A Necessary Condition for the Solvability of a Control Problem of an Asynchronous Spectrum of Linear Periodic Systems with an Irregular Feasible Set

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Let the control system be described by the equation

$$\dot{x} = f(t, x, u), \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n,$$

the right-hand side of which ensures the existence and uniqueness of solutions and is periodic or almost periodic in t. The control u takes values from a certain feasible set determined by the formulation of a specific problem [9,10], etc. The problem of selecting a control u such that a given equation will produce irregular periodic solutions whose frequency spectrum contains a given subset L is called the spectrum assignment problem for irregular oscillations (asynchronous spectrum) with a target frequency set L [3].

The solvability of the formulated problem was studied in [1, 5, 6] for linear periodic systems with control having the same period. It is quite natural to expect that the problem of controlling an asynchronous spectrum admits modified versions associated with other types of control. For example, in [4, Chapter III] the case of control synthesis in the form of feedback linear in phase variables was considered.

In this paper, we formulate for the first time the problem of controlling the asynchronous spectrum of periodic systems, where the admissible set consists of periodic functions whose period is incommensurate with the period of the system.

For a continuous ω-periodic function $f(t)$, the mean value is a constant

$$\widehat{f} = \frac{1}{\omega} \int\limits_0^{\omega} f(\tau)\, d\tau,$$

and the oscillating part is defined by the equality

$$\widetilde{f}(t) = f(t) - \widehat{f}.$$

By $\operatorname{rank}_{\text{row}} \widetilde{H}$ we denote the row rank of some periodic matrix $H(t)$, i.e., the largest number of its linearly independent columns. The column rank of this matrix $\operatorname{rank}_{\text{col}} H$ can be defined similarly. Note that in general the row and column ranks of the matrix $H(t)$ do not have to coincide. Indeed, for the matrix

$$H(t) = \begin{pmatrix} \sin t & 2\sin t \\ \cos t & 2\cos t \end{pmatrix}$$

we have $\operatorname{rank}_{\text{row}} H = 2$ and $\operatorname{rank}_{\text{col}} H = 1$. In the stationary case, the ranks introduced coincide. We say that $H(t)$ is a matrix of incomplete column rank if column rank of H is less than the number of columns.

Consider a quasiperiodic system

$$\frac{dz}{dt} = g(t,z) + h(t,z), \quad z \in \mathbb{R}^n,$$

where the vector functions g and h are periodic in the first argument with periods ω and Ω, respectively, and the ratio of these periods is irrational.

A periodic solution $z = z(t)$ with period Ω of this system is called partially irregular [2], and the frequency spectrum of z is called partially asynchronous.

Now we consider a linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

in which $A(t)$ is a continuous ω-periodic $n \times n$-matrix, B is a constant $n \times m$-matrix, u is the control. We suppose that $u(\cdot)$ is Ω-periodic m-vector continuous functions defined on the real axis and such that the numbers ω and Ω are incommensurable. Admissible sets of periodic functions of this kind will be called irregular.

Problem A. The problem of assignment for the partially asynchronous spectrum with a target set L and an irregular feasible set can be formulated in the following form: select a programmed control

$$u = U(t) \tag{1}$$

from the specified feasible set such that the system

$$\dot{x} = A(t)x + Bu(t) \tag{2}$$

has a nontrivial partially irregular solution $x = x(t)$ of period Ω with a given frequency spectrum L.

Theorem. *If assignment problem* **A** *has a solution, then the oscillating component of the coefficient matrix has an incomplete column rank, i.e.,*

$$\operatorname{rank}_{\text{col}} \widetilde{A} = n - d, \quad 1 \le d < n. \tag{3}$$

Proof. Assume that Problem A is solvable, and condition (3) does not hold. In other words, there exists an Ω-periodic vector (1) such that system (2) has a nontrivial solution $x = x(t)$ of the same period, and the matrix $\widetilde{A}(t)$ has full column rank, i.e. the equality

$$\operatorname{rank}_{\text{col}} \widetilde{A} = n \tag{4}$$

holds. Let us expand the Ω-periodic vector $x(t)$ into a Fourier series

$$x(t) \sim \sum_{m=-\infty}^{\infty} x_m \exp\Big(\frac{2\pi i m}{\Omega}\, t\Big),$$

where

$$x_m = \frac{1}{\Omega} \int_0^{\Omega} x(\tau) \exp\Big(-\frac{2\pi i m}{\Omega}\, \tau\Big)\, d\tau.$$

