On Well-Posed Initial-Boundary Value Problems in a Characteristic Rectangle for Linear Hyperbolic Systems of Second Order

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Consider the linear hyperbolic system

$$u_{xy} = P_1(x,y)u_x + P_2(x,y)u_y + P_0(x,y)u + q(x,y) \tag{1}$$

with the initial-boundary conditions

$$u(0,y) = \varphi(y) \ \text{ for } \ y \in [0,\omega_2], \quad h(u_x(x,\cdot))(x) = \psi(x) \ \text{ for } \ x \in [0,\omega_1], \tag{2}$$

where $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$, and $h : C([0,\omega_2];\mathbb{R}^n) \to C([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator.

By a solution of problem $(1),(2)$ we understand a *classical solution*, i.e., a function $u : \Omega \to R$ having continuous partial derivatives u_x, u_y and u_{xy} and satisfying system (1) and the initial–boundary conditions (2) everywhere in Ω.

Problem $(1),(2)$ do not belong to the classical boundary value problems of mathematical physics, with the exception of Darboux and Goursat initial value problems.

Initial-periodic problem is one of the most important particular cases of problem $(1),(2)$.

Beginning from the 1960ies, problems on periodic solutions in a strip, as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see [1–6, 11, 12]). These problems naturally led to the initial-boundary value problems in a rectangle with general boundary conditions. A comprehensive theory of initial-boundary value problems $(1),(2)$ with general boundary conditions was constructed in [7].

Along with problem $(1),(2)$ consider the problem

$$v' = P_1(x^*,y)v, \tag{1*}$$

$$h(v)(x^*) = 0. \tag{2*}$$

Problem $(1^*),(2^*)$ is called ***associated problem*** of problem $(1),(2)$.

Definition 1. Problem $(1),(2)$ is called well-posed if for every $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$, problem $(1),(2)$ has a unique solution $u \in C^{1,1}(\Omega;\mathbb{R}^n)$ admitting the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\big(\|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])} + \|q\|_{C(\Omega)}\big), \tag{3}$$

where M is a positive constant independent of φ, ψ and q.

Theorem 1. *Problem* $(1),(2)$ *is well-posed if and only if the associated problem* $(1^*),(2^*)$ *has only the trivial solution for every* $x^*\in[0,\omega_1]$.

Remark 1. Problem $(1),(2)$ was studied in [7]. The sufficiency part of Theorem 1 was proved in [7] (see Theorems 4.1 and 4.1′).

Remark 2. It is easy to show that estimate (3) is equivalent to the estimate

$$\|u\|_{C^{1,1}(\Omega_x)}\leq M\big(\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,x])}+\|q\|_{C(\Omega_x)}\big), \tag{4}$$

where $\Omega_x=[0,x]\times[0,\omega_2]$ and M is a positive constant independent of φ, ψ and q and $x\in[0,\omega_1]$.

For system (1), consider the initial-boundary conditions

$$u(0,y)=\varphi(y),\quad u_x(x,\gamma_1(x))=A(x)u_x(x,\gamma_2(x))+\psi(x), \tag{5}$$

where $A\in C([0,\omega_1];\mathbb{R}^{n\times n})$, $\gamma_i\in C([0,\omega_1];[0,\omega_2])$ $(i=1,2)$ and

$$\gamma_1(x)<\gamma_2(x)\ \text{ for }\ x\in[0,\omega_1]. \tag{6}$$

Initial-boundary conditions of periodic and antiperiodic types

$$u(0,y)=\varphi(y),\quad u_x(x,\gamma_1(x))=u_x(x,\gamma_2(x))+\psi(x) \tag{7}$$

and

$$u(0,y)=\varphi(y),\quad u_x(x,\gamma_1(x))=-u_x(x,\gamma_2(x))+\psi(x) \tag{8}$$

are particular cases of (5).

Corollary 1. *Let* $P_1(x,y)\equiv P_1(x)$. *Then problem* $(1),(5)$ *is well-posed if and only if*

$$\det\big(I-\exp\big((\gamma_2(x)-\gamma_1(x))P_1(x)\big)\,A(x)\big)\neq 0\ \textit{ for }\ x\in[0,\omega_1]. \tag{9}$$

Corollary 2. *Let* $P_1(x,y)\equiv P_1(x)$. *Then problem* $(1),(7)$ *is well-posed if and only if*

$$\det\big(I-\exp\big((\gamma_2(x)-\gamma_1(x))P_1(x)\big)\big)\neq 0\ \textit{ for }\ x\in[0,\omega_1]. \tag{10}$$

Corollary 3. *Let* $P_1(x,y)\equiv P_1(x)$. *Then problem* $(1),(8)$ *is well-posed if and only if*

$$\det\big(I+\exp\big((\gamma_2(x)-\gamma_1(x))P_1(x)\big)\big)\neq 0\ \textit{ for }\ x\in[0,\omega_1]. \tag{11}$$

Corollary 4. *Let* $P_1(x,y)\equiv P_1(x)$ *be a symmetric matrix function. Then problem* $(1),(8)$ *is well-posed.*

For an arbitrary matrix $P\in\mathbb{R}^{n\times n}$ set:

$$\widehat{P}=\frac{1}{2}\,(P+P^T),$$

where P^T is the transpose of the matrix P.

