Sharp Conditions for the Solvability of a Nonlocal Boundary Value Problem for Linear Second Order Functional Differential Equations

Eugene Bravyi

Perm National Research Polytechnic University, Perm, Russia
E-mail: bravyi@perm.ru

1 Introduction

We establish unimprovable conditions for the unique solvability and positive solvability of a boundary value problem, various forms of which have frequently appeared in recent literature on differential equations and their applications. The motivation for this article comes from works [3, 5, 6, 8, 12, 13, 15–18] which addressed various issues related to these boundary conditions for different modifications of the heat equation. However, to the best of the author's knowledge, unimprovable conditions for the solvability and positive solvability of the boundary value problem have not been obtained for such boundary conditions.

By unimprovable sufficient conditions for solvability, we mean necessary and sufficient conditions for all objects from a given family to possess the property we are interested in (namely, the unique solvability of the boundary value problem). If these conditions are not met, then there exists an object in this family for which the property in question fails to hold.

For the functional differential equation

$$\ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [0,1], \tag{1.1}$$

where $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator and $f \in \mathbf{L}[0,1]$, we consider the boundary value problem

$$\dot{x}(0) = 0, \tag{1.2}$$
$$x(\xi) = \alpha \dot{x}(1). \tag{1.3}$$

Here, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ denote the spaces of real-valued continuous and integrable functions on the interval $[0,1]$, endowed with their standard norms, respectively. A solution to the boundary value problem (1.1)–(1.3) is understood as a function that is absolutely continuous together with its derivative, satisfies the boundary conditions (1.2), (1.3), and satisfies equation (1.1) for almost all $t \in [0,1]$.

This problem describes the stationary state of a heated rod in a heat conduction model governed by equation (1.1). The boundary condition (1.2) corresponds to the rod being insulated at the left end $t = 0$. The nonlocal boundary condition (1.3) means that the heat flux at the right end of the rod, at $t = 1$, is determined by a special controller (thermostat) and is proportional, with a coefficient $\alpha \in \mathbb{R}$, to the temperature measured by a sensor at the point $t = \xi \in [0,1]$ on the rod. The nonstationary formulation of the linear heat conduction problem with the nonlocal boundary condition (1.3) was apparently first studied in [5, 6]. In recent years, numerous works have been devoted to problem (1.1)–(1.3) for the case where the operator T is a Nemytskii operator [15–17].

Generalizations of these models have also been considered, in particular, models with fractional derivatives [8, 12, 13, 18] and with a functional operator T [3].

Closely related boundary value problems for delay equations are considered in [2, 4]. In these works, conditions for the existence of solutions to problem (1.1)–(1.3) were found for both classical and generalized models.

Here, problem (1.1)–(1.3) is considered for families of monotone operators T with a given norm. We study such linear operators T for which either T or $-T$ is positive, meaning it maps nonnegative functions to almost everywhere nonnegative ones. The norm of such an operator T is defined by the equality $\|T\|_{\mathbf{C}\to\mathbf{L}} = \int\limits_0^1 |(T\mathbf{1})(s)|\,ds$, where $\mathbf{1}$ is the unit function.

2 Main results

For the general boundary value problem for a second-order functional-differential equation, [1] established general assertions about necessary and sufficient conditions for unique solvability in families of regular operators T (representable as a difference of positive operators). Verifying these conditions requires solving a finite-dimensional optimization problem. This is a standard issue; however, in each specific case, obtaining the necessary and sufficient conditions in explicit form requires overcoming particular difficulties that depend on the boundary conditions. Necessary and sufficient conditions for such families in explicit form have been found only in exceptional cases: for the Cauchy problem, for the periodic problem, for the two-point boundary value problem with boundary conditions $x(0) = 0$, $x(1) = 0$ or $x(0) = 0$, $\dot{x}(1) = 0$, and for some other problems, typically for first and second-order equations, see the works of A. Lomtatidze, S. Mukhigulashvili, R. Hakl, J. Sremr [7, 9–11, 14].

Here, based on the approach of [1], we obtain effective necessary and sufficient conditions for the unique solvability of the boundary value problem for families of equations with monotone operators T and conditions for the sign preservation of the boundary value problem solutions.

Let us state the unique solvability conditions. They substantially improve the known solvability conditions for this problem obtained, for example, using the contraction mapping method.

