Positive Periodic Solutions for Systems of Linear Functional Differential Inequalities

Robert Hakl

Institute of Mathematics, Czech Academy of Sciences
Brno, Czech Republic
E-mail: hakl@ipm.cz

José Oyarce

Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Chile
E-mail: oyarce@ubiobio.cl

1 Introduction

The paper [2] is devoted to study the system of functional differential inequalities

$$\mathcal{D}(\sigma)\big[u'(t)-\ell(u)(t)\big]\geq 0 \text{ for a.e. } t\in\mathbb{R}. \tag{1.1}$$

Here, $\ell : C_\omega(\mathbb{R}^n) \to L_\omega(\mathbb{R}^n)$ is a linear bounded operator, $\sigma = (\sigma_i)_{i=1}^n$ with $\sigma_i \in \{-1, 1\}$, and $\mathcal{D}(\sigma) = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$. The aim of the present paper is to establish conditions guaranteeing that there exists $c \in]0, 1[$ such that every absloutely continuous ω-periodic vector-valued function $u = (u_i)_{i=1}^n$ satisfying (1.1) belongs to a cone

$$\mathcal{K}_c^n \overset{def}{=} \big\{u\in C_\omega(\mathbb{R}^n):\ u_i(s)\geq cu_i(t) \text{ for } s,t\in\mathbb{R}\ (i=1,\dots,n)\big\}. \tag{1.2}$$

The results will be applied to a system of differential inequalities with deviating arguments, i.e., to the case when $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_\omega^n$ is defined by

$$\ell_i(v)(t) \overset{def}{=} \sum_{j=1}^n p_{ij}(t)v_j(\tau_{ij}(t)) - g_i(t)v_i(\mu_i(t)) \text{ for a.e. } t\in\mathbb{R},$$
$$v=(v_j)_{j=1}^n\in C_\omega(\mathbb{R}^n)\ \ (i=1,\dots,n),$$

where $p_{ij}, g_i \in L_\omega(\mathbb{R})$, $\tau_{ij}, \mu_i \in \mathcal{M}_\omega$ $(i, j = 1, \dots, n)$.

2 Basic notation and definitions

$C_\omega(\mathbb{R}^n)$ is the Banach space of continuous ω-periodic vector-valued functions $x : \mathbb{R} \to \mathbb{R}^n$ endowed with a norm

$$\|x\|_{C_\omega} = \max\big\{\|x(t)\|:\ t\in[0,\omega]\big\}.$$

$AC_\omega(\mathbb{R}^n)$ is a set of locally absolutely continuous ω-periodic vector-valued functions $x : \mathbb{R} \to \mathbb{R}^n$.

$L_\omega(\mathbb{R}^n)$ is a Banach space of locally Lebesgue integrable ω-periodic vector-valued functions $p : \mathbb{R} \to \mathbb{R}^n$ endowed with the norm

$$\|p\|_{L_\omega} = \int_0^\omega \|p(s)\|\,ds.$$

$\mathcal{M}_\omega$ is a set of locally Lebesgue measurable functions $\tau : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\tau(t+\omega)-\tau(t)}{\omega} \in \mathbb{Z} \ \text{ for a.e. } \ t \in \mathbb{R}.$$

$\mathcal{L}_\omega^n$ is a set of linear bounded operators $\ell : C_\omega(\mathbb{R}^n) \to L_\omega(\mathbb{R}^n)$. We write $\mathcal{L}_\omega$ instead of $\mathcal{L}_\omega^1$.

δ_{ij} is the Kronecker's symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j. \end{cases}$$

For any $\ell \in \mathcal{L}_\omega^n$, the operators $\ell_i : C_\omega(\mathbb{R}^n) \to L_\omega(\mathbb{R})$ and $\ell_{ij} \in \mathcal{L}_\omega$ $(i,j = 1,\dots,n)$ are defined as follows:

- for any $v \in C_\omega(\mathbb{R}^n)$, $\ell_i(v)$ is the i-th component of the vector-valued function $\ell(v)$;
- for any $z \in C_\omega(\mathbb{R})$ we put $\ell_{ij}(z) \overset{def}{=} \ell_i(\widehat{z})$, where $\widehat{z} = (\delta_{kj} z)_{k=1}^n$.

If $\ell \in \mathcal{L}_\omega$, then $\|\ell\|$ is the operator norm.