Theorem 2. *Let there exist* $\sigma\in\{-1,1\}$ *such that for every* $x\in[0,\omega_1]$

$$\sigma\widehat{P}_1(x,y)\ \textit{ is positive semi-definite for }\ y\in[\gamma_1(x),\gamma_2(x)], \tag{12}$$

$$\sigma(A^T(x)A(x)-I)\ \textit{ is positive semi-definite}, \tag{13}$$

and either

$$\sigma\widehat{P}_1(x,y^*)\ \textit{ is positive definite for some }\ y^*\in[\gamma_1(x),\gamma_2(x)], \tag{14}$$

or

$$\sigma(A^T(x)A(x)-I)\ \textit{ is positive definite}. \tag{15}$$

Then problem $(1),(5)$ *is well-posed.*

Let $n = 2m$, $u = (v, w)$, $v, w \in \mathbb{R}^m$, and let $\gamma_i \in C([0,\omega_1];[0,\omega_2])$ $(i = 1, 2)$ be functions satisfying inequality (6).

For the system

$$\begin{aligned} v_{xy} = Q(x,y)v_x + P_{11}(x,y)w_x + P_{2_{11}}(x,y)v_y + P_{2_{12}}(x,y)w_y \\ + P_{0_{11}}(x,y)v + P_{0_{12}}(x,y)w + q_1(x,y), \end{aligned} \tag{16}$$

$$\begin{aligned} w_{xy} = P_{12}(x,y)v_x - Q(x,y)w_x + P_{2_{21}}(x,y)v_y + P_{2_{22}}(x,y)w_y \\ + P_{0_{21}}(x,y)v + P_{0_{22}}(x,y)w + q_2(x,y). \end{aligned} \tag{17}$$

consider the following initial-boundary conditions

$$\begin{gathered} v(0,y) = \varphi_1(y), \quad w(0,y) = \varphi_2(y), \\ v_x(x,\gamma_1(x)) = A(x)v_x(x,\gamma_2(x)) + \psi_1(x), \quad w_x(x,\gamma_2(x)) = A(x)w_x(x,\gamma_1(x)) + \psi_2(x), \end{gathered} \tag{18}$$

and

$$\begin{gathered} v(0,y) = \varphi_1(y), \quad w(0,y) = \varphi_2(y), \\ v_x(x,\gamma_2(x)) = A(x)w_x(x,\gamma_2(x)) + \psi_1(x), \quad v_x(x,\gamma_1(x)) = -B(x)w_x(x,\gamma_1(x)) + \psi_2(x), \end{gathered} \tag{19}$$

where $A \in C([0,\omega_1];\mathbb{R}^{m\times m})$ and $B \in C([0,\omega_1];\mathbb{R}^{m\times m})$ are symmetric matrix functions.

Nicoletti, periodic, and anti-periodic type initial-boundary conditions

$$v(0,y) = \varphi_1(y), \quad w(0,y) = \varphi_2(y), \quad v_x(x,\gamma_1(x)) = \psi_1(x), \quad w_x(x,\gamma_2(x)) = \psi_2(x), \tag{20}$$

$$\begin{gathered} v(0,y) = \varphi_1(y), \quad w(0,y) = \varphi_2(y), \\ v_x(x,\gamma_1(x)) = v_x(x,\gamma_2(x)) + \psi_1(x), \quad w_x(x,\gamma_2(x)) = w_x(x,\gamma_1(x)) + \psi_2(x), \end{gathered} \tag{21}$$

and

$$\begin{gathered} v(0,y) = \varphi_1(y), \quad w(0,y) = \varphi_2(y), \\ v_x(x,\gamma_1(x)) = -v_x(x,\gamma_2(x)) + \psi_1(x), \quad w_x(x,\gamma_2(x)) = -w_x(x,\gamma_1(x)) + \psi_2(x) \end{gathered} \tag{22}$$

are particular cases of (18).

Theorem 3. *Let $Q \in C\big(\Omega;\mathbb{R}^{m\times m}\big)$ and $A \in C([0,\omega_1];\mathbb{R}^{m\times m})$ be symmetric matrix functions, and let there exist $\sigma \in \{-1, 1\}$ such that for every $x \in [0,\omega_1]$*

$$\sigma P_{1j}(x,y) \ \text{ is positive semi-definite for } \ y \in [\gamma_1(x),\gamma_2(x)] \ \ (j = 1, 2) \tag{23}$$

and

$$\sigma P_{1j}(x,y^*) \ \text{ is positive definite for some } \ y^* \in [\gamma_1(x),\gamma_2(x)] \ \ (j = 1, 2). \tag{24}$$

