Some Nonlinear Partial Differential and Parabolic Type Integro-Differential Models

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In the present note some systems of nonlinear partial differential equations (NPDE) and nonlinear parabolic integro-differential equations (NPIDE) are discussed. These models originate from the Maxwell system [15], which describes the diffusion of a magnetic field into a conductive material.

We study the existence, uniqueness, and long-time behavior of solutions. The linear stability of stationary solutions for the associated initial-boundary value problems is presented, and conditions for the appearance of Hopf-type bifurcations [21] are identified.

One of the main aims of this work is to investigate the following two classes of NPIDE systems:

$$\frac{\partial H}{\partial t} + \nabla \times \left[a\left(\int_0^t |\nabla \times H|^2 \, d\tau \right) \nabla \times H \right] = 0, \tag{1}$$

$$\frac{\partial H}{\partial t} - a\left(\int_0^t \int_\Omega |\nabla \times H|^2 \, dx \, d\tau \right) \Delta H = 0. \tag{2}$$

Models of the form (1) first appeared and were studied in [6].

The unique solvability of the initial-boundary value problems for NPIDE systems such as (1) was established under quite general assumptions on the function $a = a(S)$ in [5]. The proofs rely on a modified Galerkin method combined with compactness techniques [19, 23].

Based on the results of [6] and [5], models of type (2) were introduced in [17], where the author referred to them as averaged integro-differential equations (AIDE).

Systems of type (1) and (2) are complex and have been extensively investigated. The early publications quickly drew considerable interest, and soon after [6] and [5], many works appeared (see, e.g., [1, 2, 7, 8, 11, 14, 16, 18, 20]). More complete references up to 2019 may be found in [9] and [12]. Since then, attention to these problems has only grown.

In $Q = (0,1) \times (0,\infty)$, consider the initial-boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x} \Big), \quad \frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x} \Big)^2, \;\; (x,t) \in Q, \tag{3}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \;\; t \in [0,\infty), \tag{4}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \;\; x \in [0,1], \tag{5}$$

where $\psi = Const > 0$, and $U_0 = U_0(x)$, $V_0 = V_0(x)$ are the given functions. If $U(x,0) = \psi x$ and $V(x,0) = \delta_0 = Const > 0$, then the pair of functions:

$$U(x,t) = \psi x, \quad V(x,t) = \big[\delta_0^{1-\alpha} + (1-\alpha)\psi^2 t \big]^{\frac{1}{1-\alpha}}$$

is the solution of problem (3)–(5) for any $\alpha \neq 1$. However, when $\alpha > 1$, the function V blows up at the finite time $t_0 = \frac{\delta_0^{1-\alpha}}{\psi^2(\alpha-1)}$. Thus, even with smooth initial and boundary data, solutions of systems like (3) may experience finite-time blow-up. Consequently, for $\alpha > 1$, global solutions to (3)–(5) generally do not exist.

We now turn to the stability of the stationary solution of the system

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \quad \frac{\partial V}{\partial t} = -aV^\beta + bV^\gamma \Big(\frac{\partial U}{\partial x}\Big)^2, \ \ (x,t) \in Q, \tag{6}$$

with the boundary and initial conditions (4), (5). Here a, b, c, ψ are positive constants and α, β, γ are real numbers which will be specified later; $U_0(x), V_0(x)$ are known functions of their arguments; $a \neq 0$, $2\alpha + \beta - \gamma \neq 0$. The stationary solution of (6) is $(\psi x, (\frac{b}{a}\psi^2)^{\frac{1}{\beta-\gamma}})$.

Assume that a smooth solution of (6) exists with $V(x,t) \geq c > 0$.

Theorem 1. *If $a \neq 0$, $2\alpha + \beta - \gamma > 0, \beta \neq \gamma$, then stationary solution $(\psi x, (\frac{b}{a}\psi^2)^{\frac{1}{\beta-\gamma}})$ of (4)–(6) is linearly stable if and only if the following condition takes place*

$$a(\gamma - \beta)\Big(\frac{b}{a}\psi^2\Big)^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2.$$

Let $\gamma - \beta > 0$, $\beta - \alpha - 1 \neq 0$, and consider the quantity

$$\psi_c = \Big[\frac{\pi^2}{\gamma - \beta}\, a^{\frac{\gamma-\alpha-1}{\beta-\gamma}} b^{\frac{\alpha-\beta+1}{\beta-\gamma}}\Big]^{\frac{\beta-\gamma}{\beta-\alpha-1}}.$$

If $0 < \psi < \psi_c$, the stationary solution of (6) is stable; if $\psi > \psi_c$, it becomes unstable. At $\psi = \psi_c$, a Hopf-type bifurcation may occur [21]. Similar analysis for related models appear in many works (e.g., [4, 9, 10, 13]).

Next we consider NPIDEs of type (1) and (2). These models are highly nonlinear and only certain special cases have been fully investigated. Numerous studies address such systems (see, e.g., [1, 2, 5–9, 11, 12, 14, 16–18, 20, 22]).

In Q consider the initial-boundary value problem

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left[a\Big(\int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2 d\tau\Big)\frac{\partial U}{\partial x}\right], \ \ (x,t) \in Q, \tag{7}$$

$$U(0,t) = U(1,t) = 0, \ \ t \geq 0, \tag{8}$$

$$U(x,0) = U_0(x), \ \ x \in [0,1]. \tag{9}$$

Earlier results on the asymptotic behavior of solutions to (7)–(9) can be found in [8]. The following results hold.

