

Oscillatory Behavior of Solutions to Second Order Half-Linear Differential Equations

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1 Introduction

This report is a continuation of the notes [5,6] that was submitted to the QUALITDE – 2021, 2024 in which the distribution of zeros and extrema of oscillatory solutions were studied for second order half-linear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \quad (\text{HL})$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive continuously differentiable functions on $[a, \infty)$, and φ_α stands for the odd function on \mathbb{R} given by

$$\varphi_\alpha(u) = |u|^\alpha \operatorname{sgn} u = |u|^{\alpha-1}u, \quad u \in \mathbb{R}.$$

We assume that equation (HL) is oscillatory, that is, all of its nontrivial solutions are oscillatory. Let $x(t)$ be a solution of (HL) on $[a, \infty)$. Let $\{\sigma_k\}_{k=1}^\infty$ ($\sigma_k < \sigma_{k+1}$) be the sequence of all zeros of $x(t)$, and let $\{\tau_k\}_{k=1}^\infty$ ($\tau_k < \tau_{k+1}$) be the sequence of all points at which $x(t)$ takes on its local extrema. It is clear that $x'(\tau_k) = 0$ for all k . The value $|x'(\sigma_k)|$ is called the *slope* of $x(t)$ at $t = \sigma_k$, while the value $|x(\tau_k)|$ is called the *amplitude* of $x(t)$ at $t = \tau_k$.

Our primary interest is to prove the qualitative properties including amplitudes and slopes at the zero points and extrema points of oscillatory solutions of (HL), respectively. Thus, we assume that (HL) is oscillatory, so that all of its solutions are oscillatory on I . If all solutions of (HL) are

oscillatory, then equation (HL) is said to be oscillatory. There are various sufficient conditions for (HL) to be oscillatory. We here state the most representative one:

$$\int_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty \text{ and } \int_a^\infty q(t) dt = \infty. \tag{1.1}$$

If (1.1) hold, then equation (HL) is oscillatory. For other conditions, we refer the reader to the monograph by Dořlý and Řehák [1].

Let $x(t)$ be an oscillatory solution on $[a, \infty)$ of (HL), and we adopt definitions as follows.

Definition 1.1. The sequence $\{\sigma_k\}_{k=1}^\infty$ is called consecutive zero points if $\sigma_k < \sigma_{k+1}$, $x(\sigma_k) = x(\sigma_{k+1}) = 0$ and $x(t) \neq 0$ in (σ_k, σ_{k+1}) for $k = 1, 2, \dots$. Additionally, the sequence $\{\tau_k\}_{k=1}^\infty$ is called consecutive extrema points if $\tau_k < \tau_{k+1}$, $x'(\tau_k) = x'(\tau_{k+1}) = 0$ and $x'(t) \neq 0$ in (τ_k, τ_{k+1}) for $k = 1, 2, \dots$.

Definition 1.2. The values $|x(\tau_k)|$ and $|x'(\sigma_k)|$ are mentioned to as the *amplitudes* and the *slopes*, respectively, of the k -th wave of $x(t)$.

To the best of the authors' knowledge, the starting results for qualitative properties – such as amplitude and slope of oscillatory solutions – were presented by Hille [3], who utilized a pair of some quadratic forms. There results are also included in the monograph by Hartman [2]. Inspired by these results, Kusano and Yoshida [7] derived the following results for Sturm–Liouville differential equation, which corresponds to equation (HL) with $\alpha = 1$

$$(p(t)x')' + q(t)x = 0. \tag{L}$$

The purpose of our study in this talk is to investigate the sequences of zeros and extrema points of solutions of (HL). We are interested in explicit laws or rules, if any, governing the arrangement of these sequences.

2 Main Results

Theorem 2.1. *The sequence $\{\sigma_{k+1} - \sigma_k\}$ is decreasing or increasing according to $p'(t) \leq 0$ and $q'(t) \geq 0$, or $p'(t) \geq 0$ and $q'(t) \leq 0$ for $t \geq a$, respectively.*

Theorem 2.2.

