Boundary Value Problem for One Class of Nonlinear Partial Differential Equations with Iterated Wave Operator in the Principal Part

Sergo Kharibegashvili

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mail: kharibegashvili@yahoo.com

In the Euclidean space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t consider a semilinear partial differential equation with an iterated wave operator in the principal part

$$L_f u := \square^2 u + a(x)u_{tt} + b(x,t)u + f(u) = F(x,t), \tag{1}$$

where a, b, f and F are given, while u is an unknown functions

$$\square := \frac{\partial^2}{\partial t^2} - \Delta, \quad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\,, \quad n\geq 2.$$

Let $D_T:\ \psi(x)<t<\psi(x)+T$, $x\in\Omega$, denote a cylindrical domain, where Ω is a domain in $\mathbb{R}^n$ with Lipschitz boundary, and the surface $S_0:\ t=\psi(x)$, $x\in\overline{\Omega}$, is a smooth characteristic manifold, i.e.,

$$\psi\in C^1(\overline{\Omega}), \quad 1-\sum_{i=1}^{n}\Big(\frac{\partial\psi(x)}{\partial x_i}\Big)^2=0, \quad x\in\overline{\Omega}.$$

Thus, the domain D_T is bounded from below and above by characteristic manifolds $S_0:\ t=\psi(x)$, $x\in\Omega$ and $S_T:\ t=\psi(x)+T$, $x\in\Omega$, respectively, and from the side by a cylindrical surface $\Gamma:=\{(x,t)\in\mathbb{R}^{n+1}:\ \psi(x)\leq t\leq\psi(x)+T,\ x\in\partial\Omega\}$. Below we will assume that $a\in C(\overline{\Omega})$, $b\in C(\overline{D}_T)$ and $f\in C(R)$.

For equation (1), we consider the boundary value problem in the following formulation: in the cylindrical domain D_T find a solution $u=u(x,t)$ of equation (1) under the boundary conditions

$$u\big|_{S_0}=0, \quad u\big|_{S_T}=0, \tag{2}$$

$$u\big|_{\Gamma}=0, \quad \frac{\partial u}{\partial v}\Big|_{\Gamma}=0, \tag{3}$$

where $\frac{\partial}{\partial v}$ is the derivative in the direction of the outer normal to the boundary ∂D_T of domain D_T, here $v=(v_1,\dots,v_n,v_{n+1})$ is the unit vector of the outer normal to ∂D_T and it is obvious that $v_{n+1}|_\Gamma=0$.

Another boundary value problem in a cylindrical domain $G_T:\ 0<t<T$, $x\in\Omega$, bounded from below and above by surfaces $\Omega_0:\ x\in\Omega$, $t=0$ and $\Omega_T:\ x\in\Omega$, $t=T$ of spatial type, respectively, when homogeneous Dirichlet and Neumann boundary conditions for systems of type (1) are given on the entire boundary of the domain G_T, was considered in paper [6], and in the scalar case – in article [3]. We also note papers [2, 4, 5] in which boundary value problems in conical domains of different geometric structures were investigated for semilinear equation of type (1).

Let

$$\overset{\circ}{C}{}^k(\overline{D}_T) := \Big\{ u \in C^k(D_T) : \ u\big|_{S_0\cup S_T} = 0, \ u\big|_{\Gamma} = 0, \ \frac{\partial u}{\partial v}\Big|_{\Gamma} = 0 \Big\}, \quad k \geq 1. \tag{4}$$

If $v \in C^4(\overline{D}_T)$ and $\varphi \in C^2(\overline{D}_T)$, then integration by parts of the expression $\square\, v \,\square\, \varphi$ over the domain D_T gives

$$\int\limits_{D_T} \square\, v \,\square\, \varphi \, dx\, dt = \int\limits_{\partial D_T} \frac{\partial \varphi}{\partial N} \square\, v \, ds - \int\limits_{\partial D_T} \varphi \frac{\partial}{\partial N} \square\, v \, ds + \int\limits_{D_T} \varphi \,\square^2 v \, dx\, dt, \tag{5}$$

where $\frac{\partial}{\partial N} = v_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$ is the derivative in the direction of the conormal, and it is easy to see that

$$\frac{\partial \varphi}{\partial N}\Big|_{\Gamma} = -\frac{\partial \varphi}{\partial v}\Big|_{\Gamma}. \tag{6}$$

Since on the characteristic manifolds S_0 and S_T the derivative in the direction of the conormal $\frac{\partial}{\partial N}$ is an inner differential operator, then from the fact that $\varphi|_{S_0\cup S_T} = 0$ it will follow

$$\frac{\partial \varphi}{\partial N}\Big|_{\partial D_T \setminus \Gamma} = \frac{\partial \varphi}{\partial N}\Big|_{S_0\cup S_T} = 0. \tag{7}$$