Theorem 2. *If $a(S) \geq a_0 = Const > 0$ and $U_0 \in L_2(0,1)$, then solutions of (7)–(9) satisfy the following estimation*

$$\|U\| \leq C\exp(-a_0 t).$$

Theorem 3. *If $a(S) = (1+S)^p, 0 < p \leq 1$, $U_0 \in W_2^2(0,1) \cap \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem (7)–(9) the following estimate is true*

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big| \leq C\exp\Big(-\frac{t}{2}\Big).$$

Let us consider the following boundary conditions:

$$U(0,t)=0, \quad U(1,t)=\psi=Const>0, \;\; t\geq 0. \tag{10}$$

Theorem 4. *If $a(S)=(1+S)^p$, $0<p\leq 1$, $U_0\in W_2^2(0,1)$, $U_0(0)=0$, $U_0(1)=\psi$, then for the solution of problem* (7), (9), (10) *the following estimate holds*

$$\Big|\frac{\partial U(x,t)}{\partial x}-\psi\Big|\leq Ct^{-1-p}, \;\; t\geq 1.$$

These results rely on an approach similar to that in [3], where the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is investigated.

Theorems 3 and 4 highlight the contrasting stabilization behavior in the cases of homogeneous versus nonhomogeneous boundary data.

Now turn to the AIDE

$$\frac{\partial U}{\partial t}=a\bigg(\int\limits_0^t\int\limits_0^1\Big(\frac{\partial U}{\partial x}\Big)^2\,dx\,d\tau\bigg)\,\frac{\partial^2 U}{\partial x^2}\,. \tag{11}$$

The existence and uniqueness of solutions for initial-boundary value problems of type (11) were first established in [8]. We now describe their asymptotic behavior.

Consider (8), (9), (11). Earlier results on solvability and asymptotics can be found in [8].

Theorem 5. *If $a(S)=(1+S)^p$, $p>0$, $U_0\in W_2^2(0,1)\cap \overset{\circ}{W}{}_2^1(0,1)$, then for the solution of problem* (8), (9), (11) *the following estimates are true:*

$$\Big|\frac{\partial U(x,t)}{\partial x}\Big|\leq C\exp\Big(-\frac{t}{2}\Big), \quad \Big|\frac{\partial U(x,t)}{\partial t}\Big|\leq C\exp\Big(-\frac{t}{2}\Big).$$

Let us consider problem (9)–(11).

Theorem 6. *If $a(S)=(1+S)^p$, $p>0$, $\psi=Const>0$; $U_0\in W_2^2(0,1)$, $U_0(0)=0$, $U_0(1)=\psi$, then for the solution of problem* (9)–(11) *the following estimates are true:*

$$\Big|\frac{\partial U(x,t)}{\partial x}-\psi\Big|\leq Ct^{-1-p}, \quad \Big|\frac{\partial U(x,t)}{\partial t}\Big|\leq Ct^{-1-p}, \;\; t\geq 1.$$

Now, let us consider following two classes of nonlinear parabolic type multi-dimensional NPIDEs:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^n D_i\bigg[a\bigg(\int\limits_0^t|\nabla U|^q\,d\tau\bigg)|\nabla U|^{q-2}D_iU\bigg]=f(x,t), \tag{12}$$

$$\frac{\partial U}{\partial t}-a\bigg(\int\limits_0^t\int\limits_\Omega|\nabla U|^q\,dx\,d\tau\bigg)\sum_{i=1}^n D_i\big(|\nabla U|^{q-2}D_iU\big)=f(x,t), \tag{13}$$

where

$$D_i=\frac{\partial}{\partial x_i}\,, \quad \nabla=\Big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\Big), \quad |\nabla U|=\Big(\sum_{k=1}^n|D_kU|^2\Big)^{1/2},$$

$q\geq 2$, $\Omega\subset R^n$ is a bounded domain, f and a are the given functions of their arguments. The similar type equations as given here, at first were proposed in [8].

In the domain $Q_T^n = \Omega \times (0,T)$, of the variables $(x_1, x_2, \dots, x_n, t)$, where Ω is the bounded domain with a sufficiently smooth boundary $\partial\Omega$, let us consider the following initial-boundary value problem:

$$U(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,T), \tag{14}$$

$$U(x,0) = 0, \ \ x \in \Omega. \tag{15}$$

In the cylinder $Q^n = \Omega \times (0,\infty)$, we consider the following problem:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} D_i \bigg[a \bigg(\int_0^t |\nabla U|^2 \, d\tau \bigg) D_i U \bigg] = 0, \tag{16}$$

$$U(x,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,\infty), \tag{17}$$

$$U(x,0) = U_0(x), \ \ x \in \Omega. \tag{18}$$

Theorem 7. *If $a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, $U_0 \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, then for solution of problem* (16)–(18) *the following expression holds:*

$$\|U(\,\cdot\,,t)\|_{W_2^1(\Omega)} \to 0, \ \ t \to \infty.$$

The unique solvability of (12), (14), (15) and of the corresponding problem for AIDEs of type (13) was established in [8].

To conclude, many of the results in this work were obtained earlier, some in collaboration with coauthors.

It remains an important challenge to develop and analyze such models for broader classes of nonlinearities.

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