- (i) *Suppose that $(p(t)^{\frac{1}{\alpha}}q(t))' \geq 0$ for $t \geq a$.*
 - (i-a) *If $\lim_{t \rightarrow \infty} p(t) = 0$ holds, then $\lim_{k \rightarrow \infty} (\sigma_{k+1} - \sigma_k) = 0$. i.e., all solutions of (HL) are quickly oscillating;*
 - (i-b) *If $\lim_{t \rightarrow \infty} q(t) = 0$ holds, then $\lim_{k \rightarrow \infty} (\sigma_{k+1} - \sigma_k) = \infty$. i.e., all solutions of (HL) are slowly oscillating.*
- (ii) *Suppose that $(p(t)^{\frac{1}{\alpha}}q(t))' \leq 0$ for $t \geq a$.*
 - (ii-a) *If $\lim_{t \rightarrow \infty} p(t) = \infty$ holds, then $\lim_{k \rightarrow \infty} (\sigma_{k+1} - \sigma_k) = \infty$. i.e., all solutions of (HL) are slowly oscillating;*
 - (ii-b) *If $\lim_{t \rightarrow \infty} q(t) = \infty$ holds, then $\lim_{k \rightarrow \infty} (\sigma_{k+1} - \sigma_k) = 0$. i.e., all solutions of (HL) are quickly oscillating.*

In order to say something about the sequence of extrema points of oscillatory solutions $x(t)$ of (HL), we consider the equation

$$(Q(t)\varphi_{\frac{1}{\alpha}}(y'))' + P(t)\varphi_{\frac{1}{\alpha}}(y) = 0, \quad P(t) = p(t)^{-\frac{1}{\alpha}}, \quad Q(t) = q(t)^{-\frac{1}{\alpha}}. \quad (\text{HLR})$$

Equations (HL) and (HLR) are interrelated via $y = p(t)\varphi_{\alpha}(x')$ and $x = Q(t)\varphi_{\frac{1}{\alpha}}(y')$. This means that $x'(\tau_k) = 0$ implies $y(\tau_k) = 0$, and vice versa. Hence the sequence of extrema points of $y(t)$ coincides with that of zeros of the corresponding $z(t)$, and conversely. This observation suggests that Theorem 2.2 applied to (HLR) will provide information as to the monotonicity of the distances of consecutive extrema points of solutions of (HL).

Theorem 2.3. *Suppose that (HL) is oscillatory. Let $\{\tau_k\}$ denote the sequence of extrema points of its arbitrary solution.*

- (i) *The sequence $\{\tau_{k+1} - \tau_k\}$ is decreasing or increasing according to $p'(t) \leq 0$ and $q'(t) \geq 0$, or $p'(t) \geq 0$ and $q'(t) \leq 0$ for $t \geq a$.*
- (ii) *Suppose that $(p(t)\frac{1}{\alpha}q(t))' \geq 0$ for $t \geq a$.*
 - (ii-a) *If $\lim_{t \rightarrow \infty} p(t) = 0$ holds, then $\lim_{k \rightarrow \infty} (\tau_{k+1} - \tau_k) = 0$. i.e., all solutions of (HL) are quickly oscillating;*
 - (ii-b) *If $\lim_{t \rightarrow \infty} q(t) = 0$ holds, then $\lim_{k \rightarrow \infty} (\tau_{k+1} - \tau_k) = \infty$. i.e., all solutions of (HL) are slowly oscillating.*
- (iii) *Suppose that $(p(t)\frac{1}{\alpha}q(t))' \leq 0$ for $t \geq a$.*
 - (iii-a) *If $\lim_{t \rightarrow \infty} p(t) = \infty$ holds, then $\lim_{k \rightarrow \infty} (\tau_{k+1} - \tau_k) = \infty$. i.e., all solutions of (HL) are slowly oscillating;*
 - (iii-b) *If $\lim_{t \rightarrow \infty} q(t) = \infty$ holds, then $\lim_{k \rightarrow \infty} (\tau_{k+1} - \tau_k) = 0$. i.e., all solutions of (HL) are quickly oscillating.*

We present results concerning the amplitude of oscillatory solutions of (HL).

