

Robust Stability for Non-Autonomous Inclusion of Reaction-Diffusion Type

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1 Introduction

We study global solvability and attractor stability for a parabolic inclusion with multi-valued reaction-diffusion function and non-autonomous perturbations. For L^2 -disturbances, global L^2 -solutions exist. Translation-bounded disturbances yield multi-valued semiprocesses with uniform attractor. For L^∞ -disturbances, the unperturbed global attractor is asymptotically gain-stable. Global attractors are key in dissipative infinite-dimensional systems [12, 14]. Recent extensions cover non-autonomous and multi-valued cases [1–3]. Multi-valued semiflows for inclusions were introduced in [10]; their attractors studied in [4]. Uniform attractors for linear-growth inclusions appear in [11]. We prove L^2 -solvability [8] and uniform attractor for polynomial-growth cases. Robust stability follows from [5–7, 13]. The result appeared in [9].

2 Setting of the problem and the main results

In the bounded domain $\Omega \subset \mathbb{R}^m$, $m \geq 1$, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u \in f(u) + d(t, x), & (t, x) \in Q = (0, +\infty) \times \Omega, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{2.1}$$

where $d \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ is a non-autonomous disturbance function, $u_0 \in L^2(\Omega)$, and the multi-valued function f satisfies the following assumptions:

$$\begin{aligned} f : \mathbb{R} \rightarrow C_v(\mathbb{R}) \text{ is upper semicontinuous, } \exists C > 0, \alpha_1, \alpha_2 > 0, p \geq 2, \\ \text{such that } \forall s \in \mathbb{R}, \forall \xi \in f(s) \quad -C - \alpha_1|s|^p \leq \xi \cdot s \leq C - \alpha_2|s|^p. \end{aligned} \tag{2.2}$$

Here $C_v(\mathbb{R})$ denotes the set of all non-empty, convex, compact sets in \mathbb{R} , upper semicontinuity is understood in the standard sense.

Definition 1. A function $u = u(t, x)$, $(t, x) \in Q_T = (0, T) \times \Omega$ is called a (weak) solution of (2.1) on $(0, T)$, if $u \in L^2(0, T; H^1_0(\Omega))$ and there exists $l \in L^q(Q_T)$, $\frac{1}{q} + \frac{1}{p} = 1$ such that $\forall v \in C^\infty_0(\Omega)$, $\forall \eta \in C^\infty_0(0, T)$

$$\begin{aligned} - \int_0^T (u(t), v) \eta_t dt + \int_0^T (\nabla u(t), \nabla v) \eta dt = \int_0^T \int_\Omega (l(t, x) + d(t, x)) v(x) \eta(t) dx dt, \\ l(t, x) \in f(u(t, x)) \text{ almost everywhere (a.e.) on } Q_T, \end{aligned} \tag{2.3}$$

and the initial condition holds.

Here by $\|\cdot\|$ and (\cdot, \cdot) we denote the norm and the scalar product in the phase space $X = L^2(\Omega)$. We note that equality (2.3) means [12] that

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega) + L^q(Q_T)) \subset L^q(0, T; H^{-s}(\Omega)), \quad \text{where } s \geq \max \left\{ 1, m \left(\frac{1}{2} - \frac{1}{p} \right) \right\}.$$

Therefore, due to [12], $u \in \mathbb{C}([0, T]; H^{-s}(\Omega))$, and the initial condition $u|_{t=0} = u_0$ makes sense. Our second goal is to show that under additional translation-bounded assumption

$$\sup_{t \geq 0} \int_t^{t+1} \|d(s)\|^2 ds < \infty,$$

obtained solutions generate the family of multi-valued semiprocesses $\{U_\sigma\}_{\sigma \in \Sigma(d)}$, which possesses uniform attractor $\Theta_{\Sigma(d)}$. Here $U_\sigma(t, \tau, u_\tau)$ denotes the set of solutions of (2.1) with disturbances σ at the moment $t \geq \tau$ with initial data $u|_{t=\tau} = u_\tau$, and $\sigma = \sigma(t, x)$ belongs to the shift-invariant space $\Sigma(d)$. Assuming that $d \in L^\infty(\mathbb{R}_+; L^2(\Omega))$, we prove:

$$\text{dist}(\Theta_{\Sigma(d)}, \Theta) \rightarrow 0 \quad \text{as } \|d\|_\infty \rightarrow 0,$$

where Θ is the global attractor of the unperturbed system ($d \equiv 0$). It allows us to show the following robust estimate: $\forall u_0 \in L^2(\Omega)$

$$\overline{\lim}_{t \rightarrow \infty} \|U_d(t, 0, u_0)\|_\Theta \leq \gamma(\|d\|_\infty),$$

where γ belongs to the well-known comparison class \mathcal{K} , $\|d\|_\infty := \text{ess sup}_{t \geq 0} \|d(t)\|$, and for $A \subset X$

$$\|A\|_\Theta = \text{dist}(A, \Theta) = \sup_{a \in A} \inf_{\theta \in \Theta} \|a - \theta\|.$$