Definition 2.1. We say that a function $q \in L_\omega(\mathbb{R}^n)$ is *σ-positive* if $\mathcal{D}(\sigma)q(t) \geq 0$ for a.e. $t \in \mathbb{R}$.

An operator $\ell \in \mathcal{L}_\omega^n$ is said to be *σ-positive* if it transforms non-negative functions into the set of σ-positive functions, i.e., if for every $u \in C_\omega(\mathbb{R}^n)$ such that

$$u(t) \geq 0 \ \text{ for } \ t \in \mathbb{R},$$

the relation

$$\mathcal{D}(\sigma)\ell(u)(t) \geq 0 \ \text{ for a.e. } \ t \in \mathbb{R}$$

is fulfilled. A set of σ-positive operators is denoted by $\mathcal{P}_\omega^n(\sigma)$. We write $\mathcal{P}_\omega(\sigma)$ instead of $\mathcal{P}_\omega^1(\sigma)$.

Definition 2.2. Let $c \in]0,1[$. We will say that an operator $\ell \in \mathcal{L}_\omega^n$ belongs to the set $\mathcal{U}_c^n(\sigma)$ if every function $u \in AC_\omega(\mathbb{R}^n)$ satisfying (1.1) belongs to the cone $\mathcal{K}_c^n$ defined by (1.2). We write $\mathcal{U}_c(\sigma)$ instead of $\mathcal{U}_c^1(\sigma)$.

Definition 2.3. We say that an operator $\ell \in \mathcal{L}_\omega^n$ is an *irreducible* operator, if the matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n\times n}$ with

$$a_{ij} \overset{def}{=} (1-\delta_{ij})\|\ell_{ij}\| \ \ (i,j = 1,\dots,n)$$

is an irreducible matrix.

Definition 2.4. We say that an operator $\ell \in \mathcal{L}_\omega^n$ is a *σ-Metzler type* operator, if its off-diagonal elements are σ_i-positive operators, i.e., $\ell_{ij} \in \mathcal{P}_\omega(\sigma_i)$ $(i \neq j;\ i,j = 1,\dots,n)$.

3 On the set $\mathcal{U}_c^n(\sigma)$

General results of the paper [2] are closely related to those of the paper [1].

Theorem 3.1. *Let $\ell \in \mathcal{L}_\omega^n$ be an irreducible operator of σ-Metzler type with $\ell_{ii} = \ell_{ii}^+ - \ell_{ii}^-$ where $\ell_{ii}^+, \ell_{ii}^- \in \mathcal{P}_\omega(\sigma_i)$ $(i = 1,\dots,n)$. Let, moreover, there exist $c \in]0,1[$ and $\nu_i \in [0,1]$ $(i = 1,\dots,n)$ such that*

$$-\ell_{ii}^- + \nu_i \ell_{ii}^+ \in \mathcal{U}_c(\sigma_i) \ \ (i = 1,\dots,n). \tag{3.1}$$

Then

$$\ell \in \mathcal{U}_c^n(\sigma) \tag{3.2}$$

if and only if there exist $\beta = (\beta_i)_{i=1}^n \in AC_\omega(\mathbb{R}^n)$ *and* $i_0 \in \{1, \dots, n\}$ *such that*

$$\beta(t) > 0 \ \text{ for } \ t \in \mathbb{R},$$
$$\mathcal{D}(\sigma)\big[\beta'(t) - \ell(\beta)(t)\big] \geq 0 \ \text{ for a.e. } \ t \in \mathbb{R},$$
$$\sum_{j=1}^{n} \int_0^\omega \sigma_{i_0} \ell_{i_0 j}(\beta_j)(s) ds < 0.$$

Efficient conditions guaranteeing the inclusion (3.1) can be found in [2]. Theorem 3.1 introduces a necessary and sufficient condition guaranteeing the inclusion (3.2) is fulfilled. Nevertheless, the following theorem can be useful in finding efficient conditions.

