The Dirichlet Problem in a Characteristic Rectangular Domain for a Third-Order Linear Hyperbolic Equation

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On the plane of independent variables x and y consider a third-order hyperbolic equation of the form

$$\Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\alpha\Big)u_{xy}=F, \tag{1}$$

where F is a given and u is an unknown real functions, α is a given real constant.

Denote by D the rectangular bounded by the characteristics lines: $x=0$, $x=a$ and $y=0$, $y=b$, where a, b are given positive numbers.

For equation (1) consider the Dirichlet problem in the following statement: find a solution u of equation (1) in the domain D that satisfies the boundary condition

$$u\big|_{\partial D}=f, \tag{2}$$

where f is on the boundary of domain D given real function. In this direction, the following papers should be noted [1–7].

Remark 1. When considering problem (1), (2), we will search the solution u in the class $C^3(\overline{D})$ under the fulfillment of the corresponding conditions of smoothness and agreement of the data of this problem.

Theorem 1. *If $\alpha\neq 0$, then problem* (1), (2) *is posed correctly, the solution of which can be constructed in quadratures in the characteristic rectangle D; if $\alpha=0$, then problem* (1), (2) *is ill-posed, in particular, the corresponding to* (1), (2) *homogeneous problem has an infinite number of linearly independent solutions, which can be constructed in quadratures too.*

Proof. Below, for simplicity of presentation, we will assume that $a=2b$. Let us rewrite the Dirichlet characteristic boundary condition (2) in the form

$$u(x,0)=f_1(x),\ \ 0\leq x\leq 2b,\quad u(0,y)=f_2(y),\ \ 0\leq y\leq b, \tag{3}$$

$$u(2b,y)=f_3(y),\quad 0\leq y\leq b,\quad u(x,b)=f_4(x),\quad 0\leq x\leq 2b, \tag{4}$$

where the functions f_i, $i=1,2,3,4$, satisfy the following conditions of smoothness $f_1,f_4\in C^3([0,2b])$, $f_2,f_3\in C^3([0,b])$, and consistency

$$f_1(0)=f_2(0),\ \ f_2(b)=f_4(0),\ \ f_3(0)=f_1(2b),\ \ f_3(b)=f_4(2b).$$

First, for equation (1), we consider the auxiliary Goursat problem with the boundary conditions on the characteristic segments $x = 0$ and $y = 0$, when, in addition to conditions (3), the following conditions must be satisfied:

$$u_y(x,0) = \varphi(x), \quad 0 \le x \le 2b, \quad u_x(0,y) = \psi(y), \quad 0 \le y \le b, \tag{5}$$

with subsequent selection of the functions φ and ψ so that condition (4) is satisfied. It is assumed that the following conditions of smoothness $\varphi \in C^2([0,2b]), \psi \in C^2([0,b])$ and consistency

$$f_1(0) = f_2(0), \quad \psi(0) = f_1'(0), \quad \varphi(0) = f_2'(0), \quad \varphi'(0) = \psi'(0)$$

are satisfied.

Let's introduce the notation

$$v := u_{xy}.$$

Then taking into account equation (1) and condition (5) for the function v we obtain the following boundary value problem

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \alpha\Big)v = F, \quad (x,y) \in D, \tag{6}$$

$$v(x,0) = \varphi'(x), \quad 0 \le x \le 2b, \quad v(0,y) = \psi'(y), \quad 0 \le y \le b. \tag{7}$$

By direct integration the solution of problem (6), (7) is given by the formula

$$v(x,y) = \begin{cases} \exp(-\alpha y)\varphi'(x-y) + F_1(x,y), & y \le x, \\ \exp(-\alpha x)\psi'(y-x) + F_2(x,y), & y \ge x, \end{cases}$$

where

$$F_1(x,y) := \exp(-\alpha x) \int\limits_{x-y}^{x} F(\xi, \xi + y - x) \exp(\alpha \xi)\, d\xi, \quad y \le x,$$

$$F_2(x,y) := \exp(-\alpha y) \int\limits_{y-x}^{y} F(\eta + x - y, \eta) \exp(\alpha \eta)\, d\eta, \quad y \ge x.$$

The next step is the solution of Goursat's problem for the equation

$$u_{xy} = v$$

with the characteristic data (3). The solution is given by the formula

$$u(x,y) = \begin{cases} f_1(x) + f_2(y) - f_2(0) + \int\limits_0^y \exp(-\alpha\xi)\, d\xi \int\limits_\xi^y \psi'(\eta - \xi)\, d\eta \\ \qquad + \int\limits_0^y \exp(-\alpha\eta)\, d\eta \int\limits_\eta^x \varphi'(\xi - \eta)\, d\xi + F_3(x,y), & y \le x, \\ f_1(x) + f_2(y) - f_2(0) + \int\limits_0^x \exp(-\alpha\xi)\, d\xi \int\limits_\xi^y \psi'(\eta - \xi)\, d\eta \\ \qquad + \int\limits_0^x d\xi \int\limits_0^\xi \exp(-\alpha\eta)\varphi'(\xi - \eta)\, d\eta + F_4(x,y), & y \ge x, \end{cases} \tag{8}$$

where

$$F_3(x,y):=\int\limits_0^y d\xi\int\limits_\xi^y F_2(\xi,\eta)\,d\eta+\int\limits_0^y d\eta\int\limits_\eta^x F_1(\xi,\eta)\,d\xi,\ \ y\le x,$$
$$F_4(x,y):=\int\limits_0^x d\xi\int\limits_\xi^y F_2(\xi,\eta)\,d\eta+\int\limits_0^x d\xi\int\limits_0^\xi F_1(\xi,\eta)\,d\eta,\ \ y\ge x.$$