Note that if u is a weak solution of (2.1) on $(0, T)$ and, additionally, $u \in L^p(Q_T)$, then due to [1] the function $[0, T] \ni t \mapsto u(t) \in L^2(\Omega)$ is absolutely continuous and for a.e. $t \in [0, T]$

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|\nabla u(t)\|^2 = \int_{\Omega} (l(t, x) + d(t, x))u(t, x) dx. \quad (2.4)$$

If $d(t, x) \equiv d(x)$, then (2.4) implies dissipation and asymptotic compactness of the multi-valued semiflow $G : \mathbb{R}_+ \times X \rightarrow 2^X$ generated by weak solutions of (2.1) belonging to $L^p_{loc}(0, +\infty; L^p(\Omega))$ [12]. It means that the m -semiflow G possesses a global attractor, i.e., there exists a compact set $\Theta \subset X$ such that

$$\forall t > 0 \quad G(t, \Theta) = \Theta; \quad \forall r > 0 \quad \sup_{\|u_0\| \leq r} \text{dist}(G(t, u_0), \Theta) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Lemma 2.1. *Assume that $d(t, x) \equiv d(x) \in L^2(\Omega)$ and conditions (2.2) hold. Then for every $T > 0$, $u_0 \in L^2(\Omega)$ there exists at least one solution of (2.1) on $(0, T)$ in the sense of Definition 1 which belongs to $L^p(Q_T)$.*

Theorem 2.1. *Assume that condition (2.2) holds. Then for every $T > 0$, $d \in L^2(Q_T)$, $u_0 \in L^2(\Omega)$ problem (2.1) has at least one solution on $(0, T)$ which belongs to $L^p(Q_T)$, and, moreover, for every such solution the following inequality holds for a.e. $t \in (0, T)$*

$$\frac{d}{dt} \|u(t)\|^2 + \delta(\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{L^p}^p) \leq C_1(1 + \|d(t)\|^2),$$

where positive constants δ, C_1 do not depend on T, d, u_0 .

Corollary. *We can reformulate Definition 1 and speak about solution of (2.1) on (τ, T) . Then Theorem 2.1 allows us to claim that for every $\tau \geq 0$, $u_\tau \in L^2(\Omega)$ problem (2.1) has a solution on $(\tau, +\infty)$ with initial data $u|_{t=\tau} = u_\tau$, which belongs to $L^p_{loc}(\tau, +\infty; L^p(\Omega))$, i.e., there exists a function $u \in L^2_{loc}(\tau, +\infty; H^1_0(\Omega)) \cap L^p_{loc}(\tau, +\infty; L^p(\Omega)) \cap C([\tau, +\infty); L^2(\Omega))$ with $u|_{t=\tau} = u_\tau$, which is a solution of (2.1) on $(\tau, T) \forall T > \tau$.*

Therefore, for every $t \geq \tau \geq 0$, $u_\tau \in L^2(\Omega)$ we can correctly define the following set

$$U_d(t, \tau, u_\tau) = \left\{ u(t) \mid u(\cdot) \text{ is a solution of (2.1) on } (\tau, \infty) \right. \\ \left. \text{such that } u(\tau) = u_\tau \text{ and } u \in L^p_{loc}(\tau, +\infty; L^p(\Omega)) \right\}. \quad (2.5)$$

Lemma 2.2 ([7]). *Assume that an m -semiprocesses family $\{U_\sigma\}_{\sigma \in \Sigma}$ satisfies the following conditions:*

- (1) *there exists bounded set $B_0 \subset X$ such that for every bounded set $B \subset X$ there exists $T = T(B) > 0$ such that*

$$\forall t \geq T(B) \quad \forall \sigma \in \Sigma \quad U_\sigma(t, 0, B) \subset B_0;$$

- (2) *for every $\{\sigma_n\} \subset \Sigma$, $\{t_n \nearrow \infty\}$, and bounded $\{u_n\} \subset X$ every sequence $\{\xi_n \in U_{\sigma_n}(t_n, 0, u_n)\}$ is precompact in X .*

Then $\{U_\sigma\}_{\sigma \in \Sigma}$ possesses uniform attractor Θ_Σ .

