c_2.$$

In all other cases, i.e. when the linear functionals on the left-hand side of (2) are not concentrated at the points  $a$  and  $b$ , conditions (2) are said to be **nonlocal**. For example, the nonlocal conditions are

$$\sum_{k=1}^m \lambda_k u(a_k) = c_1, \quad \sum_{k=1}^n \mu_k u(b_k) = c_2, \quad (2')$$

where  $a = a_1 < \dots < a_m < b_1 < \dots < b_n = b$ .

We are interested exactly in the case where equation (1) is singular and conditions (2) are nonlocal.

Along with the nonhomogeneous problem (1),(2) it is natural to consider its corresponding homogeneous problem

$$u'' = p_1(t)u + p_2(t)u', \tag{1_0}$$

$$\int_a^b u(s) d\sigma_i(s) = 0. \tag{2_0}$$

(1), (2) is said to be a **Fredholm type problem** if it is uniquely solvable if and only if (1<sub>0</sub>), (2<sub>0</sub>) has only the trivial solution.

The singular nonlocal problem (1),(2) has not been studied previously. Namely, the question of whether it is of Fredholm type remained open. This question was first studied in [2] where the following theorem was proved.

**Theorem 1.** *If the conditions*

$$\int_a^b (t - a)(b - t)|p_i(t)| dt < +\infty \quad (i = 0, 1), \tag{3}$$

$$\int_a^b [p_2(t) \operatorname{sgn}(t - t_0)]_+ dt < +\infty, \quad t_0 \in ]a, b[ \tag{4}$$

hold, then every solution of equation (1) has finite limits at the points  $a$  and  $b$ , and problem (1),(2) is of Fredholm type.

*Remark.* Conditions (3) are unimprovable. Indeed, let us consider problem (1),(2'), where  $p_0(t) \equiv p_2(t) \equiv 0$ , while  $p_1$  is a nonnegative function such that

$$\int_a^b (t - a)(b - t)p_1(t)dt = +\infty.$$

Then it is easy to see that the corresponding to (1),(2') homogeneous problem has only the trivial solution. Problem (1),(2') itself is unsolvable. Hence, by Theorem 1, it follows that problem (1),(2') is uniquely solvable if and only if

$$\int_a^b (t - a)(b - t)p_1(t)dt < +\infty.$$

**Example.** If the conditions

$$p_i(t) = \frac{p_{0i}(t)}{(t - a)^{\lambda_i}(b - t)^{\mu_i}} \quad (i = 0, 1),$$

$$(p_2(t) - p_{02}(t))(t - t_0) \leq 0$$

hold, where  $p_{0i} : [a, b] \rightarrow \mathbb{R}$  ( $i = 0, 1$ ) are bounded measurable functions,

$$0 < \lambda_i < 2, \quad 0 < \mu_i < 2, \quad p_{02} \in L([a, b]),$$

then condition (4) is satisfied.

Our aim was to establish sufficient conditions for problem (1), (2) to be well-posed, i.e. the conditions guaranteeing the unique solvability of the problem obtained by "small" perturbations of the coefficients of equation (1) and the boundary conditions (2), and the proximity of a solution of that problem to a solution of problem (1), (2). Theorem 2 below contains exactly such conditions.

Let us denote by  $V(\sigma)$  the variation of any finite variation function  $\sigma : [a, b] \rightarrow \mathbb{R}$  on  $[a, b]$ .

Along with problem (1), (2), for any natural  $k$  we consider the problem

$$u'' = p_{1k}(t)u + p_{2k}(t)u' + p_{0k}(t), \quad (1_k)$$

$$\int_a^b u(t) d\sigma_{ik}(t) = c_{ik} \quad (i = 1, 2), \quad (2_k)$$

where  $p_{ik} \in L_{loc}([a, b])$  ( $i = 0, 1, 2$ ), and  $\sigma_{ik} : [a, b] \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) are functions of finite variations.