Theorem 1. *Let $\xi = 0$. The boundary value problem (1.1)–(1.3) has a unique solution for all positive operators T with a given norm $\mathcal{T}^+$ if and only if either the inequalities*

$$0 \le \mathcal{T}^+ \le \begin{cases} 4 + \dfrac{1}{\alpha}, & \text{if } \alpha \le -\dfrac{1}{2}, \\ \dfrac{1}{1+\alpha}, & \text{if } \alpha > -\dfrac{1}{2} \end{cases} \tag{2.1}$$

hold, or the inequalities

$$\alpha \ge \frac{1}{8}, \quad \frac{1}{\alpha} \le \mathcal{T}^+ \le \begin{cases} 4 + 8\big(\sqrt{\alpha(1+\alpha)} + \alpha\big), & \text{if } \alpha \in \Big[\dfrac{1}{8}, \alpha_1\Big], \\ 4 + \dfrac{1}{\alpha}, & \text{if } \alpha > \alpha_1 \end{cases} \tag{2.2}$$

hold, where

$$\alpha_1 = \frac{q}{24} + \frac{6}{q} - \frac{1}{12} \approx 0.19, \quad q = \big(100 + 12\sqrt{69}\big)^{1/3}.$$

Theorem 2. *Let $\xi=0$. The boundary value problem* (1.1)–(1.3) *has a unique solution for all linear operators T such that $-T$ is a positive operator with norm $\mathcal{T}^-$ if and only if either the inequalities*

$$0\le\mathcal{T}^-\le\begin{cases}-\dfrac{1}{\alpha}\,, & \text{if } \alpha\le-\dfrac{1}{4}\,,\\ 16\alpha+8, & \text{if } -\dfrac{1}{4}<\alpha\le\alpha_2,\\ 2+2\sqrt{1+\dfrac{1}{\alpha}}\,, & \text{if } \alpha>\alpha_2\end{cases}\tag{2.3}$$

hold, where

$$\alpha_2=\frac{p}{24}+\frac{1}{2p}-\frac{1}{4}\approx0.08,\ \ p=\big(108+12\sqrt{69}\big)^{1/3},$$

or the inequalities

$$\alpha\le-\frac{4}{3}\,,\quad -\frac{1}{1+\alpha}\le\mathcal{T}^-\le2+2\sqrt{1+\frac{1}{\alpha}}\tag{2.4}$$

hold.

The dependence on α of the set of values of the norms of the operators $T=T^+$ and $T=T^-$, for which problem (1.1)–(1.3) is uniquely solvable, is shown in Figures 1 and 2, respectively. The diagrams are placed at the end of the text.

Denote $\xi_0=(\sqrt{5}-2)^2\approx0.056$.

In the following theorems, the parameter ξ lies entirely to the left or entirely to the right of ξ_0.

Theorem 3. *Let $\xi\in[0,\xi_0]$, $\alpha>\xi$, $\mathcal{T}^+\ge\frac{1}{\alpha-\xi}$. The boundary value problem* (1.1)–(1.3) *has a unique solution for all linear positive operators T with norm $\mathcal{T}^+$ if and only if the inequalities*

$$\alpha\ge\frac{(3\xi+1)^2}{8(1+\xi)}$$

and

$$\frac{1}{\alpha-\xi}\le\mathcal{T}^+\le\min\bigg\{4\Big(\frac{\sqrt{\alpha}+\sqrt{\alpha+1-\xi}}{1-\xi}\Big)^2,2+\frac{1-\xi}{\alpha}+2\sqrt{1-\frac{\xi}{\alpha}}\bigg\}\tag{2.5}$$

hold.

Theorem 4. *Let $\xi\in[\xi_0,1]$, $\alpha>\xi$, $\mathcal{T}^+\ge\frac{1}{\alpha-\xi}$.*

The boundary value problem (1.1)–(1.3) *has a unique solution for all linear positive operators T with norm $\mathcal{T}^+$ if and only if the inequalities*

$$\alpha\ge\frac{\sqrt{\xi}(\sqrt{\xi}+1)^2}{2+\sqrt{\xi}}$$

and

$$\frac{1}{\alpha-\xi}\le\mathcal{T}^+\le2+\frac{1-\xi}{\alpha}+2\sqrt{1-\frac{\xi}{\alpha}}\tag{2.6}$$

hold.

We state several results on the sign preservation of solutions.