- (3) *If, additionally, the multi-valued map*

$$\Sigma \times X \ni (\sigma, u) \mapsto U_\sigma(t, 0, u)$$

has closed graph, then

$$\forall t \geq 0 \quad \Theta_\Sigma \subset \bigcup_{\sigma \in \Sigma} U_\sigma(t, 0, \Theta_\Sigma). \quad (2.6)$$

Now consider problem (2.1) under conditions (2.2) with the additional assumption that

$$\|d\|_+^2 := \sup_{t \geq 0} \int_t^{t+1} \|d(s)\|^2 ds < \infty. \quad (2.7)$$

Theorem 2.2. *Under conditions (2.2), (2.7) the m -semiprocesses family $\{U_\sigma\}_{\sigma \in \Sigma(d)}$ given by (2.5) possesses uniform attractor $\Theta_{\Sigma(d)}$, which satisfies invariance property (2.6).*

Assume that $d \in L^\infty(0, +\infty; L^2(\Omega))$, $\|d\|_\infty := \operatorname{ess\,sup}_{t \geq 0} \|d(t)\| \leq R$, $R \geq 0$ is fixed.

Theorem 2.3. *Under conditions (2.2), there exists $\gamma \in \mathcal{K}$ such that $\forall u_0 \in X$*

$$\overline{\lim}_{t \rightarrow \infty} \|U_d(t, 0, u_0)\|_\Theta \leq \gamma(\|d\|_\infty),$$

where Θ is the global attractor of the unperturbed problem (2.1).

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References

- [1] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*. American Mathematical Society Colloquium Publications, 49. American Mathematical Society, Providence, RI, 2002.
- [2] S. Dashkovskiy, P. Feketa, O. Kapustyan and I. Romaniuk, Invariance and stability of global attractors for multi-valued impulsive dynamical systems. *J. Math. Anal. Appl.* **458** (2018), no. 1, 193–218.
- [3] N. V. Gorban, O. V. Kapustyan and P. O. Kasyanov, Uniform trajectory attractor for non-autonomous reaction-diffusion equations with Carathéodory's nonlinearity. *Nonlinear Anal.* **98** (2014), 13–26.
- [4] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Structure and regularity of the global attractor of a reaction-diffusion equation with non-smooth nonlinear term. *Discrete Contin. Dyn. Syst.* **34** (2014), no. 10, 4155–4182.
- [5] O. V. Kapustyan, O. B. Kurylko and T. V. Yusyiv, Robust stability of the global attractor of the reaction-diffusion system. *Visn., Ser. Fiz.-Mat. Nauky, Kyiv. Univ. Im. Tarasa Shevchenka* **3** (2021), 46–50.
- [6] O. V. Kapustyan, O. B. Kurylko, T. V. Yusyiv and A. V. Pankov, Stability w.r.t. disturbances for the global attractor of multi-valued semiflow generated by nonlinear wave equation. *J. Optim. Differ. Equ. Appl., JODEA* **31** (2023), no. 1, 111–124.
- [7] O. V. Kapustyan, V. V. Sobchuk, T. V. Yusyiv and A. V. Pankov, Robust stability of global attractors for evolutionary systems without uniqueness. *J. Optim. Differ. Equ. Appl., JODEA* **30** (2022), no. 2, 49–61.
- [8] O. V. Kapustyan, O. M. Stanzhitskyi, J. V. Fedorenko and V. I. Kravets, Global solvability and attractors for a parabolic inclusion with upper semicontinuous polynomial right-hand part. *Nonlinear Oscillations* **27** (2024), no. 3, 368–377.
- [9] O. V. Kapustyan, T. V. Yusyiv, M. Ospanov and M. Alday, Global attractors and asymptotic gain property for non-autonomous inclusion of reaction-diffusion type. *J. Optim. Differ. Equ. Appl.* **33** (2025), no. 1, 1–14.
- [10] V. S. Melnik and J. Valero, On attractors of multivalued semi-flows and differential inclusions. *Set-Valued Anal.* **6** (1998), no. 1, 83–111.
- [11] V. S. Melnik and J. Valero, On global attractors of multivalued semiprocesses and nonautonomous evolution inclusions. *Set-Valued Anal.* **8** (2000), no. 4, 375–403.
- [12] J. C. Robinson, *Infinite-Dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001.
- [13] J. Schmid, O. Kapustyan and S. Dashkovskiy, Asymptotic gain results for attractors of semi-linear systems. *Math. Control Relat. Fields* **12** (2022), no. 3, 763–788.
- [14] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997.