Exponential Stability of Impulsive Stochastic Systems with Delay in Terms of Positive-Inverse Matrices

Ramazan I. Kadiev[1,2]

[1]*Dagestan Research Center of the Russian Academy of Sciences, Makhachkala, Russia*
[2]*Department of Mathematics, Dagestan State University, Makhachkala, Russia*
E-mail: kadiev_r@mail.ru

Arcady Ponosov

Department of Mathematical Sciences and Technology,
Norwegian University of Life Sciences, P.O. Box 5003, N-1432 Ås, Norway
E-mail: arkadi@nmbu.no

The work is aimed to contribute to stochastic stability analysis, based on the well-known W-method, a general description of which is offered in the monograph [1]. In [2,3,5], stability of linear deterministic delay equations was studied by means of this method, while some results, obtained by this method in the stochastic case, can be found in the papers [6–12, 14]. This approach shows efficiency of the analysis for many popular classes of deterministic and stochastic delay equations including high-order, impulsive and fractional ones. Moreover, as it shown in [11] and [14], the method can be successfully adjusted to the nonlinear case. It is also essential to note that the second Lyapunov method may be difficult to use in the case of many stochastic delay equations, in particular, those with random impulses. As it is shown in the presentation, these difficulties can be overcome by the W-method.

With this work, we continue stability analysis of stochastic delay equations with random impulses started in [8], but unlike that paper we concentrate on the nonlinear case, where the standard W-transform, as it is outline in [2], see also [3], cannot be applied. Therefore, it was suggested in [4] to incorporate inverse-positive matrices into this transform, which opened for asymptotic analysis of nonlinear delay equations.

The presentation is organized as follows. We start with some preliminary information and the notation. Then we introduce the system of nonlinear stochastic delay equations with impulses we intend to study and formulate the assumptions to be used in the main theorem. This theorem describes sufficient conditions of exponential $2p$-stability ($p \geq 1$) of the main system. Some particular cases of the general system are formulated as corollaries.

1 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis, where Ω is set of elementary probability events, $\mathcal{F}$ is a σ-algebra of all events on Ω, $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous family of σ-subalgebras of $\mathcal{F}$, P is a probability measure on $\mathcal{F}$; all the above σ-algebras are assumed to be complete w.r.t. P, i.e. containing all subsets of zero measure. The expectation (the integral w.r.t. the measure P) is denoted by E. An $\mathcal{B}(I)\otimes\mathcal{F}$-measurable stochastic process $\xi(t) = \xi(t, \omega)$, $t \in I$, is called $\mathcal{F}_t$-adapted if $\xi(t, \cdot)$ is $\mathcal{F}_t$-measurable for all $t \in I$.

We will use the following notation:

- $|\cdot|$ is an arbitrary yet fixed norm in R^n, $\|\cdot\|$ being the associated matrix norm;

- ν is the Lebesgue measure on $[0,+\infty)$;
- $\|\cdot\|_X$ is the norm in a normed space X;
- p is an arbitrary real number satisfying $1\le p<\infty$;
- $(\mathcal{B}_1,\dots,\mathcal{B}_m)$ is the standard m-dimensional Brownian motion (i.e. the scalar Brownian motions $\mathcal{B}_l$ are all independent);
- k^n stands for the linear space of all n-dimensional, $\mathcal{F}_0$-measurable random values;
- $k^n_q=\{\alpha:\ \alpha\in k^n,\ \|\alpha\|_{k^n_q}\equiv(E|\alpha|^q)^{1/q}<\infty\}$.

The two lemmas below, the proof of which can be found in [9], contain inequalities which are used in this presentation.

Lemma 1.1.
$$\bigg(E\bigg|\int\limits_0^t f(s)\,d\mathcal{B}_i(s)\bigg|^{2p}\bigg)^{1/2p}\le c_p\bigg(E\bigg(\int\limits_0^t|f(s)|^2\,ds\bigg)^p\bigg)^{1/2p} \tag{1.1}$$
for any $\mathcal{F}_t$-adapted stochastic process $f(s)$ $(0\le s\le t)$, any $t>0$ and any component $\mathcal{B}_i(s)$ $(1\le l\le m)$ of the Brownian motion $\mathcal{B}$.

