On Lyapunov and Krasovskii Stability/Instability of Pendulum Equation with Non-Constant Coefficients

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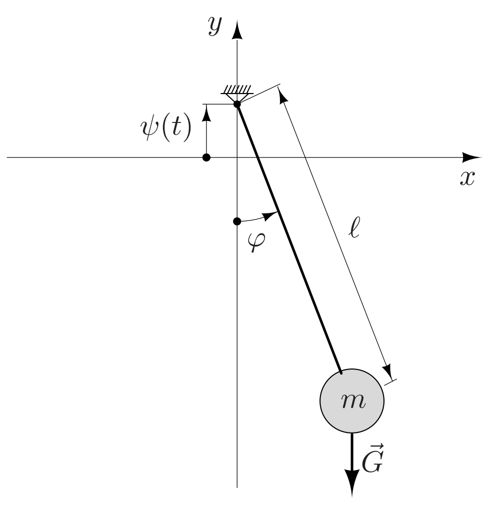
In this note, which is based on the paper [12], we discuss stability of two geometrically distinct equilibria $u_* = 0$ and $u^* = \pi$ of the equation

$$u'' + q(t)u' + r(t)\sin u = 0, \tag{0.1}$$

where $q, r : \mathbb{R} \to \mathbb{R}$ are continuous T-periodic functions. This equation can be understood as a generalisation of the equation of motion of free damped pendulum consisting of a point mass m attached to the massless rod of the length ℓ, whose suspension point oscillates vertically. The pendulum is a system with one degree of freedom, described by the coordinate φ, and its equation of motion is of the form



$$\varphi'' + \frac{b}{m}\,\varphi' + \Big(\frac{\mathrm{g}}{\ell} + \frac{\psi''(t)}{\ell}\Big)\sin\varphi = 0, \tag{0.2}$$

where $b \geq 0$ is the damping coefficient, g denotes the gravitational acceleration, and $\psi \in C^2(\mathbb{R})$ is a T-periodic function determining the oscillations of the suspension point.

If the suspension point is fixed, i.e., if $\psi(t) \equiv const.$ on $\mathbb{R}$, then the situation is simple, because equation (0.2) becomes autonomous (with constant coefficients). One can apply a standard technique of the dynamical systems theory to show that the lower equilibrium $\varphi_* = 0$ is stable and the the upper equilibrium $\varphi^* = \pi$ is unstable.

If $\psi(t) \not\equiv const.$ on $\mathbb{R}$, then the situation is much more complicated. The complications are not connected with the linearizations of (0.2) at its equilibria, but with the use of stability/instability criteria. The linearizations obtained are non-autonomous and thus, verifying all the hypotheses of the stability/instability criteria is far from being trivial. It is well known that, without some additional assumptions, stability/instability of the linearized equation does not guarantee stability/instability of the corresponding solution to the original non-linear equation (see, e. g., [10, Section 3]).

1 History of the problem

The history of the problem goes back to the beginning of the 20th century to works of Stephenson (see, e.g., the paper [2] for overview and historical background). Stephenson considered "small" deflections from the upper equilibrium of the pendulum with harmonic oscillations of the suspension point. Therefore, he studied stability and the approximate solutions to the linear second-order differential equations with a non-constant coefficient

$$\theta'' \pm \Big(\frac{\mathrm{g}}{\ell} - \frac{A}{\ell}\,\Omega^2 \sin(\Omega t)\Big)\theta = 0,$$

which are known as Mathieu equations. In the first half of the 20th century, these problems were studied by many mathematicians such as Van der Pol, Strutt, Hirsh, Erdleyi, Lowenstern, etc. However, the works of Bogolyubov (published in 1950) and Kapitza (published in 1951) became really important for the field of non-linear dynamics.