Therefore, by virtue of (4)–(7), if $\varphi \in \overset{\circ}{C}{}^2(\overline{D}_T)$, then from (5) we obtain

$$\int\limits_{D_T} \square\, v \,\square\, \varphi \, dx\, dt = \int\limits_{D_T} \varphi \,\square^2 v \, dx\, dt. \tag{8}$$

Let us now assume that $u \in \overset{\circ}{C}{}^4(\overline{D}_T)$ is the classical solution to problem (1), (2), (3). Then, multiplying both parts of equation (1) by an arbitrary function $\varphi \in \overset{\circ}{C}{}^2(\overline{D}_T)$ and, integrating by parts over the domain D_T, by virtue of (8), we obtain

$$\int\limits_{D_T} \big[\square\, v \,\square\, \varphi - a(x) u_t \varphi_t + b(x,t) u \varphi + f(u)\varphi \big] \, dx\, dt = \int\limits_{D_T} F \varphi \, dx\, dt \quad \forall\, \varphi \in \overset{\circ}{C}{}^2(\overline{D}_T). \tag{9}$$

In a certain sense, the converse statement is also true, i.e., if $u \in \overset{\circ}{C}{}^4(\overline{D}_T)$ satisfies the integral equality (9) for any $\varphi \in \overset{\circ}{C}{}^2(\overline{D}_T)$, then standard reasoning implies that u is a solution of equation (1) in the domain D_T and, by virtue of the definition of the space $\overset{\circ}{C}{}^4(\overline{D}_T)$, satisfies the boundary conditions (2) and (3). Below, we will use equality (9) as the basis of definition of a weak generalized solution of problem (1), (2), (3) in a certain Hilbert space under certain conditions imposed on the growth rate of the nonlinearity of the function $f(u)$.

The following bilinear form

$$(u,v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \square\, u \,\square\, v \Big] \, dx\, dt \tag{10}$$

defines a scalar product in the space $\overset{\circ}{C}{}^2(\overline{D}_T)$. Let $\overset{\circ}{W}_{\square}(D_T)$ denote the Hilbert space obtaining by completion the space $\overset{\circ}{C}{}^2(\overline{D}_T)$ with the norm corresponding to the scalar product (10).

Now we present the conditions imposed on the function $f(u)$ from (1):

$$f \in C(\mathbb{R}), \quad |f(u)| \leq M_1 + M_2 |u|^{\alpha}, \quad \alpha = const \geq 0, \quad u \in \mathbb{R}, \tag{11}$$

where $M_i = const \geq 0$, $i = 1, 2$, and

$$0 \leq \alpha = const < \frac{n+1}{n-1}\,. \tag{12}$$

Remark 1. The embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is a linear compact operator for $1 < q < \frac{2(n+1)}{n-1}$ and $n > 1$ [7]. At the same time, the Nemitskii operator $K : L_q(D_T) \to L_2(D_T)$, acting according to the formula $Ku = f(u)$, where $u \in L_q(D_T)$ and the function f satisfies conditions (11) and (12), is continuous and bounded for $q \geq 2\alpha$ [1]. Therefore, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \geq 2\alpha$. In this case, the operator

$$K_0 = KI : W_2^1(D_T) \to L_2(D_T)$$

is continuous and compact. It follows in particular that if $u \in \overset{\circ}{W}_{\Box}(D_T) \subset W_2^1(D_T)$, then $f(u) \in L_2(D_T)$. Here $W_2^1(D_T)$ is the well-known Sobolev space consisting of elements of $L_2(D_T)$ that have generalized partial derivatives of the first order from $L_2(D_T)$.

Definition 1. Let conditions (11), (12) and $F \in L_2(D_T)$ be satisfied. The function $u \in \overset{\circ}{W}_{\Box}(D_T)$ is called a weak generalized solution of problem (1), (2), (3), if the integral equality (9) is valid for any function $\varphi \in \overset{\circ}{W}_{\Box}(D_T)$, i.e.,

$$\int\limits_{D_T} \big[\Box v \Box \varphi - a(x) u_t \varphi_t + b(x,t) u \varphi + f(u)\varphi\big]\, dx\, dt = \int\limits_{D_T} F\varphi \, dx\, dt \quad \forall\, \varphi \in \overset{\circ}{W}_{\Box}(D_T). \tag{13}$$

Note that, by virtue of Remark 1, the integral $\int\limits_{D_T} f(u)\varphi\, dx\, dt$ is defined correctly, since from $u \in \overset{\circ}{W}_{\Box}(D_T)$ follows that $f(u) \in L_2(D_T)$ and, consequently, $f(u)\varphi \in L_1(D_T)$.

We introduce a bilinear form

$$(u, v)_1 = \int\limits_{D_T} \big[\Box v \Box \varphi - a(x) u_t \varphi_t + b(x,t) uv\big]\, dx\, dt \tag{14}$$

in the space $\overset{\circ}{W}_{\Box}(D_T)$ induced by the left-hand side of equality (9).