Lemma 1.2. *Let $g(s)$ be a scalar function which is square integrable on $[0,\infty)$, $f(s)$ be an $\mathcal{F}_t$-adapted stochastic process satisfying $\sup\limits_{s\ge0}(E|f(s)|^{2p})^{1/2p}<\infty$. Then*
$$\sup_{t\ge0}\bigg(E\bigg|\int\limits_0^t g(s)f(s)\,ds\bigg|^{2p}\bigg)^{\frac{1}{2p}}\le\sup_{t\ge0}\bigg(\int\limits_0^t|g(s)|\,ds\bigg)\sup_{t\ge0}\big(E|f(t)|^{2p}\big)^{\frac{1}{2p}} \tag{1.2}$$
and
$$\sup_{t\ge0}\bigg(E\bigg|\int\limits_0^t(g(s))^2(f(s))^2\,ds\bigg|^{p}\bigg)^{\frac{1}{2p}}\le\sup_{t\ge0}\bigg(\int\limits_0^t(g(s))^2\,ds\bigg)^{\frac12}\sup_{t\ge0}\big(E|f(t)|^{2p}\big)^{\frac{1}{2p}}. \tag{1.3}$$

The definition below is central for the presentation.

Definition 1.1. A non-singular $n\times n$-matrix $B=(b_{ij})^n_{i,j=1}$ will be called totally inverse-positive if the entries of its inverse matrix B^{-1} are strictly positive.

Some explicit characterizations of totally inverse-positive matrices can be found in the paper [13]. It is evident, yet important, that this property is stable under small perturbations (in any matrix norm).

2 Formulation of the problem

We study exponential stability of the following system of Itô delay differential equations with impulses
$$\begin{aligned} dx(t)=\Big[-\sum_{j=1}^N A^j(t)x(h^j(t))\,dt+F(t,x(h^0_1(t),\dots,x(h^0_{m_0}(t)))\Big]\,dt&\\ +\sum_{i=1}^m G^i\big(t,x(h_1(t)),\dots,x(h_{m_i}(t))\big)\,d\mathcal{B}_i(t)\ \ (t\ge0,\ \ j=1,\dots,N),& \\ x(\mu_\theta)=B^\theta x(\mu_\theta-0)\ \ (\theta=1,2,3,\dots)\ \text{ almost surely}& \end{aligned} \tag{2.1}$$

equipped with the initial conditions

$$x(t) = \varphi(t) \ \ (t < 0), \tag{2.2}$$

and

$$x(0) = b. \tag{2.3}$$

Remark 2.1. According to [1], we separate the initial conditions for $t < 0$ and $t = 0$, as we do not require the continuity of the function $\varphi(t)$. This function is assumed to be only bounded and measurable, so that changing its value at countably many points does not change the solution of Eq. (2.1). On the other hand, it can be easily checked by examples that changing the value of $x(0) = b$ usually changes the solution of Eq. (2.1). That is, the space of initial functions φ and the space of initial values b have different topologies.

The following assumptions are put on (2.1)–(2.3):

(1) $x = col(x_1, \dots, x_n)$ is an unknown n-dimensional $\mathcal{F}_t$-adapted stochastic process defined for $t \geq -\sigma$;

(2) $A^j(t) = [a^j_{sk}(t)]$ $(t \geq 0,\ j = 1, \dots, N,\ s, k = 1, \dots, n)$ are $n \times n$-matrices, the entries of which are progressively measurable stochastic processes with almost surely locally integrable trajectories;

(3) $F(t, u) = (F_s(t, u_1, \dots, u_{m_0}))^n_{s=1},\ F(\,\cdot\,, 0) = 0$, is a Carathéodory function for $t \geq 0,\ u_j = col(u^j_1, \dots, u^j_n) \in R^n$ $(j = 1, \dots, m_0)$ such that

$$|F_s(t, u)| \leq \sum_{j=1}^{m_0} \sum_{l=1}^{n} \overline{F}^{\,j}_{sl} |u^j_l| \ \ (t \geq 0, \ \ s = 1, \dots, n),$$

ν-almost everywhere and $\overline{F}^{\,j}_{sl} \geq 0$ $(s, l = 1, \dots, n,\ j = 1, \dots, m_0)$;