Theorem 5. *Let* $\alpha > 1/2$ *and* $\xi = 0$. *Suppose the conditions of Theorem 1 are satisfied. Then all solutions of problem* (1.1)–(1.3) *preserve their sign for every nonnegative function* $f \in L[0,1]$ *satisfying*

$$\mu \operatorname*{ess\,sup}_{x\in[0,1]} f(x) \le \operatorname*{ess\,inf}_{x\in[0,1]} f(x),$$

where $\mu \in [0,1]$, *if and only if*

$$\mathcal{T}^+ \le 1 + \sqrt{\mu}. \tag{2.7}$$

Moreover, if conditions (2.1) *hold, then the solutions are nonnegative, and if conditions* (2.2) *hold, then the solutions are nonpositive.*

Theorem 6. *Let* $\xi = 0$. *All solutions of the boundary value problem* (1.1)–(1.3) *are nonnegative for every nonnegative function* $f \in L[0,1]$ *and for every positive operator* $T : C[0,1] \to L[0,1]$ *with a given norm* $\mathcal{T}^+$ *if and only if either the inequalities*

$$\alpha \le -1, \;\; \mathcal{T}^+ \le 1 + \frac{1}{\alpha}\,,$$

or the inequalities

$$\alpha \ge 0, \;\; \mathcal{T}^+ \le \frac{1}{1+\alpha}$$

hold.

All solutions of the boundary value problem (1.1)–(1.3) *are nonpositive for every nonnegative function* $f \in L[0,1]$ *and for every positive operator* $T : C[0,1] \to L[0,1]$ *with a given norm* $\mathcal{T}^+$ *if and only if the inequalities*

$$\alpha \ge 1, \;\; \frac{1}{\alpha} \le \mathcal{T}^+ \le 1$$

hold.

Theorem 7. *Let* $\xi = 0$. *All solutions of the boundary value problem* (1.1)–(1.3) *are nonnegative for every nonnegative function* $f \in L[0,1]$ *and for every operator* $T : C[0,1] \to L[0,1]$ *such that* $-T$ *is positive with norm* $\mathcal{T}^-$ *if and only if either the inequalities*

$$\alpha \le -1, \;\; \mathcal{T}^- \le \min\left\{1 + \frac{1}{\alpha}, -\frac{1}{\alpha}\right\},$$

or the inequalities

$$\alpha \ge 0, \;\; \mathcal{T}^- \le \min\{1, 4\alpha\}$$

hold.

All solutions of the boundary value problem (1.1)–(1.3) *are nonpositive for every nonnegative function* $f \in L[0,1]$ *and for every operator* $T : C[0,1] \to L[0,1]$ *such that* $-T$ *is positive with norm* $\mathcal{T}^-$ *if and only if the inequalities*

$$\alpha \le -\frac{3+\sqrt{5}}{2}\,, \;\; \frac{-1}{1+\alpha} \le \mathcal{T}^+ \le 1 + \frac{1}{\alpha}$$

hold.

All conditions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6) and the conditions of Theorems 6, 7 are unimprovable in the sense that if they are not satisfied, then there exists a corresponding operator T with the given norm for which the property under study fails to hold.

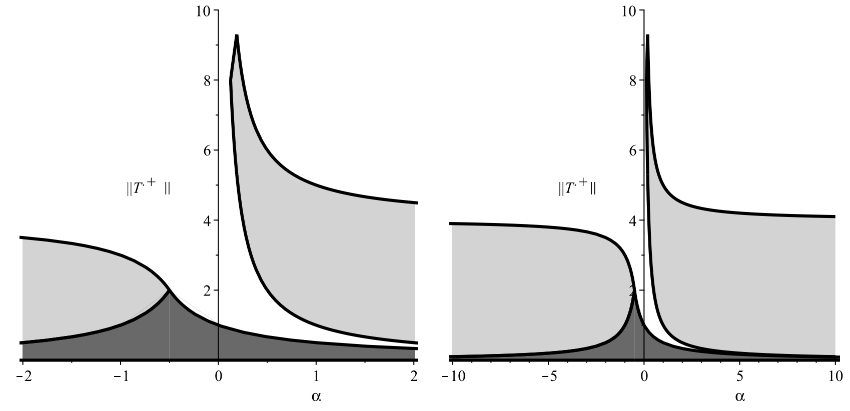

Figure 1. For each α, the set of norms of positive operators T^+ for which, by Theorem 1, any problem (1.1)–(1.3) with $T = T^+$ is uniquely solvable is highlighted in color. If the solvability is proved using the contraction mapping principle, the region is shaded black. For convenience, the graphs are shown in different scales.

