On Qualitative Properties of Minimizing Function in Smooth Parabolic Control Problem

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1 Introduction

Let us consider the parabolic problem

$$u_t = (a(x)u_x)_x + b(x)u_x + h(x)u, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ 0 < t < T, \tag{1.2}$$

$$u(x,0) = 0, \ \ 0 < x < 1, \tag{1.3}$$

here the real functions a, b and h are smooth in $\overline{Q}_T$, $0 < a_0 \le a(x) \le a_1 < \infty$.

Let us pose the following extremum problem: by controlling the temperature φ at the left end of the segment (the function ψ are assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z \in W_2^1(0,T)$ over the entire time interval $(0,T)$ (pointwise observation). This problem arises in the mathematical modelling of climate control in industrial greenhouses [5,6]. Extremum problems for parabolic equations were considered in various papers [14,16–19], but as usual, there were problems with final or distributed observation. We also use quite different methods of investigation.

Now, continuing the research in [1,3,4,7–13], we consider some special quality functional, which is in demand in applications, providing, among other things, uniform proximity of the solution and the objective function, implemented by the norm in the space $W_2^1(0,T)$. Since in applied problems the control and observation time T is sufficiently large, the influence of the initial function is relatively small and can be neglected, setting the initial function equal to zero.

As well as in [15, p. 6], we denote the Banach space

$$V_2^{1,0}(Q_T) = \big\{u \in W_2^{1,0}(Q_T), \ u(\,\cdot\,,t) \in C([0,T] \to L_2(0,1))\big\},$$

where $W_2^{1,0}(Q_T)$ is the Sobolev space with the norm

$$\|u\|^2_{W_2^{1,0}(Q_T)} = \int\limits_{Q_T} (u_x^2 + u^2)\,dx\,dt,$$

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(\,\cdot\,,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}.$$

Let $\widetilde{W}_2^1(Q_T)$ be set of all functions $\eta \in W_2^1(Q_T)$ satisfying the conditions $\eta(\,\cdot\,,T)=0$, $\eta(0,\,\cdot\,)=0$. Moreover, we define the spaces

$$\widehat{W}_{2,0}^k(0,T) = \big\{y \in W_2^k(0,T),\ y(0)=0\big\}$$

with the norm $\|\cdot\|_{W_2^k(0,T)}$. The space $W_k^2(0,T)$ is the Sobolev space with the norm

$$\|y\|^2_{W_2^k(0,T)} = \int\limits_0^T \Big(\sum_{j=0}^k \big(y^{(k)}(t)\big)^2\Big)\,dt.$$

Definition. A function $u \in V_2^{1,0}(Q_T)$ satisfying the condition $u(0,t)=\varphi(t)$ and the equality

$$\int\limits_{Q_T} (a(x)u_x\eta_x - b(x)u_x\eta - h(x)u\eta - u\eta_t)\,dx\,dt = \int\limits_0^T a(1,t)\psi(t)\eta(1,t)\,dt$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$ is called a weak solution to problem (1.1)–(1.3).

2 Main Results

Theorem 2.1 ([9])**.** *If $\varphi,\psi \in \widehat{W}_{2,0}^2(0,T)$, then problem* (1.1)–(1.3) *has a unique weak solution $u \in V_2^{1,0}(Q_T)$ with $u_t \in V_2^{1,0}(Q_T)$, and the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} + \|u_t\|_{V_2^{1,0}(Q_T)} \le C_1\big(\|\varphi\|_{W_2^2(0,T)} + \|\psi\|_{W_2^2(0,T)}\big)$$

holds with some constant C_1, independent of φ and ψ.

Denote by $\Phi \subset \widehat{W}_{2,0}^2(0,T)$ the nonempty set of control functions φ and let $Z \subset \widehat{W}_{2,0}^1(0,T)$ be the nonempty set of objective functions z. Consider the functional

$$\mathfrak{J}[z,\varphi] = \|u_\varphi(x_0,t) - z(t)\|^2_{W_2^1(0,T)},\quad \varphi\in\Phi,\quad z\in Z,$$

where u_φ is the solution to problem (1.1)–(1.3) with the given control function φ. Considering the function z to be fixed, we denote

$$\mathfrak{m}[z,\Phi] = \inf_{\varphi\in\Phi} \mathfrak{J}[z,\varphi] \tag{2.1}$$

to pose the minimization problem.