(4) $G^i(t, u) = (G^i_s(t, u_1, \dots, u_{m_i}))^n_{s=1},\ G^i(\,\cdot\,, 0) = 0$, is a Carathéodory function for $t \geq 0,\ u^j = col(u^j_1, \dots, u^j_n) \in R^n$ $(j = 1, \dots, m_i)$ such that

$$|G^i_s(t, u)| \leq \sum_{j=1}^{m_i} \sum_{l=1}^{n} \overline{G}^{ij}_{sl} |u^j_l| \ \ (t \geq 0, s = 1, \dots, n),$$

ν-almost everywhere and $\overline{G}^{ij}_{sl} \geq 0$ $(s, l = 1, \dots, n,\ i = 1, \dots, m,\ j = 1, \dots, m_i)$;

(5) h^j $(j = 1, \dots, N)$, h^i_l $(i = 0, \dots, m,\ l = 1, \dots, m_i)$ are Borel measurable scalar functions defined on $[0, \infty)$, which for some constants τ^j $(j = 1, \dots, N)$, τ^i_l $(i = 0, \dots, m, l = 1, \dots, m_i)$ satisfy the inequalities

$$0 \leq t - h^j(t) \leq \tau^j \ \ (t \geq 0, \ \ j = 1, \dots, N),$$
$$0 \leq t - h^i_l(t) \leq \tau^i_l \ \ (t \geq 0, \ \ i = 0, \dots, m, \ \ l = 1, \dots, m_i),$$

ν-everywhere;

(6) $b = col(b_1, \dots, b_n)$ is an n-dimensional $\mathcal{F}_0$-measurable random variable belonging to the space $b \in k^n$;

(7) $\varphi = col(\varphi_1, \dots, \varphi_n)$ is an n-dimensional $\mathcal{F}_0$-measurable stochastic process with essentially bounded trajectories, defined on the interval $[-\sigma, 0)$, where $\sigma = \max\{\tau^j, \tau^i_l :\ j = 1, \dots, N,\ i = 0, \dots, m,\ l = 1, \dots, m_i\}$;

(8) μ_θ is a strictly increasing sequence of stopping times $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, $\lim\limits_{\theta\to\infty} \mu_k = \infty$ almost surely;

(9) B^θ is a diagonal $n \times n$-matrix, the diagonal entries $b^\theta_{ii} (i = 1, \dots, n)$ of which are real numbers for all $\theta = 1, 2, 3, \dots$;

(10) for any initial conditions (2.2), (2.3), satisfying the above assumptions, there exists a unique (up to the natural equivalence of stochastic processes) strong, continuous and $\mathcal{F}_t$-adapted solution $x(t, b, \varphi)$ of problem (2.1)–(2.3), i.e. a solution defined on the given stochastic basis.

Remark 2.2. It can be proven [7] that under assumptions (1)–(7) and Lipschitz conditions on the functions F, G^i $(i = 1, \dots, m)$, the initial value problem (2.1)–(2.3), indeed, has property (9).

Definition 2.1. We say that equation (2.1) is globally exponentially q-stable $(0 < q < \infty)$ with respect to the initial data, i.e. the initial value x_0 and the "prehistory" function φ, if there are positive numbers K, λ such that all solutions $x(t, b, \varphi)$ of the initial value problem (2.1)–(2.3) satisfy

$$\big(E|x(t,b,\varphi)|^q\big)^{1/q} \le K \exp\{-\lambda t\}\Big(E|b|^q + \operatorname*{ess\,sup}_{t<0}\, (E|\varphi(t)|^q)^{1/q}\Big) \ \ (t \ge 0). \tag{2.4}$$

3 Main results

The conditions below are used in the main theorem.

(A) For each $s = 1, \dots, n$ there exist a nonempty subset $I_s \subset \{1, \dots, N\}$ and positive real numbers ρ, σ, $\overline{b}_s$, $\overline{a}_s$, $\overline{a}^j_{sk}$ $(j = 1, \dots, N,\ s, k = 1, \dots, n)$, for which the following estimates hold true:

- $|b^\theta_{ss}| \le \overline{b}_s$ $(s = 1, \dots, n,\ \theta = 1, 2, \dots)$;
- $\rho \le \mu_{\theta+1} - \mu_\theta \le \sigma$ almost surely for all $\theta = 1, 2, \dots$;
- $|a^j_{sk}(t)| \le \overline{a}^j_{sk}$ $(t \ge 0,\ j = 1, \dots, N,\ s, k = 1, \dots, n)$ $P \times \nu$-almost everywhere;
- $\sum\limits_{k\in I_s} a^k_{ss}(t) \ge \overline{a}_s$ $(t \ge 0,\ s = 1, \dots, n)$ $P \times \nu$-almost surely.