Theorem 2.4.

- (i) *If $(p(t)\frac{1}{\alpha}q(t))' \geq 0$, then the amplitudes of oscillatory solutions of (HL) are decreasing or constant.*
- (ii) *If $(p(t)\frac{1}{\alpha}q(t))' \leq 0$, then the amplitudes of oscillatory solutions of (HL) are increasing or constant.*

Remark. If $p'(t) \equiv q'(t) \equiv 0$, then we see that the amplitudes of oscillatory solutions of (HL) are constant for $t \geq a$.

3 Examples

This section concludes with illustrative examples that concretely demonstrate the main results stated above.

Example 3.1. We consider the following half-linear differential equation

$$(e^{-\gamma t} \varphi_\alpha(x'))' + \mu e^{\sigma t} \varphi_\alpha(x) = 0, \quad t \geq 0 \tag{E_1}$$

which is a special case of (HL) with $p(t) = e^{-\gamma t}$ and $q(t) = \mu e^{\sigma t}$ for some positive constants γ, σ and μ . Since simple calculation shows that

$$\int_0^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \int_0^\infty e^{\frac{\gamma}{\alpha} t} dt = \infty \quad \text{and} \quad \int_0^\infty q(t) dt = \mu \int_0^\infty e^{\sigma t} dt = \infty,$$

equation (E₁) is oscillatory. Moreover, as in this case $p'(t) \leq 0$ and $q'(t) \geq 0$, it follows that the both of sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ of zero and extrema points of any solution of (E₁) are decreasing by Theorems 2.1 and 2.3(i). Here, we need to consider the following three cases separately:

- (i) $\alpha\sigma > \gamma$,
- (ii) $\alpha\sigma < \gamma$,
- (iii) $\alpha\sigma = \gamma$.

(i) Let $\alpha\sigma > \gamma$, i.e., $\sigma - \frac{\gamma}{\alpha} > 0$. Then, since $(p(t)^{\frac{1}{\alpha}} q(t))' \geq 0$ and $\lim_{t \rightarrow \infty} p(t) = 0$, all solutions of (E₁) are quickly oscillating by Theorems 2.2(i-a) and 2.3(ii-a).

(ii) Let $\alpha\sigma < \gamma$, i.e., $\sigma - \frac{\gamma}{\alpha} < 0$. Also in this case, all solutions of (E₁) are quickly oscillating by Theorems 2.2(ii-b) and 2.3(iii-b) because $(p(t)^{\frac{1}{\alpha}} q(t))' \leq 0$ and $\lim_{t \rightarrow \infty} q(t) = \infty$.

(iii) Let $\alpha\sigma = \gamma$, i.e., $\sigma = \frac{\gamma}{\alpha}$. Then,

$$\lim_{t \rightarrow \infty} p(t)^{\frac{1}{\alpha}} q(t) = 1 \quad \text{and} \quad (p(t)^{\frac{1}{\alpha}} q(t))' = 0$$

hold. Therefore, we see that the amplitudes of oscillatory solutions of (E₁) are constant by Theorem 2.4.

