The Asymptotic Representation of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of Second Order Differential Equations with the Product of a Regularly and a Rapidly Varying Functions in Some Critical Case

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We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation the constant α_0 is responsible for the sign of the equation, functions $p : [a, \omega[\to]0, +\infty[$, $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is some one-sided neighborhood of Y_i.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of the index σ_1 [8, pp. 10-15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y') \neq 0 \;\text{ as }\; y' \in \Delta_{Y_0}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \varphi_0(y') \in \{0, +\infty\}, \quad \lim_{\substack{y' \to Y_0 \\ y' \in \Delta_{Y_0}}} \frac{\varphi_0(y')\varphi_0''(y')}{(\varphi_0'(y'))^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 [1]. So equation (1) is the second order differential equation that contains the product of a regularly varying function of unknown function and a rapidly varying function of its first derivative in its right-hand side. In previous works [3] we obtained results for this kind of the equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative.

Let's consider the following class of solutions for equation (1).

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the following conditions take place

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \;\; (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

This class of solutions was introduced in the work by V. M. Evtukhov and A. M. Samoilenko [6] for nth-order Emden–Fowler type differential equations and was later specified for the second-order case. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, each such solution belongs to one of four disjoint sets depending on the value of λ_0:

$$\lambda_0 \in \mathbb{R} \setminus \{0, 1\}, \;\; \lambda_0 = 0, \;\; \lambda_0 = 1, \;\; \lambda_0 = \pm\infty.$$

In this work, we investigate equation (1) with respect to the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions. This class of solutions for equations of the form (1) is among the most difficult to study, since the second derivative cannot be explicitly expressed in terms of the first derivative.

For the $P_\omega(Y_0, Y_1, \pm\infty)$-solutions under consideration, in [6] it was established the following a priori asymptotic relations:

$$\frac{\pi_\omega(t)y'(t)}{y(t)} = [1 + o(1)], \quad \frac{\pi_\omega(t)y''(t)}{y'(t)} = o(1) \ \text{ as } \ t \uparrow \omega, \tag{3}$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty. \end{cases}$$

From conditions (3) it follows that $P_\omega(Y_0, Y_1, \pm\infty)$-solutions are regularly varying functions of order 1 as $t \uparrow \omega$, and their first derivatives are slowly varying functions in the same limit. Some results concerning $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) in the case $\lambda_0 = \pm\infty$ were obtained in [3] and [4].

The aim of this work is to establish necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1), and also to obtain asymptotic representations as $t \uparrow \omega$ for these solutions and their first derivatives in a certain critical case that was not considered in [3] and [4].

To formulate the main results, we introduce the following definitions.

Definition 2. Let $Y \in \{0, \infty\}$, and let Δ_Y be a one–sided neighborhood of Y. A continuously differentiable function $L : \Delta_Y \to]0, +\infty[$ is called a normalized slowly varying function as $y \to Y$ $(y \in \Delta_Y)$ (see [7, pp. 2–3]) if

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

Definition 3. We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S as $z \to Y$, if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to]0; +\infty[$ the next relation is valid

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y \ \ (z \in \Delta_Y).$$

Conditions S is satisfied by functions $\ln|y|$, $|\ln|y||^\mu$ $(\mu \in R)$, $\ln|\ln|y||$ and many others.

Let us put in the following notations

$$\mu_0 = \operatorname{sign} \varphi_0'(y'), \quad \upsilon_0 = \operatorname{sign} y_0^0, \quad \upsilon_1 = \begin{cases} 1, & \text{if } \ \Delta_{Y_0} = [y_0^0, Y_0[\,, \\ -1, & \text{if } \ \Delta_{Y_0} =]Y_0, y_0^0]. \end{cases}$$

According to the definition of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of the differential equation (1), the numbers υ_0, υ_1 and α_0 uniquely determine the signs of any such solution, as well as the signs of its first and second derivatives in some neighborhood of ω. Note that the conditions

$$\upsilon_0 \cdot \upsilon_1 < 0 \ \text{ for } \ Y_0 = 0, \quad \upsilon_0 \cdot \upsilon_1 > 0 \ \text{ for } \ Y_0 = \pm\infty,$$

and

$$\upsilon_1 \cdot \alpha_0 < 0 \ \text{ for } \ \lim_{t \uparrow \omega} y'(t) = 0, \quad \alpha_0 \cdot \upsilon_1 > 0 \ \text{ for } \ \lim_{t \uparrow \omega} y'(t) = \pm\infty$$

are necessary for the existence of such solutions.

Then, taking into account (3), the conditions

$$\upsilon_0 \cdot \upsilon_1 \pi_\omega(t) > 0 \ \text{ for } \ t \in [a, \omega[$$

and

$$Y_0 = 0 \ \text{ for } \ \omega < +\infty, \quad Y_0 = \pm\infty \ \text{ for } \ t \in [a, \omega[$$

are also necessary for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$–solutions.