Theorem 3.2. *Let* $\ell \in \mathcal{L}_\omega^n$ *be an irreducible operator of* σ*-Metzler type with* $\ell_{ii} = \ell_{ii}^+ - \ell_{ii}^-$ *where* $\ell_{ii}^+, \ell_{ii}^- \in \mathcal{P}_\omega(\sigma_i)$ $(i = 1, \dots, n)$*. Let, moreover, there exist* $c \in \,]0, 1[$ *and* $\nu_i \in [0, 1]$ $(i = 1, \dots, n)$ *such that* (3.1) *is fulfilled. Let, in addition, there exist a matrix* $B = (b_{ij})_{i,j=1}^n \in \mathbb{R}_+^{n \times n}$ *such that*

$$r(B) < 1 \tag{3.3}$$

and

$$(1 - \nu_i) \int_{t-\omega}^{t} G_i(t, s) \ell_{ii}^+(1)(s) \, ds \leq b_{ii} \ \text{ for } \ t \in \mathbb{R} \quad (i = 1, \dots, n),$$
$$\int_{t-\omega}^{t} G_i(t, s) \ell_{ij}(1)(s) \, ds \leq b_{ij} \ \text{ for } \ t \in \mathbb{R} \quad (i \neq j; i, j = 1, \dots, n),$$

where G_i *is the Green's function to* ω*-periodic problem for*

$$v'(t) = -\ell_{ii}^-(v)(t) + \nu_i \ell_{ii}^+(v)(t) \ \text{ for a.e. } \ t \in \mathbb{R}.$$

Then (3.2) *holds.*

A number of efficient conditions guaranteeing the inclusion (3.2) can be found in [2]. As an example we formulate here one result for operators with deviating arguments, i.e., for the case when $\ell = (\ell_i)_{i=1}^n \in \mathcal{L}_\omega^n$ is defined by

$$\ell_i(v)(t) \stackrel{def}{=} \sum_{j=1}^{n} p_{ij}(t) v_j(\tau_{ij}(t)) - g_i(t) v_i(\mu_i(t)) \ \text{ for a.e. } \ t \in \mathbb{R},$$
$$v = (v_j)_{j=1}^n \in C_\omega(\mathbb{R}^n) \ \ (i = 1, \dots, n), \tag{3.4}$$

where $p_{ij}, g_i \in L_\omega(\mathbb{R})$, $\sigma_i p_{ij}(t) \geq 0$, $\sigma_i g_i(t) \geq 0$ for a.e. $t \in \mathbb{R}$, $\tau_{ij}, \mu_i \in \mathcal{M}_\omega$ $(i, j = 1, \dots, n)$.

Corollary 3.1. *Let* $\sigma_i g_i(t) \geq 0$, $\sigma_i p_{ij}(t) \geq 0$ *for a. e.* $t \in \mathbb{R}$ $(i, j = 1, \dots, n)$*, and let*

$$\sigma_i(g_i(t) - p_{ii}(t)) \geq 0 \ \text{ for a.e. } \ t \in \mathbb{R} \ \ (i = 1, \dots, n),$$
$$\int_0^\omega \sigma_i(g_i(s) - p_{ii}(s)) \, ds > 0 \ \ (i = 1, \dots, n),$$
$$p_{ii}(t)(\tau_{ii}(t) - \mu_i(t)) \geq 0 \ \text{ for a.e. } \ t \in \mathbb{R} \ \ (i = 1, \dots, n).$$

Let, moreover,

$$\int_0^{\omega} \sigma_i \left(g_i(s) - p_{ii}(s) + p_{ii}(s) \int_{\mu_i(s)}^{\tau_{ii}(s)} g_i(\xi)\, d\xi \right) ds < 1 \quad (i = 1, \dots, n),$$

and let (3.3) *hold where* $B = (b_{ij})_{i,j=1}^{n} \in \mathbb{R}_+^{n \times n}$, $b_{ii} = 0$ $(i = 1, \dots, n)$, *and*[1]

$$b_{ij} = \frac{1 + L_i}{(1 - L_i^+) L_i} \int_0^{\omega} \sigma_i \left[p_{ii}(s) \int_{\mu_i(s)}^{\tau_{ii}(s)} p_{ij}(\xi)\, d\xi + p_{ij}(s) \right] ds \quad (i \neq j; \ \ i, j = 1, \dots, n),$$

with

$$L_i \overset{def}{=} - \int_0^{\omega} \sigma_i \widetilde{\ell}_{ii}(1)(s)\, ds, \qquad L_i^+ \overset{def}{=} \int_0^{\omega} \sigma_i \widetilde{\ell}_{ii}^{+}(1)(s)\, ds \quad (i = 1, \dots, n),$$