Theorem 2.2 ([9])**.** *If the set Φ is closed, convex and bounded, then for any $z \in Z$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$\mathfrak{m}[z,\Phi] = \mathfrak{J}[z,\varphi_0].$$

We study qualitative properties of the minimizer φ_0 as an element of the set Φ. In the following theorems we will assume that Φ is closed, convex and bounded.

Theorem 2.3. *If $\mathfrak{m}[z,\Phi] > 0$, then $\varphi_0 \in \partial\Phi$.*

Theorem 2.4. *If $\Phi_2 \subset \operatorname{Int}\Phi_1$ and $\mathfrak{m}[z,\Phi_1] > 0$, then $\mathfrak{m}[z,\Phi_1] < \mathfrak{m}[z,\Phi_2]$.*

3 Proofs

At first let us prove Theorem 2.3.

Proof. Let us suppose that $\varphi_0 \in \Phi$, $\mathfrak{J}[z,\varphi_0] = \mathfrak{m}[z,\Phi]$ (see Theorem 2.2),

$$\mathfrak{J}[z,\varphi_0] > 0, \tag{3.1}$$

and the relation $\varphi_0 \in \partial\Phi$ is not true. Then

$$\varphi_0 \in \operatorname{Int}\Phi. \tag{3.2}$$

It follows from (3.1) and the dense controllability of problem (1.1)–(1.3), (2.1) from $\widehat{W}^2_{2,0}(0,T)$ to $\widehat{W}^1_{2,0}(0,T)$ [9, Theorem 2.3] that for any $z \in \widehat{W}^1_{2,0}(0,T)$ we have the equality $\mathfrak{m}[z,\widehat{W}^2_{2,0}(0,T)] = 0$. Therefore, there exists a function $\varphi_1 \in W^1_2(0,T)$ such that

$$\mathfrak{J}[z,\varphi_1] < \frac{\mathfrak{m}[z,\Phi]}{4}\,.$$

Let $\varphi_2 = (1-\alpha)\varphi_0 + \alpha\varphi_1$, $0 \le \alpha \le 1$. By (3.2), for some $\alpha_0 \in (0,1)$ we have $\varphi_2 \in \Phi$. Now, we obtain

$$\begin{aligned}\sqrt{\mathfrak{J}[z,\varphi_2]} &= \|u_{\varphi_2}(x_0,t) - z(t)\|_{W^1_2(0,T)} = \big\|(1-\alpha_0)u_{\varphi_0}(x_0,t) + \alpha_0 u_{\varphi_1}(x_0,t) - z(t)\big\|_{W^1_2(0,T)}\\ &\le (1-\alpha_0)\|u_{\varphi_0}(x_0,t) - z(t)\|_{W^1_2(0,T)} + \alpha_0\|u_{\varphi_1}(x_0,t) - z(t)\|_{W^1_2(0,T)}\\ &< (1-\alpha_0)\sqrt{\mathfrak{m}[z,\Phi]} + \frac{\alpha_0}{2}\sqrt{\mathfrak{m}[z,\Phi]} = \Big(1 - \frac{\alpha_0}{2}\Big)\sqrt{\mathfrak{m}[z,\Phi]}\,.\end{aligned}$$

Hence,

$$\mathfrak{J}[z,\varphi_2] < \Big(1 - \frac{\alpha_0}{2}\Big)^2 \mathfrak{m}[z,\Phi] < \mathfrak{m}[z,\Phi],$$

and φ_0 is not a minimizer of $\mathfrak{J}[z,\varphi]$ on Φ. This contradiction proves Theorem 2.3. □

Now let us prove Theorem 2.4.

Proof. It follows from the inclusion $\Phi_2 \subset \Phi_1$ that $\mathfrak{m}[z,\Phi_1] \le \mathfrak{m}[z,\Phi_2]$. Suppose,

$$\mathfrak{m}[z,\Phi_1] = \mathfrak{m}[z,\Phi_2]. \tag{3.3}$$

By Theorem 2.2 and (3.3) we have the unique minimizer $\varphi_0 \in \Phi_1 \cap \Phi_2$ such that $\mathfrak{J}[z,\varphi_0] = \mathfrak{m}[z,\Phi_1] = \mathfrak{m}[z,\Phi_2]$. Additionally, it follows from Theorem 2.2 that

$$\varphi_0 \in \partial\Phi_1 \cap \partial\Phi_2.$$

But the relation $\Phi_2 \subset \operatorname{Int}\Phi_1$ means $\partial\Phi_1 \cap \partial\Phi_2 = \varnothing$. □

Acknowledgements

The reported study was partially supported by RSF, project # 25-11-00133.

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