In [4] the following theorem is proved.

Theorem 1. *Let $\sigma_1 \neq 1$, and suppose the function $\varphi_1(y')|y'|^{-\sigma_1}$ satisfies condition S as $y' \to Y_1$ ($y' \in \Delta_{Y_1}$). Then every $P_\omega(Y_0, Y_1, \pm\infty)$–solution of* (1) *can be written as*

$$y(t) = \pi_\omega(t) L(t). \tag{4}$$

Here $L : [t_0, \omega[\to \mathbb{R}$ is twice continuously differentiable, and satisfies the conditions

$$y_0^0 \pi_\omega(t) L(t) > 0, \quad L'(t) \neq 0 \ \text{ for } \ t \in [t_1, \omega[\,, \tag{5}$$

$$\lim_{t \uparrow \omega} L(t) \in \{0, \pm\infty\}, \quad \lim_{t \uparrow \omega} \pi_\omega(t) L(t) = Y_0, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) L'(t)}{L(t)} = 0. \tag{6}$$

If the limit

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) L''(t)}{L'(t)}$$

exists, then

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) L''(t)}{L'(t)} = -1, \ \ \alpha_0 L'(t) > 0. \tag{7}$$

Moreover, the next asymptotic representation as $t \uparrow \omega$ takes place

$$p(t) = \frac{\alpha_0 L'(t)}{|\pi_\omega(t) L(t)|^{\sigma_1} \theta_1(\pi_\omega(t)) \varphi_0(L(t)[1 + \frac{\pi_\omega(t) L'(t)}{L(t)}])} \, [1 + o(1)]. \tag{8}$$

Theorem 1 contains the necessary conditions for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$–solutions of equation (1).

To formulate the sufficient conditions, we introduce the following definition.

Definition 4. We say that condition N holds for some $L(t) : [t_0, \omega[\to R$ ($t_0 \in [a, \omega[$), satisfying (4)–(7), if and only if

$$p(t) = \frac{\alpha_0 L'(t)}{|\pi_\omega(t) L(t)|^{\sigma_1} \theta_1(\pi_\omega(t)) \varphi_0(L(t)[1 + \frac{\pi_\omega(t) L'(t)}{L(t)}])} \, [1 + r(t)],$$

where $r(t) \to 0$ as $t \uparrow \omega$.

Let

$$\theta_1(y) = \varphi_1(y)|y|^{-\sigma_1}, \quad X(t) = L(t) \cdot e_1(t),$$

$$H(t) = \frac{L^2(t) \varphi_0'(X(t))}{\pi_\omega(t) L'(t) \varphi_0(X(t))}\,, \quad q_1(t) = \frac{(\frac{\varphi_0'(y)}{\varphi_0(y)})'}{(\frac{\varphi_0'(y)}{\varphi_0(y)})^2}\bigg|_{y = X(t)}, \quad q_2(t) = y \frac{(\frac{\varphi_0'(y)}{\varphi_0(y)})'}{(\frac{\varphi_0'(y)}{\varphi_0(y)})}\bigg|_{y = X(t)},$$

$$e_1(t) = 1 + \frac{\pi_\omega(t) L'(t)}{L(t)}\,, \quad e_2(t) = 2 + \frac{\pi_\omega(t) L''(t)}{L'(t)}\,.$$

From (2) and (6) it follows that the following statements are true.

1)

$$\lim_{t\uparrow\omega} e_1(t) = \lim_{t\uparrow\omega} e_2(t) = 1, \tag{9}$$

$$\lim_{t\uparrow\omega} H(t) = \pm\infty, \quad \lim_{t\uparrow\omega} q_1(t) = 0; \tag{10}$$

2) If the limit

$$\lim_{t\uparrow\omega} \frac{L(t)}{L'(t)} \cdot \frac{H'(t)}{|H(t)|^{3/2}}$$

exists, then

$$\lim_{t\uparrow\omega} \frac{L(t)}{L'(t)} \cdot \frac{H'(t)}{|H(t)|^{3/2}} = 0. \tag{11}$$

Indeed, statements (9) follow directly from conditions (6) and (7). Statements (10) follow from the validity of the corresponding relations.

Statement (11) is proved analogously to the corresponding result presented in the work by Evtukhov V. M. and Chernikova A. H. [5].

In this paper, unlike [3] and [4], we consider the case when the condition

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)} |H(t)|^{1/2} = 0 \tag{12}$$

holds.

