Geometric Regularization of Fractional Differential Equations

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1 Introduction

Fractional differential equations (FDEs) have become an indispensable tool for modeling nonlocal phenomena in viscoelasticity, anomalous diffusion, and control theory [3, 6]. While classical operators (Riemann–Liouville, Caputo) often introduce singularities at the origin, the recently introduced Atangana-Baleanu derivative in the Caputo sense (ABC) utilizes a non-singular Mittag–Leffler kernel, making it particularly suitable for initial value problems [1].

The classic theory of fractional calculus has been widely researched in recent years [4, 7]. However, existing analytical methods – such as Picard iterations, Adomian decomposition or HAM – are primarily developed for finite intervals $[0, T]$. The extension of these methods to the unbounded domain $[0, \infty)$ presents significant challenges: integral operators accumulate "memory" divergence, and standard norms fail to control the asymptotic behavior of solutions.

In this paper, we fill the gap between classical analysis and geometric topology. We argue that the correct setting for global FDE analysis is not merely a Banach space of functions, but a *compact manifold with boundary.* By employing weighted Sobolev spaces, we effectively perform a geometric compactification of the domain. This allows us to apply techniques from the geometric theory of differential equations [5, 8] to establish global existence, uniqueness, and asymptotic stability of solutions.

2 Analytic approach: weighted Sobolev spaces

2.1 Problem statement

Consider the nonlinear Cauchy problem for the ABC-fractional derivative of order $\alpha \in (0, 1)$:

$$ {}^{ABC}D^{\alpha}_{0+}u(x) = f(x, u(x)), \ \ x > 0, \quad u(0) = u_0, \tag{2.1}$$

where $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function. The ABC-fractional derivative is defined as:

$$ {}^{ABC}D^{\alpha}u(x) = \frac{\beta(\alpha)}{1-\alpha} \int\limits_0^x u'(s) E_\alpha\Big(-\frac{\alpha}{1-\alpha}\,(x-s)^\alpha\Big)\, ds,$$

where E_α is the Mittag–Leffler function and $\beta(\alpha)$ is a normalization constant.

To handle the non-compactness of the domain, we introduce the weighted Sobolev space $H^1_\lambda(\mathbb{R}_+)$ characterized by the exponential weight $w(x) = e^{\lambda x}$ for some parameter $\lambda > 0$. The norm is defined as:

$$\|u\|_{H^1_\lambda} = \bigg(\int\limits_0^\infty e^{2\lambda x}\big(|u(x)|^2 + |u'(x)|^2\big)\, dx\bigg)^{1/2}. \tag{2.2}$$

This norm imposes a strict decay condition at infinity, ensuring the compactness of the fractional integration operator.

2.2 Main qualitative results

We generalize the convergence results obtained for finite intervals in [2] to the unbounded case.

Theorem 2.1. *Let $f(x,u)$ be continuous and satisfy the weighted Lipschitz condition:*

$$\|f(\,\cdot\,,u) - f(\,\cdot\,,v)\|_{L^2_\lambda} \leq L\|u-v\|_{L^2_\lambda}, \quad \forall\, u,v \in H^1_\lambda(\mathbb{R}_+),$$

where L is the Lipschitz constant. Then, for any $\alpha \in (0,1)$, there exists a critical weight parameter $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the problem (2.1) admits a unique solution $u \in H^1_\lambda(\mathbb{R}_+)$. Moreover, the Picard iteration sequence $\{u_n\}$ defined by

$$u_{n+1}(x) = u_0 + {}^{AB}I^{\alpha}_{0+} f(x, u_n(x))$$

converges in the topology of $H^1_\lambda(\mathbb{R}_+)$ with rate controlled by

$$\|u_{n+1} - u_n\|_{H^1_\lambda} \leq K^n \|u_1 - u_0\|_{H^1_\lambda}, \;\; K = K(\alpha, L, \lambda) < 1. \tag{2.3}$$

A crucial requirement for the future geometric interpretation is the continuous dependence of the solution on the parameters of the equation.

Theorem 2.2. *Let $u(x;\alpha,u_0)$ denote the solution to (2.1). The mapping*

$$\Psi : (0,1) \times \mathbb{R} \to H^1_\lambda(\mathbb{R}_+), \;\; (\alpha, u_0) \mapsto u(\,\cdot\,;\alpha,u_0)$$

is continuous. Specifically, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|\alpha_1 - \alpha_2| < \delta$ and $|u_{0,1} - u_{0,2}| < \delta$, then

$$\big\|u(\,\cdot\,;\alpha_1,u_{0,1}) - u(\,\cdot\,;\alpha_2,u_{0,2})\big\|_{H^1_\lambda} < \epsilon.$$

This theorem ensures that the solution family forms a smooth family of curves in the functional space, parameterized by (α, u_0). It is this continuous structure that allows geometric interpretation in terms of fiber bundles.

3 Geometric approach: explicit compactification

The introduction of the weight $e^{\lambda x}$ induces a natural coordinate transformation that maps the unbounded ray $[0,\infty)$ to a compact manifold with boundary.

3.1 The compactified manifold

Consider the diffeomorphism $\Phi : [0,\infty) \to (0,1]$ defined by

$$\xi = e^{-\lambda x}.$$

This map compactifies the physical domain into the manifold with boundary $M \cong [0,1]$, where $\xi = 1$ corresponds to $x = 0$, $\xi = 0$ corresponds to $x = +\infty$.

