On the Anti-Perron Effect in the Class of Linear Exponentially Decaying Perturbations

E. A. Barabanov

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: bar@im.bas-net.by

V. V. Bykov, A. V. Ravcheev

Lomonosov Moscow State University, Moscow, Russia
E-mails: vvbykov@gmail.com; rav4eev@yandex.ru

For a given positive integer n, we denote by $\widetilde{\mathcal{M}}^n$ the set of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with piecewise continuous functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, and by $\mathcal{M}^n$ its subset consisting of systems with coefficients bounded on the time half-axis $\mathbb{R}_+$.

The Lyapunov exponents of system (1) are denoted by $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$.

One of the key questions in the theory of Lyapunov exponents is the following: how are the Lyapunov exponents of the solutions of the perturbed system

$$\dot{x} = A(t)x + f(t, x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{2}$$

where the perturbation f is "small" in one sense or another, related to the Lyapunov exponents of the original system?

When investigating this question, it is natural to distinguish the case when the perturbation f is of higher order with respect to x, i.e., satisfies the condition

$$\lim_{x\to 0} \sup_{t\in\mathbb{R}_+} \frac{\|f(t, x)\|}{\|x\|^m} = 0 \tag{3}$$

for some $m > 1$, and the case when the perturbation $f(t, x)$ is linear in x, i.e., $f(t, x) = Q(t)x$, where the function $Q\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is "small" in some sense.

O. Perron constructed [6] an example of a two-dimensional system (1) with both Lyapunov exponents negative, and of its nonlinear perturbation f satisfying condition (3) for $m = 2$, such that all solutions of the perturbed system (2), except for a certain one-parameter family of them, have positive characteristic exponents.

In [7], the author introduced a two-dimensional system (1) with Lyapunov exponents zero and $a > 0$, along with an exponentially decaying linear perturbation Q, resulting in Lyapunov exponents a and $\gamma > a$ for the perturbed system. Thus, in this example both Lyapunov exponents jump upward under the perturbation.

Perron's examples [6,7] served as a starting point for numerous studies on the influence of various classes of linear and nonlinear perturbations on the Lyapunov exponents of solutions to systems from $\mathcal{M}^n$, and the results obtained in this direction constitute a substantial part of the modern theory of Lyapunov exponents. The effect of changing the values of the Lyapunov exponents of a system from $\mathcal{M}^n$ under certain "small" perturbations was called the Perron effect in the monograph [5, Chapter 4]. Subsequently, this term — the Perron effect — came to be used only for the situation (which was considered by O. Perron) where perturbations do not decrease the exponents of the solutions of the original system, while the opposite effect came to be called the anti-Perron effect [2–4].

In [1], families of parametrically perturbed systems are considered:

$$\dot{x} = A(t)x + Q(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \ \ \mu \in M,$$

where $A \in \mathcal{M}^n$, M is a metric space, $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is a continuous function satisfying the condition

$$\overline{\lim_{t\to+\infty}} \sup_{\mu\in M} \frac{1}{t} \ln \|Q(t,\mu)\| < 0.$$

For the class of pairs consisting of the spectrum of the original system and the spectrum of the perturbed system, under the condition that all exponents of the original system do not exceed the corresponding exponents of the perturbed one, a complete description from the viewpoint of descriptive set theory is obtained.

A similar question for parametrically perturbed systems A with unbounded coefficients is investigated in [8]. There, in particular, it is established that the class under consideration does not change if we require the perturbation Q to decay faster than any exponential.

Thus, the papers [1, 8] contain a complete description of the linear parametric Perron effect for systems from the spaces $\mathcal{M}^n$ and $\widetilde{\mathcal{M}}^n$, respectively.