Let us also introduce the following notation:

$$\overline{F}_{sl} := \sum_{j=1}^{m_0} \overline{F}^j_{sl}, \quad \overline{G}_{sl} := \sum_{i=1}^{m}\sum_{j=1}^{m_i} \overline{G}^{ij}_{sl} \ \ (s, l = 1, \dots, n),$$

$$L_{1s} := \frac{\max\{1, \overline{b}_s\}(1 - \exp\{-\overline{a}_s\sigma\})}{\overline{a}_s(1 - \exp\{-\overline{a}_s\rho\}\overline{b}_s)} \ \ (s = 1, \dots, n),$$

$$L_{2s} := \Big(\frac{\max\{1, \overline{b}_s^{\,2}\}(1 - \exp\{-2\overline{a}_s\sigma\})}{2\overline{a}_s(1 - \exp\{-2\overline{a}_s\rho\}\overline{b}_s^{\,2})}\Big)^{1/2} \ \ (s = 1, \dots, n).$$

The $n \times n$-matrix C is defined as

$$c_{ss} = 1 - L_{1s}\Big[\sum_{k\in I_s} \tau^k \overline{a}^k_{ss}\Big(\sum_{j=1}^{N} \overline{a}^j_{ss} + \overline{F}_{ss} + \frac{c_p}{\sqrt{\tau^k}}\,\overline{G}_{ss}\Big) + \sum_{j=1, j\notin I_s}^{N} \overline{a}^j_{ss} + \overline{F}_{ss}\Big] - c_p\overline{G}_{ss}L_{2s} \ \ (s = 1, \dots, n)$$

and

$$c_{sl} = L_{1s}\Big[\sum_{k\in I_s} \tau^k \overline{a}^k_{ss}\Big(\sum_{j=1}^{N} \overline{a}^j_{sl} + \overline{F}_{sl} + \frac{c_p}{\sqrt{\tau^k}}\,\overline{G}_{ss}\Big) + \sum_{j=1}^{N} \overline{a}^j_{sl} + \overline{F}_{sl}\Big] - c_p\overline{G}_{sl}L_{2s} \ \ (s, l = 1, \dots, n,\ \ s \ne l).$$

Theorem 3.1. *If Conditions* **(A)** *are fulfilled and C is totally inverse-positive, then Eq.* (2.1) *is globally exponentially $2p$-stable in the sense of Definition 2.1.*

As an example, we consider

$$
\begin{aligned}
dx(t) &= -A^1(t)x(h^1(t))\,dt + \sum_{i=1}^{m}\sum_{j=1}^{m_i} A^{ij}(t)x(h^i_jt))\,d\mathcal{B}_i(t) \ \ (t \ge 0),\\
x(\mu_\theta) &= B^\theta x(\mu_\theta - 0) \ \ (\theta = 1,2,3,\dots) \ \text{ almost surely,}
\end{aligned} \tag{3.1}
$$

where $A^1 = (a^1_{sk})^n_{s,k=1}$ are $n\times n$-matrices, the entries of which are progressively measurable stochastic processes with almost surely locally integrable trajectories, while $A^{ij} = (a^{ij}_{sk})^n_{s,k=1}$ $(i = 1,\dots,m$, $j = 1,\dots,m_i$ are $n \times n$-matrices, the entries of which are progressively measurable stochastic processes with almost surely locally square-integrable trajectories. Assume that there exist positive real numbers $\overline{a}^1_{sk}$ $(s,k = 1,\dots,n)$ and $\overline{a}^{ij}_{sk}$ $(i = 1,\dots,m,\ j = 1,\dots,m_i,\ s,k = 1,\dots,n)$ such that $|a^1_{sk}(t)| \le \overline{a}^1_{sk}$ $(s,k = 1,\dots,n)$ and $|a^{ij}_{sk}(t)| \le \overline{a}^{ij}_{sk}$ $(t \ge 0,\ i = 1,\dots,m,\ j = 1,\dots,m_i,\ s,k = 1,\dots,n)$ $P \times \nu$-almost everywhere, the delay functions $h^1(t)$ and $h^i_j(t)$ $(i = 1,\dots,m,\ j = 1,\dots,m_i)$ satisfy the same assumptions as before, while the other parameters coincide with those defined for Eq. (2.1).