where $\widetilde{\ell}_{ii} = \widetilde{\ell}_{ii}^{+} - \widetilde{\ell}_{ii}^{-}$, *and* $\widetilde{\ell}_{ii}^{+}$ *and* $\widetilde{\ell}_{ii}^{-}$ *are given by*

$$\widetilde{\ell}_{ii}^{+}(v)(t) \overset{def}{=} p_{ii}(t) \int_{\mu_i(t)}^{\tau_{ii}(t)} p_{ii}(s) v(\tau_{ii}(s))\, ds \ \ \text{for a.e. } t \in \mathbb{R}, \ \ v \in C_{\omega}(\mathbb{R}) \ \ (i = 1, \dots, n)$$

and

$$\widetilde{\ell}_{ii}^{-}(v)(t) \overset{def}{=} (g_i(t) - p_{ii}(t)) v(\mu_i(t)) + p_{ii}(t) \int_{\mu_i(t)}^{\tau_{ii}(t)} g_i(s) v(\mu_i(s))\, ds$$
$$\text{for a.e. } t \in \mathbb{R}, \quad v \in C_{\omega}(\mathbb{R}) \ \ (i = 1, \dots, n),$$

respectively. Then, there exists $c \in]0, 1[$ *such that* (3.2) *holds with* ℓ *given by* (3.4)*.*

Remark. Let $\widehat{\tau}_{ii}, \widehat{\mu}_i \in \mathcal{M}_{\omega}$ $(i = 1, \dots, n)$ be such that

$$\frac{\widehat{\tau}_{ii}(t) - \tau_{ii}(t)}{\omega} \in \mathbb{Z} \ \text{ and } \ \frac{\widehat{\mu}_i(t) - \mu_i(t)}{\omega} \in \mathbb{Z} \ \text{ for a.e. } t \in \mathbb{R} \ \ (i = 1, \dots, n).$$

Then, it is clear that if τ_{ii}, resp. μ_i are replaced by $\widehat{\tau}_{ii}$, resp. $\widehat{\mu}_i$ $(i = 1, \dots, n)$ in the assumptions of Corollary 3.1, the assertion remains still valid.

4 Positive solutions to homogeneous systems

In this section, we discuss the properties of an operator

$$\ell + \lambda \widetilde{\ell},$$

where $\lambda \in \mathbb{R}_+$ and $\widetilde{\ell} \in \mathcal{P}_{\omega}^{n}(\sigma)$ is a nonzero operator, under the assumption that

$$\ell \in \mathcal{U}_c^n(\sigma). \tag{4.1}$$

[1]Note that the assumptions guarantee $L_i > 0$ and $0 \leq L_i^+ < 1$ $(i = 1, \dots, n)$.

In particular, we consider the system of linear functional differential equations

$$u'(t) = \ell(u)(t) + \lambda\widetilde{\ell}(u)(t) \text{ for a.e. } t \in \mathbb{R}, \tag{4.2}$$

and we ask when (4.2) has a positive ω-periodic solution. Obviously, the inclusion (4.1) guarantees that the only ω-periodic solution to the system (4.2) is the zero solution provided $\lambda = 0$. In what follows, we will show that there exists a unique finite positive λ^* such that (4.2) with $\lambda = \lambda^*$ has a positive ω-periodic solution.

Theorem 4.1. *Let $\ell \in \mathcal{L}_\omega^n$ be an irreducible operator of σ-Metzler type with $\ell_{ii} = \ell_{ii}^+ - \ell_{ii}^-$ where $\ell_{ii}^+, \ell_{ii}^- \in \mathcal{P}_\omega(\sigma_i)$ $(i = 1, \dots, n)$. Let, moreover, there exists $c \in]0, 1[$ such that (4.1) holds, and let $\widetilde{\ell} \in \mathcal{P}_\omega^n(\sigma)$ be a nonzero operator. Then, there exists a finite threshold $\lambda^* > 0$ such that*

(i)
$$\ell + \lambda\widetilde{\ell} \in \mathcal{U}_c^n(\sigma)$$
provided $0 \le \lambda < \lambda^*$;

(ii) *there is a positive ω-periodic solution $u_* \in \mathcal{K}_c^n$ to (4.2) provided $\lambda = \lambda^*$;*

(iii)
$$\ell + \lambda\widetilde{\ell} \notin \mathcal{U}_c^n(\sigma)$$
provided $\lambda > \lambda^*$.