In [2], which focuses on the anti-Perron effect, the following question is posed: is it true that if $\lambda_1(A) > 0$, then $\lambda_n(A+Q) > 0$ for any $A, Q \in \mathcal{M}^n$, where Q decays exponentially at infinity, i.e., for some $C > 0$ and $\sigma > 0$ satisfies the estimate

$$\|Q(t)\| \le Ce^{-\sigma t}, \ \ t \in \mathbb{R}_+? \tag{4}$$

We have obtained a negative answer to this question. Applying the method of [1] yields the following result.

Theorem 1. *For any numbers $n \ge 2$, $\sigma > 0$ and n-tuples $\alpha_1 \le \dots \le \alpha_n$ and $\beta_1 \le \dots \le \beta_n$, satisfying the condition $\alpha_i \ge \beta_i$, $i = \overline{1,n}$, there exist systems $A, Q \in \mathcal{M}^n$ such that*

$$\lambda_i(A) = \alpha_i, \quad \lambda_i(A+Q) = \beta_i, \ \ i = \overline{1,n},$$

and for some $C > 0$ inequality (4) holds.

Remark 1. The stated theorem strengthens the result of [3], where a similar result was established in the case $\alpha_1 > 0$, $\beta_n < 0$ and $Q(t) \to 0$ as $t \to +\infty$.

If we abandon the condition of bounded coefficients, then the statement of Theorem 1 can be strengthened as follows.

Theorem 2. *For any $n \ge 2$ and n-tuples $\alpha_1 \le \dots \le \alpha_n$ and $\beta_1 \le \dots \le \beta_n$, satisfying the condition $\alpha_i \ge \beta_i$, $i = \overline{1,n}$, and an arbitrary function $\Theta \in C(\mathbb{R}_+, \mathbb{R})$, there exist systems $B \in \widetilde{\mathcal{M}}^n$ and $Q \in \mathcal{M}^n$ such that*

$$\lambda_i(B) = \alpha_i, \quad \lambda_i(B+Q) = \beta_i, \ \ i = \overline{1,n},$$

and for some $C > 0$ the following holds:

$$\|Q(t)\| \le C \exp(-\Theta(t)t), \ \ t \in \mathbb{R}_+.$$

References

[1] E. A. Barabanov and V. V. Bykov, Description of the linear Perron effect under parametric perturbations exponentially vanishing at infinity. (Russian) *Tr. Inst. Mat. Mekh.* **25** (2019), no. 4, 31–43.

[2] N. A. Izobov and A. V. Il'in, On the existence of linear differential systems with all positive characteristic exponents of the first approximation and with exponentially decaying perturbations and solutions. (Russian) *Differ. Uravn.* **57** (2021), no. 11, 1450–1457; translation in *Differ. Equ.* **57** (2021), no. 11, 1426–1433.

[3] N. A. Izobov and A. V. Il'in, Linear version of the anti-Perron effect of change of positive characteristic exponents to negative ones. (Russian) *Differ. Uravn.* **58** (2022), no. 11, 1443–1452; translation in *Differ. Equ.* **58** (2022), no. 11, 1439–1449.

[4] N. A. Izobov and A. V. Il'in, Existence of an anti-Perron effect of change of positive exponents of the linear approximation system to negative ones under perturbations of a higher order of smallness. (Russian) *Differ. Uravn.* **59** (2023), no. 12, 1599–1605; translation in *Differ. Equ.* **59** (2023), no. 12, 1591–1597.

[5] G. A. Leonov, Chaotic dynamics and classical theory of motion stability. *NITs RKhD, Izhevsk, Moscow*, 2006.

[6] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen. (German) *Math. Z.* **32** (1930), no. 1, 703–728.

[7] O. Perron, Die Ordnungszahlen linearer Differentialgleichungssysteme. (German) *Math. Z.* **31** (1930), no. 1, 748–766.

[8] A. V. Ravcheev, Description of a linear Perron effect under parametric perturbations of a linear differential system with unbounded coefficients. (Russian) *Differ. Uravn.* **57** (2021), no. 11, 1464–1473; translation in *Differ. Equ. 57* (2021), no. 11, 1441–1450.