Then problem (16), (17), (18) *is well-posed.*

Corollary 5. *Let $P_{11} \in C(\Omega;\mathbb{R}^{m\times m})$ and $P_{12} \in C(\Omega;\mathbb{R}^{m\times m})$ be positive definite symmetric matrix functions, and let $Q \in C(\Omega;\mathbb{R}^{m\times m})$ be a symmetric matrix function. Then problem* (16), (17), (20) *is well-posed.*

Corollary 6. *Let $P_{11} \in C(\Omega;\mathbb{R}^{m\times m})$ and $P_{12} \in C(\Omega;\mathbb{R}^{m\times m})$ be positive definite symmetric matrix functions, and let $Q \in C(\Omega;\mathbb{R}^{m\times m})$ be a symmetric matrix function. Then problem* (16), (17), (21) *is well-posed.*

Corollary 7. *Let $P_{11} \in C(\Omega;\mathbb{R}^{m\times m})$ and $P_{12} \in C(\Omega;\mathbb{R}^{m\times m})$ be positive definite symmetric matrix functions, and let $Q \in C(\Omega;\mathbb{R}^{m\times m})$ be a symmetric matrix function. Then problem* (16), (17), (22) *is well-posed.*

Theorem 4. *Let $Q \in C(\Omega;\mathbb{R}^{m\times m})$ be a symmetric matrix function, let there exist $\sigma \in \{-1,1\}$ such that for every $x \in [0,\omega_1]$ $\sigma A(x)$ is positive semi-definite, $\sigma B(x)$ is positive semi-definite, and*

$$\sigma P_{1j}(x,y) \text{ is positive semi-definite for } y \in [\gamma_1(x),\gamma_2(x)] \ \ (j=1,2).$$

*Furthermore, let for every $x \in [0,\omega_1]$ **one of the following three** conditions hold:*

$$\sigma P_{1j}(x,y^*) \text{ is positive definite for some } y^* \in [\gamma_1(x),\gamma_2(x)] \ \ (j=1,2); \tag{25}$$

$$\sigma A(x) \text{ is positive definite}; \tag{26}$$

$$\sigma B(x) \text{ is positive definite}. \tag{27}$$

Then problem $(16),(17),(19)$ is well-posed.

Lastly consider the following one-dimensional and two-dimensional cases of problem $(1),(2)$:

$$u_{xy} = p_1(x,y)u_x + p_2(x,y)u_y + p_0(x,y)u + q(x,y), \tag{28}$$

$$u(0,y)=\varphi(y), \quad \int\limits_{\gamma_1(x)}^{\gamma_2(x)} f(x,t)u_x(x,t)\,dt = \psi(x), \tag{29}$$

and

$$\begin{aligned} v_{xy} &= g(x,y)v_x + p_{11}(x,y)w_x + p_{2_{11}}(x,y)v_y + p_{2_{12}}(x,y)w_y \\ &\qquad\qquad + p_{0_{11}}(x,y)v + p_{0_{12}}(x,y)w + q_1(x,y), \end{aligned} \tag{30}$$

$$\begin{aligned} w_{xy} &= p_{12}(x,y)v_x - g(x,y)w_x + p_{2_{21}}(x,y)v_y + p_{2_{22}}(x,y)w_y \\ &\qquad\qquad + p_{0_{21}}(x,y)v + p_{0_{22}}(x,y)w + q_2(x,y), \end{aligned} \tag{31}$$

$$\begin{gathered} v(0,y)=\varphi_1(y), \quad w(0,y)=\varphi_2(y), \\ \int\limits_{\gamma_1(x)}^{\gamma_2(x)} f_1(x,t)v_x(x,t)\,dt = \psi_1(x), \quad \int\limits_{\gamma_1(x)}^{\gamma_2(x)} f_2(x,t)w_x(x,t)\,dt = \psi_2(x), \end{gathered} \tag{32}$$

where $f, f_1, f_2 \in C(\Omega)$, and $\gamma_i \in C([0,\omega_1];[0,\omega_2])$ $(i=1,2)$ are functions satisfying inequality (6).

Theorem 5. *Let $f(x,y) \geq 0$ and let*

$$\int\limits_{\gamma_1(x)}^{\gamma^2(x)} f(x,t)\,dt > 0$$

for every $x \in [0,\omega_1]$. Then problem $(28),(29)$ is well-posed.

Theorem 6. *Let $f_i(x,y) \geq 0$ $(i=1,2)$, and let*

$$\int\limits_{\gamma_1(x)}^{\gamma^2(x)} f_i(x,t)\,dt > 0 \quad (i=1,2) \tag{33}$$

for every $x \in [0,\omega_1]$. Furthermore, let there exist $\sigma \in \{-1,1\}$ such that for every $x \in [0,\omega_1]$

$$\sigma p_{1i}(x,y) \geq 0 \text{ for } y \in [\gamma_1(x),\gamma_2(x)] \ \ (i=1,2), \tag{34}$$

$$\sigma p_{1i}(x,y^*) > 0 \text{ for some } y^* \in [\gamma_1(x),\gamma_2(x)] \ \ (i=1,2). \tag{35}$$

Then problem $(30),(31),(32)$ is well-posed.

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