Time-Averaging Approach to Minimax Estimation for Parabolic Problems with Nonlinear Observations

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1 Introduction

Minimax estimation of functionals arising in partial differential equations (PDEs) has been actively studied for several decades. A general formulation of minimax estimation problems for operator equations in Hilbert spaces was first proposed by O. Nakonechnyi [10]. In the parabolic case, O. Kapustian and O. Nakonechnyi investigated minimax problems for pointwise observations in boundary value problems [4] and later constructed optimal bounded controls for parabolic systems with rapidly oscillating coefficients [3]. Further developments concerned guaranteed mean-square estimates based on homogenization techniques: in particular, O. Nakonechnyi, O. Kapustian, and A. Chikrii established the existence of a guaranteed linear estimate and applied averaging methods to obtain approximate minimax estimators for parabolic equations with fast oscillations [11]. Related questions for hyperbolic systems were considered in [7], where minimax estimation was extended to linear hyperbolic PDEs with uncertain data.

In a broader context, the monograph of T. Basar and P. Bernhard [1] provides a general H^∞-control framework and related minimax design problems in a dynamic game setting, forming an important part of the modern basis of minimax control theory. Minimax-robust filtering for stochastic sequences, including models with stationary increments and cointegration, was studied by M. Luz and M. Moklyachuk [9], thereby extending the minimax estimation paradigm beyond PDEs. Important ideas involving averaging and robustness also appear in the theory of impulsive differential equations, as presented in the monograph by A. Samoilenko and M. Perestyuk [12]. Furthermore, A. Bensoussan's book [2] gives a systematic treatment of perturbation and averaging methods in optimal control, which has significantly influenced the analysis of minimax estimation and control for distributed parameter systems.

In addition to estimation problems, closely related results concern control problems for parabolic systems. In [5], an approximate bounded synthesis method was proposed for distributed control of parabolic models. Averaging techniques have also been successfully applied in this setting: in particular, O. Kapustyan et al. recently employed a time-averaging approach to an optimal control problem for a parabolic differential inclusion with rapidly oscillating coefficients on a finite time interval [6]. These works illustrate how averaging, in time or with respect to parameters, can simplify the analysis and design of control strategies for complex systems.

In this paper, we address the problem of minimax estimation of functionals depending on the solution of a parabolic boundary value problem under a non-linear observation. The use of time averaging in the observation model distinguishes our approach from earlier studies that relied primarily on spatial homogenization techniques.

2 Setting of the problem and the main results

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(n \geq 1)$. In the cylinder $Q_T = \Omega \times (0,T)$ we consider the problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A\varphi = f_1(x,t), \\ \varphi\big|_{\partial\Omega} = 0, \\ \varphi\big|_{t=0} = f_0(x), \end{cases} \tag{2.1}$$

where $A = -\operatorname{div}(a(x)\nabla)$, $a \in L^\infty(\Omega)$ is a symmetric matrix satisfying the uniform ellipticity and boundedness conditions: $\exists\, c_1 > 0,\ c_2 > 0\ \forall \eta \in \mathbb{R}^n\ \forall x \in \mathbb{R}^n$

$$c_1 \sum_{i=1}^n \eta_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \leq c_2 \sum_{i=1}^n \eta_i^2.$$

It is known that for fixed $f_1 \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$ there exists a unique solution in the space

$$W(0,T) = \big\{ y \in L^2(0,T; H_0^1(\Omega)) \mid\ y_t \in L^2(0,T; H^{-1}(\Omega)) \big\}.$$

The observed function is

$$y^\varepsilon(t) = \int_\Omega C\Big(x, \frac{t}{\varepsilon}, \varphi(x,t)\Big)\varphi(x,t)\,dx + f_2(t), \tag{2.2}$$

where $C(x,\tau,\xi) : \Omega \times (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a given measurable function; there exists a function $C_1 \in L^\infty(\Omega)$ such that $\forall \xi \in \mathbb{R}$ and a.e. $(x,\tau) \in \Omega \times (0,\infty)$

$$|C(\tau,x,\xi)| \leq C_1(x); \tag{2.3}$$

there exists a function $C_0 \in L^\infty(\Omega)$ such that $\forall r > 0$

$$\frac{1}{T}\int_0^T C(x,s,\xi)\,ds \longrightarrow C_0(x),\ \ T \to \infty\ \text{ in } L^2(\Omega) \text{ uniformly on } |\xi| \leq r.$$

The functions $f_0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$, $f_2 \in L^2(0,T)$ are unknown and belong to a convex closed set G of the space $L^2(\Omega) \times L^2(Q_T) \times L^2(0,T)$, namely,

$$\{f_0, f_1, f_2\} \in G = \Big\{ \|f_0\|^2_{L^2(\Omega)} + \|f_1\|_{L^2(Q_T)} + \|f_2\|_{L^2(0,T)} \leq 1 \Big\}. \tag{2.4}$$

The main problem is to estimate the functional of the solutions of problem (2.1)

$$l(\varphi) = \int_0^T \int_\Omega a(x,t)\varphi(x,t)\ dx\,dt, \tag{2.5}$$

where $a \in L^2(Q_T)$, in the class of linear functionals of the observations

$$\widehat{\widehat{l}}(\varphi) = \int_0^T y^\varepsilon(t)u(t)\,dt, \tag{2.6}$$

where $u \in L^2(0,T)$ is the solution of the problem

$$J_\varepsilon(u) = \sup_{\{f_0,f_1,f_2\}\in G} \big(l(\varphi) - \widehat{\widehat{l}}(\varphi)\big)^2 \longrightarrow \inf. \tag{2.7}$$

We consider the quantity

$$\sigma_\varepsilon = \inf_u J_\varepsilon(u), \tag{2.8}$$

which is called the error of the minimax estimation.

First, it was proved that problem (2.1), (2.2), (2.4)–(2.8) admits a solution. Namely, we have the following result.

Theorem 2.1. *Let $f_0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$, $f_2 \in L^2(0,T)$. Assume that problem* (2.1) *has a unique solution in the space $W(0,T)$. Let condition* (2.3) *take place. Then for every $\varepsilon > 0$ there exists a solution to problem* (2.1), (2.2), (2.4)–(2.8)*, that is there exists such function $\widehat{u}^\varepsilon \in L^2(0,T)$ that minimizes the minimax estimation error* (2.8)*.*

And, further, it was constructed an approximate minimax estimate for the functional (2.5). Namely, we have the following result.

Theorem 2.2. *The estimate*

$$\widehat{l}(\varphi) = \int_0^T u^0(t) y^\varepsilon(t)\,dt$$

is an approximate minimax estimate for problem (2.1), (2.2), (2.4)–(2.8)*. Moreover, the errors satisfy*

$$\sigma_\varepsilon \to \sigma_0 \ \ as \ \ \varepsilon \to 0,$$
$$\widehat{\sigma}_\varepsilon \longrightarrow \sigma_0 \ \ as \ \ \varepsilon \to 0,$$

where σ_ε is given by (2.8)*,*

$$\sigma_0 = \inf_u J^0(u), \quad J^0(u) = \|u\|^2_{L^2(0,T)} + \|z(x,0)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(Q_T)}$$

is the explicit form of the functional for corresponding averaged problem, and

$$\widehat{\sigma}_\varepsilon = J_\varepsilon(u^0). \tag{2.9}$$

Furthermore, for sufficiently small $\varepsilon > 0$ the following holds: for every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\big|\sigma_\varepsilon - \widehat{\sigma}_\varepsilon\big| < \eta,$$

i.e., the errors (2.8) *and* (2.9) *are close for all sufficiently small $\varepsilon > 0$.*

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