Given the resulting expansion, we write

$$\Big(A(t) - \frac{1}{\omega}\int_0^{\omega} A(\tau)\, d\tau\Big) x(t) \sim \Big(A(t) - \frac{1}{\omega}\int_0^{\omega} A(\tau)\, d\tau\Big) \sum_{m=-\infty}^{\infty} x_m \exp\Big(\frac{2\pi i m}{\Omega} t\Big). \tag{5}$$

Let us expand the ω-periodic matrix function into a Fourier series

$$A(t)-\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\,d\tau\sim\sum_{k=-\infty,\;k\neq0}^{\infty}A_k\exp\Big(\frac{2\pi ik}{\omega}t\Big), \tag{6}$$

where

$$A_k=\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\exp\Big(-\frac{2\pi ik}{\omega}\tau\Big)\,d\tau.$$

Using expansion (6), we apply the properties of formal operations on Fourier series [8, p. 39] to (5)

$$\Big(A(t)-\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\,d\tau\Big)x(t)\sim\sum_{k,m=-\infty,\;k\neq0}^{\infty}A_kx_m\exp\Big(\frac{2\pi ik}{\omega}+\frac{2\pi im}{\Omega}\Big)t. \tag{7}$$

Let us write (7) as a series

$$\Big(A(t)-\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\,d\tau\Big)x(t)\sim\sum_{j=-\infty}^{\infty}c_j\exp(2\pi i\lambda_j)t \tag{8}$$

with the coefficients

$$c_j=\sum_{\nu_k+\mu_m=\lambda_j}A_kx_m,\quad \nu_k=\frac{k}{\omega}\,,\quad \mu_m=\frac{m}{\Omega}\,. \tag{9}$$

We show that for each pair of indices k and m, the index j is the only one that ensures the equality

$$\nu_k+\mu_m=\lambda_j.$$

This means that each of the sums (9) will consist of only one term for any values of the indices k and m. Let us assume that this is not true, i.e. there exist two pairs of indices k_1, m_1 and k_2, m_2 ($k_1\neq k_2$, $m_1\neq m_2$) such that the equality

$$\nu_{k_1}+\mu_{m_1}=\nu_{k_2}+\mu_{m_2}$$

is satisfied. Therefore, by (9) we obtain

$$\frac{\omega}{\Omega}=\frac{k_1-k_2}{m_1-m_2}\,.$$

Since the indices k_1, k_2, m_1, m_2 are integers, the right-hand side of this equality is a rational number. Consequently, the periods ω and Ω are commensurable. We have obtained a contradiction. This means that the assumption made is incorrect, and each coefficient c_j of series (8) defined by relation (9) consists of only one term.

Since quasiperiodic system (2) has a partially irregular solution $x(t)$ and the ratio ω/Ω is irrational, then according to [7] the vector $x(t)$ satisfies the identity

$$\Big(A(t)-\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\,d\tau\Big)=\widetilde{A}(t)x(t)\equiv0. \tag{10}$$

Taking into account identity (10), we have the representation

$$0\equiv\Big(A(t)-\frac{1}{\omega}\int\limits_0^{\omega}A(\tau)\,d\tau\Big)x(t)\sim\sum_{j=-\infty}^{\infty}c_j\exp(2\pi i\lambda_j) \tag{11}$$

for series (8).

From the series expansion (11), based on the uniqueness theorem for almost periodic functions [8, p. 51], it follows that all the coefficients of this series are zero. Therefore,

$$A_k x_m = 0 \tag{12}$$

for all $k = \pm 1, \pm 2, \dots$ and $m = 0, \pm 1, \pm 2, \dots$.

By assumption there exists $x(t) \not\equiv 0$. Hence a Fourier coefficient of x has a nonzero component x_s ($s \in \{0, N\}$). Let us apply the properties of formal operations on Fourier series [8, p. 39] to one of the terms of representation (5) for $m = s$, i.e. consider the product

$$\Big(A(t) - \frac{1}{\omega}\int_0^{\omega} A(\tau)\, d\tau\Big) x_s \sim \sum_{k=-\infty,\, k\neq 0}^{\infty} A_k x_s \exp\Big(\frac{2\pi i k}{\Omega}\, t\Big).$$

As shown above, according to (12), all products $A_k x_s = 0$ ($k = \pm 1, \pm 2, \dots$). Therefore, due to the periodicity of the function $\widetilde{A}(t) x_s$, based on [8, p. 51] and the adopted notation, we have the identity

$$\widetilde{A}(t) x_s \equiv 0, \quad x_s \neq 0,$$

from which it follows that $\widetilde{A}(t)$ is a matrix of incomplete column rank. We have a contradiction with assumption (4). □

Acknowledgments

This work was carried out at the Institute of Mathematics of the National Academy of Sciences of Belarus with the support of the Belarusian Republican Foundation for Basic Research (Project # Φ 25KI-015).

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