In this case, taking (11) into account,

$$\frac{\pi_\omega(t)L'(t)}{L(t)} q_1(t)H(t) = -1 + o(1), \quad t \uparrow \omega.$$

From the form of the functions H, q_1, and q_2, it follows that

$$\lim_{t\uparrow\omega} q_2(t) = -1,$$

which, in view of (7) and (8), implies that the function $\frac{\varphi_0'(y')}{\varphi_0(y')}$ is regularly varying of order -1 as $y' \to Y_0$, $(y' \in \Delta_{Y_0})$,

$$H(t) = \frac{L(t)}{\pi_\omega(t)L'(t)} \cdot \frac{X(t)\varphi_0'(X(t))}{\varphi_0(X(t))} \cdot \frac{1}{e_1(t)}, \quad \frac{\varphi_0(X(t))\varphi_0''(X(t))}{(\varphi_0'(X(t)))^2} = 1 + \frac{\left(\frac{\varphi_0'(X(t))}{\varphi_0'(X(t))}\right)'}{\frac{\varphi_0'(X(t))}{\varphi_0'(X(t))}}.$$

Thus, the case when condition (12) holds is in a certain sense critical.

We further assume that the finite or infinite limit

$$\lim_{t\uparrow\omega} \left(1 + q_1(t) + \frac{X(t)}{e_2(t)L'(t)}\right) \cdot (e_1(t) - 1) \cdot e_1(t) \cdot H(t) = \gamma_1$$

exists.

Note that the sign of γ_1 must coincide with the sign of $\alpha_0 \cdot \upsilon_0 \cdot \upsilon_1 \cdot \mu_0(t)$.

The following theorem takes place.

Theorem 2. *Let $\sigma_1 \neq 1$, the function θ_1 satisfy condition S, and conditions N and* (12) *hold. Then the differential equation* (1) *has at least one $P_\omega(Y_0, Y_1, \pm\infty)$–solution for which the following asymptotic representations as $t \uparrow \omega$ hold:*

$$y(t) = \pi_\omega(t) \cdot L(t)\,(1 + o(1)), \tag{13}$$

$$y'(t) = X(t) + \frac{\varphi_0(X(t))}{\varphi_0'(X(t))} \cdot |H(t)|^{1/2} \cdot o(1). \tag{14}$$

Moreover, if $\omega < +\infty$, then when one of the conditions

$$\textit{either}\ \ 1 < \gamma_1 < 0, \ \ \textit{or}\ \ \gamma_1 = 0 \ \ \textit{and}\ \ \alpha_0 \cdot \mu_0 > 0,$$

is satisfied, equation (1) *has a one-parameter family of $P_\omega(Y_0, Y_1, \pm\infty)$–solutions with asymptotics* (13)*,* (14)*.*

If $\omega = +\infty$, then there exists a two-parameter family of such solutions in the cases

$$\textit{either}\ \ 1 < \gamma_1 < 0, \ \ \textit{or}\ \ \gamma_1 = 0 \ \ \textit{and}\ \ \alpha_0 \cdot \mu_0 < 0$$

and a one-parameter family in the cases

$$-\infty < \gamma_1 < -1 \ \ \textit{or}\ \ 0 < \gamma_1 < +\infty.$$

To prove Theorem 2, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system is such that the determinant of the matrix is nonzero, the sum of the elements of the first row is equal to 0, and the sum of the elements of the second row is nonzero. Therefore, all the conditions of Theorem 2.7 from [6] are satisfied for the system.

According to this theorem, the system has at least one solution

$$\{\omega_i\}_{i=1}^2 : [t^*, +\infty[\to \mathbb{R}^2 \ \ (t^* \geq t_1),$$

which tends to zero as $t \uparrow \omega$. These solutions correspond to the solutions

$$y : [t^*, +\infty[\to \mathbb{R} \ \ (t^* \geq t_1)$$

of equation (1), which admit the asymptotic representations (13), (14) as $t \uparrow \omega$.

If $\omega < +\infty$ and $\alpha_0\mu_0 \upsilon_0 \upsilon_1 < 0$, and also if $\omega = +\infty$ and $\alpha_0\mu_0\upsilon_0\upsilon_1 > 0$, then there exists a one-parameter family of such $P_\omega(Y_0, Y_1, \pm\infty)$-solutions.

If $\omega = +\infty$ and $\alpha_0\mu_0\upsilon_0\upsilon_1 < 0$, then there exists a two-parameter family of such $P_\omega(Y_0, Y_1, \pm\infty)$-solutions.

From the form of the representations (13), (14), it follows that the obtained solutions are indeed $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1).

Thus, this paper studies second-order differential equations of the form (1) whose right-hand side contains the product of a regularly varying nonlinear function of the unknown function and a rapidly varying nonlinear function of its derivative, as the arguments tend to zero or to infinity.

We obtain necessary (Theorem 1) and sufficient (Theorem 2) conditions for the existence of slowly varying $P_\omega(Y_0, Y_1, \pm\infty)$-solutions in a certain critical case (12), as well as asymptotic representations for these solutions and their first derivatives.

Under additional restrictions on the coefficients of the characteristic equation of the associated equivalent system of quasilinear equations, we establish conditions for the existence of one-parameter and two-parameter families of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions.

It is worth noting that earlier similar results were obtained for equations in which, on the contrary, the right-hand side is the product of a rapidly varying function of the unknown function and a regularly varying function of its derivative. For the equations studied in this work, the obtained results are new.

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