We introduce

**Definition.** Problem (1), (2) is said to be well-posed if it has a unique solution  $u$  and for every sequence of locally integrable functions  $(p_{ik})_{k=1}^{\infty}$  ( $i = 0, 1, 2$ ), for every sequence of finite variation functions  $(\sigma_{ik})_{k=1}^{\infty}$  ( $i = 1, 2$ ), and for every sequence of numbers  $(c_{ik})_{k=1}^{\infty}$  ( $i = 1, 2$ ), satisfying the conditions

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{t_0}^t (s-a)(b-s)(p_{ik}(s) - p_i(s)) ds &= 0 \quad (i = 0, 1) \quad \text{uniformly on } ]a, b[, \\ \lim_{k \rightarrow \infty} \int_{t_0}^t (p_{2k}(s) - p_2(s)) ds &= 0 \quad \text{uniformly on } ]a, b[, \\ \limsup_{k \rightarrow \infty} \int_a^b (s-a)(b-s)|p_{1k}(s) - p_1(s)| ds < +\infty, & \quad \limsup_{k \rightarrow \infty} \int_a^b |p_{2k}(s) - p_2(s)| ds < +\infty, \quad (5) \\ \lim_{k \rightarrow +\infty} V(\sigma_{ik} - \sigma) &= 0, \quad \lim_{k \rightarrow +\infty} c_{ik} = c_i \quad (i = 1, 2), \end{aligned}$$

there exists a natural number  $k_0$  such that for  $k \geq k_0$  problem  $(1_k), (2_k)$  has a unique solution  $u_k$  and

$$\begin{aligned} \lim_{k \rightarrow +\infty} (u_k(t) - u(t)) &= 0 \quad \text{uniformly on } [a, b], \\ \lim_{k \rightarrow +\infty} [(t-a)(b-t)(u'_k(t) - u'(t))] &= 0 \quad \text{uniformly on } ]a, b[. \end{aligned}$$

Note that conditions (5) cannot be omitted from Theorem 2 (see, [1], pp. 39–40, Remark 1.4). The following theorem is true.

**Theorem 2.** *If conditions (3), (4) are satisfied and problem  $(1_0), (2_0)$  has only the trivial solution, then problem (1), (2) is well-posed.*

At the end, we consider the differential equation

$$u'' = p_1(t)u + p_0(t) \quad (6)$$

with the nonlocal conditions

$$\sum_{k=1}^m \lambda_k u(a_k) = c_1, \quad \sum_{k=1}^n \mu_k u(b_k) = c_2, \tag{7}$$

$$\sum_{k=1}^m \lambda_k u(a_k) = \sum_{k=1}^n \mu_k u(b_k) + c_1, \quad \sum_{k=1}^l \nu_k u(t_k) = c_2. \tag{8}$$

Theorems below, which are proved based on Theorems 1 and 2, contain efficient conditions guaranteeing the well-posedness of problems (6), (7) and (6), (8).

**Theorem 3.** *Let*

$$\int_a^b (t-a)(b-t)|p_i(t)| dt < +\infty \quad (i = 0, 1), \tag{9}$$

$$a = a_1 < \dots < a_m < b_1 < \dots < b_n = b,$$

$$\lambda_k > 0 \quad (k = 1, \dots, m), \quad \mu_k > 0 \quad (k = 1, \dots, n), \tag{10}$$

and

$$\int_a^b (t-a)(b-t)[p_1(t)]_- dt < b-a.$$

Then problem (6), (7) is well-posed.

**Theorem 4.** *Let along with (9) and (10) the conditions*

$$a = a_1 < \dots < a_m < t_k < b_1 < \dots < b_n = b,$$

$$\nu_k > 0 \quad (k = 1, \dots, l),$$

$$\int_a^{b_1} (t-a)(b_1-t)[p_1(t)]_- dt \leq b_1 - a,$$

$$\int_{a_m}^b (t-a_m)(b-t)[p_1(t)]_- dt \leq b - a_m$$

hold. Then problem (6), (8) is well-posed.

## References

- [1] I. Kiguradze, *Initial and Boundary Value Problems for Systems of Ordinary Differential Equations*, I. (Russian) Metsniereba, Tbilisi, 1997.
- [2] I. Kiguradze, Fredholm type nonlocal boundary value problems for second order linear singular differential equations. *Mem. Differ. Equ. Math. Phys.* (accepted).