Putting $C^{(1)} = (c_{sl})$, where

$$
c_{ss} = 1 - L_{1s}(\overline{a}^1_{ss})^2\tau^1 - c_pL_{2s}\Big[\sum_{i=1}^{m}\sum_{j=1}^{m_i}\sqrt{\tau^1}\,\overline{a}^1_{ss}\overline{a}^{ij}_{ss} + \sum_{i=1}^{m}\sum_{j=1}^{m_i}\overline{a}^{ij}_{ss}\Big] \ \ (s = 1,\dots,n), \tag{3.2}
$$

$$
c_{sl} = -L_{1s}\big[\overline{a}^1_{ss}\overline{a}^1_{sl}\tau^1 + \overline{a}^1_{sl}\big] - c_pL_{2s}\Big[\sum_{i=1}^{m}\sum_{j=1}^{m_i}\sqrt{\tau^1}\overline{a}^1_{ss}\overline{a}^{ij}_{sl} + \sum_{i=1}^{m}\sum_{j=1}^{m_i}\overline{a}^{ij}_{sl}\Big] \ \ (s,l = 1,\dots,n,\ s \ne l), \tag{3.3}
$$

we obtain

Corollary 3.1. *If $C^{(1)}$ is a totally inverse-positive matrix, the Eq.* (3.1) *is exponentially $2p$-stable in the sense of Definition 2.1.*

It is worth mentioning that Theorem 3.1 provides new stability results for impulsive deterministic systems as well. Consider e.g.

$$
\begin{aligned}
dx(t) &= -\sum_{j=1}^{N} A^j x(t - h^j)\,dt \ \ (t \ge 0),\\
x(\mu_\theta) &= B^\theta x(\mu_\theta - 0) \ \ (\theta = 1,2,3,\dots),
\end{aligned} \tag{3.4}
$$

where $A^j = (a^j_{sl})^n_{s,l=1}$ $(j = 1,\dots,N)$ are real $n\times n$-matrices, h^j $(j = 1,\dots,N)$ are nonnegative real numbers, μ_θ $(\theta = 0,1,2,\dots)$ are real numbers satisfying $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, $\lim\limits_{\theta\to\infty}\mu_k = \infty$, and B^θ $(\theta = 1,2,3,\dots)$ are real diagonal $n\times n$-matrices with the diagonal entries b^θ_{ii} $(i = 1,\dots,n)$.

Assume that

- $\sum\limits_{j=1}^{N} a^j_{ss} = a_s > 0$ $(s = 1,\dots,n)$,
- there exist positive numbers ρ, σ, $\overline{b}_s$ $(s = 1,\dots,n)$ such that $|b^\theta_{ss}| \le \overline{b}_s$ $(s = 1,\dots,n$, $\theta = 1,2,\dots)$ and $\rho \le \mu_{\theta+1} - \mu_\theta \le \sigma$ $(\theta = 1,2,\dots)$,

- $c_{ss} = 1 - L_{1s} \sum_{k=1}^{N} \sum_{j=1}^{N} h^k |a_{ss}^k| \, |a_{ss}^j|$ $(s = 1, \dots, n)$,
- $c_{sl} = -L_{1s} \Big[\sum_{k=1}^{N} \sum_{j=1}^{N} h^k |a_{ss}^k| \, |a_{sl}^j| + \sum_{j=1}^{N} |a_{sl}^j| \Big]$ $(s, l = 1, \dots, n,\ s \neq l)$.

Corollary 3.2. *Eq.* (3.4) *is exponentially stable with respect to initial data if the $n \times n$-matrix $C^{(2)} = (c_{sl})$ is a totally inverse-positive matrix.*

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