Remark 2. Under the assumption that

$$b\big|_{\overline{D}_T} \geq 0, \ \ a\big|_{\Omega} \leq -c_0, \ \ c_0 = const > 0, \tag{15}$$

it is proved that the bilinear form (14) is a scalar product and is equivalent to the scalar product (10) in the Hilbert space $\overset{\circ}{W}_{\Box}(D_T)$. The latter allows us to equivalently reduce the boundary value problem (1), (2), (3) to the functional equation

$$u = {L_0}^{-1}(-f(u) + F)$$

in the space $\overset{\circ}{W}_{\Box}(D_T)$, where $L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}_{\Box}(D_T)$ is a linear bounded operator corresponding to the problem for $f = 0$.

Remark 3. The fulfillment of only conditions (11), (12) imposed on the nonlinear function $f = f(u)$ is not yet sufficient for the solvability of problem (1), (2), (3) in the sense of Definition 1. However, if we require the fulfillment of certain conditions concerning the behavior of the nonlinear function $f = f(u)$ in the neighborhood of infinity, for example, the following condition

$$\lim_{|u|\to\infty} \inf \frac{f(u)}{u} \geq 0, \tag{16}$$

then for the weak generalized solution $u \in \overset{\circ}{W}_{\square}(D_T)$ of problem $(1), (2), (3)$ there is an a priori estimate

$$\|u\|_{\overset{\circ}{W}_{\square}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2$$

with positive constants c_1 and c_2 independent of u and F. The latter, taking into account Remarks 1 and 2, allows us to prove the existence of at least one weak generalized solution of problem $(1), (2), (3)$ in the space $\overset{\circ}{W}_{\square}(D_T)$ in the sense of Definition 1.

Theorem 1. *Let conditions* (11), (12), (15) *and* (16) *be satisfied. Then for any $F \in L_2(D_T)$ there exists at least one weak generalized solution of problem $(1), (2), (3)$ in the space $\overset{\circ}{W}_{\square}(D_T)$ in the sense of Definition 1.*

As for the uniqueness of a solution of problem $(1), (2), (3)$, it is a consequence of the monotonicity of the function $f = f(u)$.

Theorem 2. *Let conditions* (11), (12), (15) *and*

$$(f(u) - f(v))(u - v) \geq 0 \quad \forall\, u, v \in \mathbb{R} \tag{17}$$

be satisfied. Then for any $F \in L_2(D_T)$ problem $(1), (2), (3)$ cannot have more than one weak generalized solution in the space $\overset{\circ}{W}_{\square}(D_T)$ in the sense of Definition 1.

From these theorems follows the following theorem.

Theorem 3. *Let conditions* (11), (12), (15)–(17) *be satisfied. Then for any function $F \in L_2(D_T)$ problem $(1), (2), (3)$ has a unique weak generalized solution in the space $\overset{\circ}{W}_{\square}(D_T)$ in the sense of Definition 1.*

As the following theorem shows, if (17) is violated, problem $(1), (2), (3)$ may not have a solution.

Theorem 4. *Let the function $f = f(u)$ satisfy conditions* (11), (12) *and*

$$f(u) \leq -|u|^{\beta} \quad \forall\, u \in \mathbb{R}, \quad \beta = const > 1,$$

and the function $F = \mu F_0$, where $F_0 \in L_2(D_T)$, $F > 0$ in the domain D_T, $\mu = const > 0$. Then there exists a number $\mu_0 = \mu_0(F_0, \beta)$ such that problem $(1), (2), (3)$ may not have a weak generalized solution in the space $\overset{\circ}{W}_{\square}(D_T)$ in the sense of Definition 1.

References

[1] S. Fučík and A. Kufner, *Nonlinear Differential Equations*. Studies in Applied Mechanics, 2. Elsevier Scientific Publishing Co., Amsterdam–New York, 1980.

[2] S. Kharibegashvili, Boundary value problems for some classes of nonlinear wave equations. *Mem. Differential Equations Math. Phys.* **46** (2009), 1–114.

[3] S. Kharibegashvili and B. Midodashvili, On the existence, uniqueness and nonexistence of solutions to a boundary value problem for a quasilinear hyperbolic equation. (Russian) *Ukraïn. Mat. Zh.* **71** (2019), no. 8, 1123–1132; translation in *Ukrainian Math. J.* **71** (2020), no. 8, 1282–1293.

[4] S. Kharibegashvili and B. Midodashvili, Solvability of characteristic boundary-value problems for nonlinear equations with iterated wave operator in the principal part. Electron. *J. Differential Equations* **2008**, no. 72, 12 pp.

[5] S. Kharibegashvili and B. Midodashvili, On one boundary value problem for a nonlinear equation with the iterated wave operator in the principal part. *Georgian Math. J.* **15** (2008), no. 3, 541–554.

[6] S. Kharibegashvili and B. Midodashvili, On the solvability of one boundary value problem for one class of higher-order semilinear hyperbolic systems. *Lith. Math. J.* **62** (2022), no. 3, 360–371.

[7] O. A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics.* (Russian) Izdat. "Nauka", Moscow, 1973.