Using (8) and conditions (4), we get

$$\begin{aligned}
&f_1(x)+f_2(b)-f_2(0)+\int\limits_0^x \exp(-\alpha\xi)\,d\xi\int\limits_\xi^b \psi'(\eta-\xi)\,d\eta\\
&\qquad+\int\limits_0^x d\xi\int\limits_0^\xi \exp(-\alpha\eta)\varphi'(\xi-\eta)\,d\eta+F_4(x,b)=f_4(x),\ \ 0\le x\le b,\\
&f_1(x)+f_2(b)-f_2(0)+\int\limits_0^b \exp(-\alpha\xi)\,d\xi\int\limits_\xi^b \psi'(\eta-\xi)\,d\eta\\
&\qquad+\int\limits_0^b \exp(-\alpha\eta)\,d\eta\int\limits_\eta^x \varphi'(\xi-\eta)\,d\xi+F_3(x,b)=f_4(x),\ \ b\le x\le 2b,\\
&f_1(2b)+f_2(y)-f_2(0)+\int\limits_0^y \exp(-\alpha\xi)\,d\xi\int\limits_\xi^y \psi'(\eta-\xi)\,d\eta\\
&\qquad+\int\limits_0^y \exp(-\alpha\eta)\,d\eta\int\limits_\eta^{2b} \varphi'(\xi-\eta)\,d\xi+F_3(2b,y)=f_3(y),\ \ 0\le y\le b.
\end{aligned}\tag{9}$$

Based on the structure of system (9) and differentiating this system with respect to the corresponding variables x and y, respectively, we obtain the equivalent system

$$\begin{aligned}
&\exp(-\alpha x)\psi(b-x)+\int\limits_0^x \exp(-\alpha\eta)\varphi'(x-\eta)\,d\eta\\
&\qquad=f_4'(x)-f_1'(x)-F_{4x}(x,b)+\exp(-\alpha x)f_1'(0),\ \ 0\le x\le b,
\end{aligned}\tag{10}$$

$$\int\limits_0^b \exp(-\alpha\eta)\varphi'(x-\eta)\,d\eta=f_4'(x)-f_1'(x)-F_{3x}(x,b),\quad b\le x\le 2b,\tag{11}$$

$$\begin{aligned}
&\int\limits_0^y \exp(-\alpha\xi)\psi'(y-\xi)\,d\xi+\exp(-\alpha y)\varphi(2b-y)\\
&\qquad=f_3'(y)-f_2'(y)-F_{3y}(2b,y)+\exp(-\alpha y)f_2'(0),\ \ 0\le y\le b.
\end{aligned}\tag{12}$$

Making a suitable replacement of the variables ξ and η in the integrands of system (10)–(12), we obtain

$$\psi(b-x)+\int\limits_0^x \exp(\alpha\tau)\varphi'(\tau)\,d\tau=\exp(\alpha x)\big[f_4'(x)-f_1'(x)-F_{4x}(x,b)\big]+f_1'(0),\ \ 0\le x\le b, \tag{13}$$

$$\int\limits_x^{x-b} \exp(\alpha\tau)\varphi'(\tau)\,d\tau=\exp(\alpha x)\big[f_4'(x)-f_1'(x)-F_{3x}(x,b)\big],\ \ b\le x\le 2b, \tag{14}$$

$$\int\limits_0^y \exp(\alpha\tau)\psi'(\tau)\,d\tau+\varphi(2b-y)=\exp(\alpha y)\big[f_3'(y)-f_2'(y)-F_{3y}(2b,y)\big]+f_2'(0),\ \ 0\le y\le b. \tag{15}$$

Similarly, based on the structure of system (13)–(15) and differentiating this system with respect to the corresponding variables x and y we obtain an equivalent system of the equations with respect to the unknown functions φ' and ψ' with a shift

$$\exp(\alpha x)\varphi'(x)-\psi'(b-x)=h_1(x),\ \ 0\le x\le b, \tag{16}$$

$$\exp(-\alpha b)\varphi'(x-b)-\varphi'(x)=h_2(x),\ \ b\le x\le 2b, \tag{17}$$

$$\exp(\alpha y)\psi'(y)-\varphi'(2b-y)=h_3(x),\ \ 0\le y\le b, \tag{18}$$

where h_i, $i=1,2,3$ are the known functions.

From (17) and (18) it follows that

$$\exp(-\alpha b)\varphi'(x)-\varphi'(x+b)=h_2(x+b),\ \ 0\le x\le b \tag{19}$$

and

$$\psi'(b-x)=\exp\big[\alpha(x-b)\big]\,\big[\varphi'(x+b)+h_3(b-x)\big],\ \ 0\le x\le b, \tag{20}$$

respectively.

Substituting (20) into (16), we obtain

$$\varphi'(x)-\exp(-\alpha b)\varphi'(x+b)=\exp(-\alpha x)h_1(x)-\exp(-\alpha b)h_3(b-x),\quad 0\le x\le b. \tag{21}$$

An analysis of system (19), (21) allows us to conclude that the theorem stated above is valid. At the same time, it is possible to constructively describe the set of the right-hand sides of equation (1) for which the original problem (1), (2) is solvable. □

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