Theorem 4.1 gives us a possible method how to calculate the precise value of λ^*. Indeed, define an operator $A : C_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n)$ by

$$A(v)(t) = \int\limits_{t-\omega}^{t} G(t,s)\widetilde{\ell}(v)(s)\,ds \text{ for } t \in \mathbb{R}, \tag{4.3}$$

where $G : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ is the Green's matrix-valued function to ω-periodic problem for

$$u'(t) = \ell(u)(t) \text{ for a.e. } t \in \mathbb{R}. \tag{4.4}$$

Then

$$u_*(t) = \lambda^* A(u_*)(t) \text{ for } t \in \mathbb{R},$$

and consequently, according to Krasnosel'skii theory, the value $1/\lambda^*$ is equal to a spectral radius of the operator A. Therefore, according to Gelfand's formula,

$$\lambda^* = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\|A^n\|}}\,. \tag{4.5}$$

However, even in the case when the Green's matrix-valued function G is known, the approximation of λ^* by using (4.3) and (4.5) is quite difficult. Therefore, in order to find an interval containing the threshold λ^* one can use Theorem 4.2 formulated below, which is also applicable in the cases when the Green's matrix-valued function G to the ω-periodic problem for (4.4) cannot be explicitly expressed.

Theorem 4.2. *Let $\ell \in \mathcal{L}_\omega^n$ be an irreducible operator of σ-Metzler type with $\ell_{ii} = \ell_{ii}^+ - \ell_{ii}^-$ where $\ell_{ii}^+, \ell_{ii}^- \in \mathcal{P}_\omega(\sigma_i)$ $(i = 1, \dots, n)$. Let, moreover, there exists $c \in]0, 1[$ such that (4.1) holds, and let*

$\widetilde{\ell} \in \mathcal{P}^n_\omega(\sigma)$ be a nonzero operator. Let, in addition, there exist $\lambda_1 > 0$, $\lambda_2 > 0$, $u_1, u_2 \in AC_\omega(\mathbb{R}^n)$, $j_0 \in \{1, \dots, n\}$, and $s_0 \in \mathbb{R}$ such that

$$u_1(t) > 0 \ \text{ for } \ t \in \mathbb{R}, \quad u_{2j_0}(s_0) > 0,$$
$$\mathcal{D}(\sigma)\big[u_1'(t) - \ell(u_1)(t) - \lambda_1 \widetilde{\ell}(u_1)(t)\big] \geq 0 \ \text{ for a.e. } \ t \in \mathbb{R},$$
$$\mathcal{D}(\sigma)\big[u_2'(t) - \ell(u_2)(t) - \lambda_2 \widetilde{\ell}(u_2)(t)\big] \leq 0 \ \text{ for a.e. } \ t \in \mathbb{R}.$$

Then

$$\lambda_1 \leq \lambda^* \leq \lambda_2,$$

where λ^* is the threshold appearing in Theorem 4.1.

In the paper we also describe an algorithm how to assure that a given operator belongs to the set $\mathcal{U}^n_c(\sigma)$ for a suitable $c \in \,]0, 1[\,$. The algorithm runs as follows: We express the given operator in the form $\ell + \lambda\widetilde{\ell}$, where ℓ is an irreducible operator σ-Metzler type, $\widetilde{\ell}$ is a σ-positive operator, and λ is a parameter. By using the theoretical results, we verify that the operator ℓ belongs to the set $\mathcal{U}^n_c(\sigma)$. Then, using the MATLAB code ddesd, we numerically construct a solution to the initial value problem with constant initial functions for the equation (4.2) and different values of λ, and then we choose the one that is closest to the periodic solution (this will be for values of λ close to λ^*). We interpolate the readings of the numerical solution by a periodic function, which we then use as an upper and lower test function (functions u_1 and u_2 in Theorem 4.2).

The above algorithm for finding the interval for the threshold value λ^* is illustrated with several examples in the paper.

References

[1] R. Hakl, I. Kiguradze and B. Půža, Upper and lower solutions of boundary value problems for functional differential equations and theorems on functional differential inequalities. *Georgian Math. J.* **7** (2000), no. 3, 489–512.

[2] R. Hakl and J. Oyarce, Positive periodic solutions for systems of linear functional differential inequalities. *J. Math. Anal. Appl.* **553** (2026), no. 2, 15 January 2026, Article no. 129882; `https://doi.org/10.1016/j.jmaa.2025.129882`