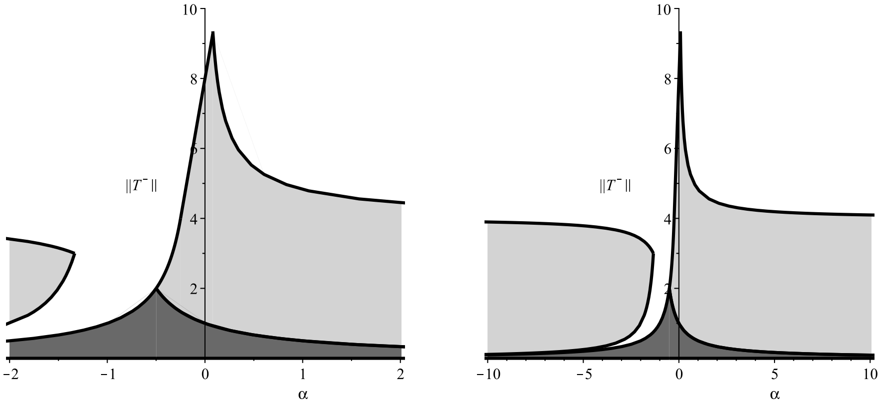

Figure 2. For each α, the set of norms of positive operators T^- for which, by Theorem 2, any problem (1.1)–(1.3) with $T = -T^-$ is uniquely solvable is highlighted in color. If the solvability is proved using the contraction mapping principle, the region is shaded black. For convenience, the graphs are shown in different scales.

Acknowledgements

The author was supported by the Ministry of Science and Higher Education of the Russian Federation (the state task FSNM-2023-0005).

References

[1] E. I. Bravyi, *Solvability of Boundary Value Problems for Linear Functional Differential Equations.* (Russian) Regular and Chaotic Dynamic, Moscow–Izhevsk, 2011.

[2] A. Calamai and G. Infante, Nontrivial solutions of boundary value problems for second-order functional differential equations. *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 3, 741–756.

[3] A. Calamai and G. Infante, Nontrivial solutions of a parameter-dependent heat-flow problem with deviated arguments. *Topological methods for delay and ordinary differential equations – with applications to continuum mechanics*, 141–150, Adv. Mech. Math., 51, *Adv. Contin. Mech., Birkhäuser/Springer, Cham*, 2024.

[4] M. A. Domínguez-Pérez and R. Rodríguez-López, Multipoint boundary value problems of Neumann type for functional differential equations. *Nonlinear Anal. Real World Appl.* **13** (2012), no. 4, 1662–1675.

[5] P. Guidotti and S. Merino, Hopf bifurcation in a scalar reaction diffusion equation. *J. Differential Equations* **140** (1997), no. 1, 209–222.

[6] P. Guidotti and S. Merino, Gradual loss of positivity and hidden invariant cones in a scalar heat equation. *Differential Integral Equations* **13** (2000), no. 10-12, 1551–1568.

[7] R. Hakl, A. Lomtatidze and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations.* Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis. Mathematica 10. Masaryk University, Brno, 2002.

[8] X. Hao and L. Zhang, Positive solutions of a fractional thermostat model with a parameter. *Symmetry* **11** 2019, no. 1, p pp.

[9] A. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5** (1995), 125–126.

[10] A. Lomtatidze and S. Mukhigulashvili, On a two-point boundary value problem for second order functional differential equations, I. *Mem. Differential Equations Math. Phys.* **10** (1997), 125–128.

[11] A. Lomtatidze and S. Mukhigulashvili, On a Two-Point Boundary Value Problem for Second Order Functional Differential Equations, II. *Mem. Differential Equations Math. Phys.* **10** (1997), 150–152.

[12] J. J. Nieto and J. Pimentel, Positive solutions of a fractional thermostat model. *Bound. Value Probl.* **2013**, 2013:5, 11 pp.

[13] C. Shen, H. Zhou and L. Yang, Existence of positive solutions of a nonlinear differential equation for a thermostat model. *Math. Methods Appl. Sci.* **41** (2018), no. 16, 6145–6154.

[14] J. Šremr, Solvability conditions of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators. *Math. Bohem.* **132** (2007), no. 3, 263–295.

[15] J. R. L. Webb, Optimal constants in a nonlocal boundary value problem. *Nonlinear Anal.* **63** (2005), no. 5-7, 672–685.

[16] J. R. L. Webb, Existence of positive solutions for a thermostat model. *Nonlinear Anal. Real World Appl.* **13** (2012), no. 2, 923–938.

[17] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach. *J. London Math. Soc. (2)* **74** (2006), no. 3, 673–693.

[18] L. Yang, H. Zhou and C. Shen, Multiple non-negative solutions of a boundary value problem of the fractional thermostat model. *J. Nonlinear Funct. Anal.* **2019** (2019), Article ID 41, 12 pp.