On Precise Lower Estimates of the First Eigenvalue of a Sturm–Liouville Problem with Weighted Integral Conditions on the Potential

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Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \ \ x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all non-negative locally integrable on $(0,1)$ functions such that the following integral conditions hold:

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \ \ \gamma \neq 0, \tag{3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{4}$$

This paper is devoted to the study of estimates for the first eigenvalue of a Sturm–Liouville problem with integral conditions on the potential initiated by Y. V. Egorov and V. A. Kondratiev in [1], where the equation

$$y'' + \lambda Q(x)y = 0$$

with the Dirichlet boundary conditions and a non-negative summable on $[0,1]$ potential Q satisfying the condition $\|Q\|_{L_{\gamma}(0,1)} = 1$, $\gamma \neq 0$, was considered.

A function y is a *solution* of problem (1), (2) if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

It was proved that if condition (4) does not hold, then there is no non-trivial solution of problem (1), (2) [2, Theorem 1].

If $\gamma < 0$, $\alpha \leq 2\gamma - 1$ or $\beta \leq 2\gamma - 1$, then the set $T_{\alpha,\beta,\gamma}$ is empty; for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty [8, Chapter 1, § 2, Theorem 3]. Since for $\gamma < 0$, $\alpha \leq 2\gamma - 1$ or $\beta \leq 2\gamma - 1$ there exists no function Q satisfying (3) and (4) taken together, for these parameter values problem (1)–(4) is not considered.

Consider the functional

$$R[Q,y] = \frac{\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}.$$

If condition (4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$ [3]. It was proved [2], [3] that for any $Q\in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q)=\inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

In this paper we describe precise estimates for

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

Theorem 1 ([5])**.** *If $\gamma>1$, $\alpha,\beta<2\gamma-1$, then there exist functions $Q_*\in T_{\alpha,\beta,\gamma}$ and $u\in H_0^1(0,1)$, $u>0$ on $(0,1)$ such that $m_{\alpha,\beta,\gamma}=R[Q_*,u]$. Moreover, u satisfies the equation*

$$u''+mu=-x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}} \tag{5}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx=1. \tag{6}$$

In addition, if $\gamma>1$, $\alpha,\beta\leqslant\gamma$, then $m_{\alpha,\beta,\gamma}=0$ [5].

Remark ([4], [5]). If $\gamma\geqslant1$, $\alpha>\gamma$ or $\beta>\gamma$, then $m_{\alpha,\beta,\gamma}<0$. If $\gamma<0$, $\alpha,\beta>2\gamma-1$; $0<\gamma<1$, $-\infty<\alpha,\beta<+\infty$ or $\gamma\geqslant1$, $\alpha>2\gamma-1$ or $\beta>2\gamma-1$, then $m_{\alpha,\beta,\gamma}=-\infty$.

Theorem 2 ([6])**.** *For $\alpha,\beta<1$, there exists a point $x_0\in(0,1)$ such that*

$$m_{\alpha,\beta,1}=m,$$

where m is a solution to the equation

$$\tan\sqrt{m}\,(1-x_0)=\frac{\sqrt{m}\,\sin\sqrt{m}\,x_0}{K\sin\sqrt{m}\,x_0-\sqrt{m}\,\cos\sqrt{m}\,x_0}\,,\ \ K={x_0}^{-\alpha}(1-x_0)^{-\beta}.$$

Moreover, $m_{\alpha,\beta,1}$ is attained at the potential $K\delta(x-x_0)$, and accurate to a constant factor,

$$y=\begin{cases}\dfrac{\sqrt{m}\,\cos\sqrt{m}\,(1-x_0)}{\cos\sqrt{m}\,(K\sin\sqrt{m}\,x_0-\sqrt{m}\,\cos\sqrt{m}\,x_0)}\,\sin\sqrt{m}\,x, & x\in[0,x_0],\\[2mm] \dfrac{\sin\sqrt{m}\,(1-x)}{\cos\sqrt{m}}\,, & x\in(x_0,1]\end{cases}$$

is a corresponding eigenfunction.

Theorem 3. *If $\gamma=1$, $\alpha=\beta=1$, then $m_{1,1,1}=0$.*

Proof. Let $\gamma=1$, $\alpha,\beta=1$. For any $Q\in T_{1,1,1}$ and $y\in H_0^1(0,1)$, we have

$$\int_0^1 Q(x)y^2\,dx\leqslant\sup_{[0,1]}\frac{y^2}{x(1-x)}\int_0^1 Q(x)x(1-x)\,dx\leqslant\int_0^1 y'^2\,dx,$$

and $m_{\alpha,\beta,\gamma}\geqslant0$.

For $\lambda=0$, from the result of [8, Chapter 1, §1, Lemma 2], we obtain that if Q satisfies (4), then the equation $y''+Q(x)y=0$ has a non-trivial solution satisfying zero boundary conditions. For example, for $Q(x)\equiv6$ on $[0,1]$, one of the corresponding solutions is

$$y=\begin{cases}\sin\sqrt{6}\,x, & x\in\Big[0,\dfrac12\Big],\\[2mm] \sin\sqrt{6}\,(1-x), & x\in\Big(\dfrac12,1\Big],\end{cases}$$

and therefore, $m_{1,1,1}=0$. □

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