3.2 Transformation of the differential operator

The first-order differential operator transforms according to the chain rule:

$$\frac{d}{dx} = \frac{d\xi}{dx}\,\frac{d}{d\xi} = -\lambda\xi\,\frac{d}{d\xi}\,.$$

This transformation reveals that the standard derivative on the half-line becomes a *b-vector field* (or *totally characteristic vector field*) in the sense of Melrose's b-calculus [5]. The vector field $V = -\lambda\xi\partial_\xi$ vanishes at the boundary $\xi = 0$, indicating that the dynamics "slows down" and becomes singular at the horizon.

3.3 Pseudo-differential structure

Under compactification, the ABC operator transforms into an element of the b-calculus algebra $\Psi_b^0(M)$ [5]. The operator becomes degenerate at the boundary:

$$\mathcal{L}^\alpha \widetilde{u}(\xi) = \int_\xi^1 K(\xi,\tau)\widetilde{u}(\tau)\frac{d\tau}{\tau} - f(\widetilde{u}).$$

This allows us to define the *indicial operator* $I(\mathcal{L}^\alpha)$ by freezing coefficients at $\xi = 0$. The roots of the corresponding indicial polynomial determine the exact asymptotic decay rates $u(x) \sim e^{-\lambda x}x^{-\gamma}$.

3.4 Transformation of the ABC-operator

We derive the explicit form of the Atangana-Baleanu integral operator in the compactified coordinate ξ. Recall the fractional integration operator:

$${}^{AB}I^\alpha u(x) = \frac{1-\alpha}{\beta(\alpha)}u(x) + \frac{\alpha}{\beta(\alpha)\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u(t)\,dt.$$

Under the substitution $t = -\frac{1}{\lambda}\ln\tau$ and $x = -\frac{1}{\lambda}\ln\xi$, the kernel transforms as:

$$(x-t)^{\alpha-1} = \lambda^{1-\alpha}\Big(\ln\frac{\tau}{\xi}\Big)^{\alpha-1},$$

and the differential element becomes $dt = -\frac{1}{\lambda\tau}\,d\tau$. Thus, the integral part takes the form:

$$\mathcal{I}^\alpha \widetilde{u}(\xi) = \frac{\alpha\lambda^{-\alpha}}{\beta(\alpha)\Gamma(\alpha)}\int_\xi^1 \Big(\ln\frac{\tau}{\xi}\Big)^{\alpha-1}\widetilde{u}(\tau)\,\frac{d\tau}{\tau}\,, \tag{3.1}$$

where

$$\widetilde{u}(\xi) = u\Big(-\frac{1}{\lambda}\ln\xi\Big).$$

Remark 3.1. The kernel $K(\xi,\tau) = (\ln\tau - \ln\xi)^{\alpha-1}\tau^{-1}$ in (3.1) exhibits a logarithmic singularity at the diagonal $\tau = \xi$, which is characteristic of pseudo-differential operators in the b-calculus. The factor τ^{-1} introduces an additional singularity at the boundary $\tau \to 0^+$, which is precisely compensated by the exponential decay condition inherent in the weighted space $H^1_\lambda(\mathbb{R}_+)$.

3.5 Geometric asymptotics and transversality

The weighted norm condition $u \in H^1_\lambda(\mathbb{R}_+)$ imposes a specific boundary behavior in the compactified coordinate.

Proposition 3.1 (Boundary condition at ∞). *A solution $u(x)$ belongs to $H^1_\lambda(\mathbb{R}_+)$ if and only if the corresponding function $\widetilde{u}(\xi) = u(-\frac{1}{\lambda}\ln\xi)$ on M satisfies the boundary condition:*

$$\lim_{\xi\to 0^+}\xi^{-1}|\widetilde{u}(\xi)| = 0, \quad \lim_{\xi\to 0^+}\xi^{-1}|\widetilde{u}'(\xi)| = 0.$$

Geometrically, this means the integral curve intersects the boundary divisor D_∞ transversally in the weighted tangent bundle structure induced by the singular metric $g = \xi^{-2}\,d\xi^2$.

This result provides a geometric criterion for selecting physical solutions: they are exactly those integral curves of the vector field generated by (3.1) that satisfy the exponential decay imposed by the weight on the boundary.

4 Connection between analytical and geometric approaches

The connection between the analytical and geometric approaches is established through the weight parameter λ.

In the **analytical sense**, λ is a regularization parameter ensuring the convergence of the norm (2.2) and controlling the spectral radius in Theorem 2.1 via equation (2.3).

In the **geometric sense**, λ defines both:

- the chart of the compactification (the scale of the logarithmic map $\xi = e^{-\lambda x}$);
- the singular metric structure $g = \xi^{-2}\, d\xi^2$ on the compactified manifold.

The convergence of the Picard method in H^1_λ implies that the solution trajectory approaches the boundary divisor D_∞ monotonically without oscillation. This is precisely the transversality condition of Proposition 3.1. The geometric structure thus allows us to classify solutions based on their asymptotic behavior by studying the vector field flow near $\xi = 0$ in the singular metric.

5 Conclusion

A combined analytical and geometric framework for solving FDEs on unbounded domains was presented. By moving to weighted Sobolev spaces, we ensure the well-posedness of problem statements, while the geometric interpretation provides insight into the asymptotic nature of solutions.

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