Example 3.2. Let us consider the half-linear differential equation

$$((t + \tau)^\gamma \varphi_\alpha(x'))' + \mu(t + \tau)^\sigma \varphi_\alpha(x) = 0, \quad t \geq 0, \tag{E_2}$$

which is a special case of (HL) with

$$p(t) = (t + \tau)^\gamma \quad \text{and} \quad q(t) = \mu(t + \tau)^\sigma$$

for constants γ and σ satisfying $\gamma \leq \alpha$ and $\sigma \geq -1$. Elementary calculations show that

$$\int_0^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \int_0^\infty (t + \tau)^{-\frac{\gamma}{\alpha}} dt = \infty \quad \text{and} \quad \int_0^\infty q(t) dt = \int_0^\infty (t + \tau)^\sigma dt = \infty.$$

Thus, equation (E₂) is oscillatory. Moreover, since

$$p(t)^{\frac{1}{\alpha}} q(t) = (t + \tau)^{\frac{\gamma}{\alpha} + \sigma} \quad \text{and} \quad (p(t)^{\frac{1}{\alpha}} q(t))' = \left(\frac{\gamma}{\alpha} + \sigma\right) (t + \tau)^{\frac{\gamma}{\alpha} + \sigma - 1},$$

we classify and observe as follows:

- (i) $\frac{\gamma}{\alpha} + \sigma \geq 0$,
- (ii) $\frac{\gamma}{\alpha} + \sigma < 0$.

(i-a) Suppose that $-1 \leq \sigma \leq 0$ and $\gamma \leq \alpha$, i.e., $-\alpha\sigma \leq \gamma \leq \alpha$, $\gamma \geq 0$. Then, since $p'(t) \geq 0$ and $q'(t) \leq 0$, it follows that the both of sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ of zero and extrema points of any solution of (E_2) are increasing by Theorem 2.1 and 2.3(i). In addition, we see that all solutions of (E_2) are slowly oscillating according to Theorem 2.3(ii-b), and also the amplitudes of oscillatory solutions of (E_2) are decreasing or constant by Theorem 2.4(i).

(i-b) Suppose that $\sigma \geq 0$ and $\gamma \leq \alpha$, i.e., $-\alpha\sigma \leq \gamma \leq \alpha$.

(i-b-1) Suppose that $0 \leq \gamma \leq \alpha$. Then, because $p'(t) \geq 0$ and $q'(t) \geq 0$, we find that the amplitudes of oscillatory solutions of (HL) are decreasing or constant by Theorem 2.4(i).

(i-b-2) Suppose that $-\alpha\sigma \leq \gamma \leq 0$. Then, since $p'(t) \leq 0$ and $q'(t) \geq 0$, the both of sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ of zero and extrema points of any solutions of (E_2) are decreasing by Theorem 2.1 and 2.3(i). We also see that all solutions of (E_2) are quickly oscillating, and the amplitudes of oscillatory solutions of (E_2) are decreasing or constant by Theorems 2.3(ii-a) and 2.4(i), respectively.

(ii-a) Suppose that $-1 \leq \sigma \leq 0$ and $\gamma \leq \alpha$, i.e., $\gamma \leq \alpha$.

(ii-a-1) Suppose that $0 \leq \gamma \leq \alpha$. Then, since $p'(t) \geq 0$ and $q'(t) \leq 0$, the both of sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ of zero and extrema points of any solutions of (E_2) are increasing by Theorem 2.1 and 2.3(i). It also follows that all solutions of (E_2) are slowly oscillating according to Theorems 2.2 and 2.3. Moreover, the amplitudes of oscillatory solutions of (E_2) are increasing or constant by Theorem 2.4(ii).

(ii-a-2) Suppose that $\gamma < 0$. In this case, the amplitudes of oscillatory solutions of (E_2) are increasing or constant by Theorem 2.4(ii).

(ii-b) Suppose that $\sigma \geq 0$. i.e., $\gamma < -\alpha\sigma$. Then, since $p'(t) \leq 0$ and $q'(t) \geq 0$, the both of sequences $\{\sigma_{k+1} - \sigma_k\}$ and $\{\tau_{k+1} - \tau_k\}$ of zero and extrema points of any solutions of (E_2) are decreasing by Theorem 2.1 and 2.3(i). We also find that all solutions of (E_2) are quickly oscillating according to Theorems 2.2(ii-b) and 2.3(iii-b). Furthermore, the amplitudes of oscillatory solutions of (E_2) are increasing or constant by Theorem 2.4(ii).

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