In the paper [3], N. N. Bogolyubov applied the method of averaging to the study of stability of the upper equilibrium $\varphi^* = \pi$ of the pendulum with a harmonically oscillating suspension point, namely, of the equation

$$\varphi'' + \frac{b}{m}\,\varphi' + \Big(\frac{\mathrm{g}}{\ell} - \frac{A}{\ell}\,\Omega^2 \sin(\Omega t)\Big) \sin\varphi = 0, \tag{1.1}$$

where $A > 0$ is the amplitude of oscillations and $\Omega > 0$ is their angular frequency. Bogolyubov shows in [3] that if

$$\frac{A}{\ell} \ll 1 \;\text{ and }\; \Omega > \sqrt{2}\,\frac{\ell}{A}\sqrt{\frac{\mathrm{g}}{\ell}}\,, \tag{1.2}$$

then the equilibrium $\varphi^* = \pi$ of the pendulum equation (1.1) is stable. Moreover, it follows from his results that if (1.2) is satisfied, then the lower equilibrium $\varphi_* = 0$ of the pendulum equation is also stable and equation (1.1) possesses approximate "quasistatic solutions"

$$\varphi_{1,2}(t) \approx \alpha_{1,2} - \frac{A}{\ell}\,\sin(\alpha_{1,2})\sin(\Omega t),$$

where $\alpha_{1,2} = \pm\arccos\big(-\frac{2\mathrm{g}\ell^2}{A^2\Omega^2}\big)$.

P. L. Kapitza studied in [7] stability of the upper equilibrium of the undamped pendulum with a harmonically oscillating suspension point. His approach is based on the physical reasoning together with the method of averaging, whereas he derived the same condition for stability of the upper equilibrium of the pendulum as Bogolyubov. Denote by θ the angle between the rod of the pendulum and the positive semi-axis of the y-axis (tj. $\theta = \pi - \varphi$). Kapitza claims in [7] (see also [8]) that if (1.2) is satisfied, then the upper equilibrium $\theta^* = 0$ and the lower equilibrium $\theta_* = \pi$ of the pendulum are stable and, moreover, there are two the so-called "quasistatic balances" of the pendulum given by the formula

$$\theta_{1,2}(t) \approx \lambda_{1,2} + \frac{A}{\ell}\,\sin(\lambda_{1,2})\sin(\Omega t),$$

where $\lambda_{1,2} = \pm\arccos\big(\frac{2\mathrm{g}\ell}{A^2\Omega^2}\big)$.

2 Basic Notions

Consider the second-order differential equation

$$u'' = h(t, u, u'), \tag{2.1}$$

where $h : [a, \infty[\times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

Definition 2.1. A point $\widetilde{u} \in \mathbb{R}$ is called an equilibrium of equation (2.1) if $h(t, \widetilde{u}, 0) \equiv 0$ on $[a, \infty[$.

As usual, we reduce the question of stability/instability of the equilibria of (2.1) into the question of stability/instability of constant solutions to the planar system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= h(t, y_1, y_2). \end{aligned} \tag{2.2}$$

Definition 2.2. The solution $y_0 : [a, \infty[\to \mathbb{R}^2$ to system (2.2) is said to be:

(1) Lyapunov stable if for any $\varepsilon > 0$ and any $t_0 \geq a$, there exists $\delta(\varepsilon, t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(\varepsilon, t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\|y(t) - y_0(t)\| \leq \varepsilon \ \text{ for } \ t \geq t_0.$$

Otherwise, it is said to be Lyapunov unstable.

(2) Krasovskii stable if for any $t_0 \geq a$, there exist $\delta(t_0) > 0$ and $R(t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\|y(t) - y_0(t)\| \leq R(t_0)\|y(t_0) - y_0(t_0)\| \ \text{ for } \ t \geq t_0.$$

Otherwise, it is said to be Krasovskii unstable.

(3) Attractive if for any $t_0 \geq a$, there exists $\delta(t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\lim_{t \to \infty} \|y(t) - y_0(t)\| = 0.$$

(4) Asymptotically stable, if it is both stable and attractive.

It is clear that if the solution $y_0 : [a, \infty[\to \mathbb{R}^n$ to system (2.2) is Krasovskii stable, then it is Lyapunov stable as well. The converse implication does not hold in general.

Definition 2.3. A solution $u_0 : [a, \infty[\to \mathbb{R}$ to equation (2.1) is said to be Lyapunov (Krasovskii) stable (resp. attractive) if the corresponding solution $y_0 = (u_0, u_0')$ to system (2.2) is Lyapunov (Krasovskii) stable (resp. attractive).

By Lyapunov (Krasovskii) stability (resp. attractivity) of an equilibrium $\widetilde{u}$ of equation (2.1) we understand Lyapunov (Krasovskii) stability (resp. attractivity) of the corresponding constant solution $u_0(t) := \widetilde{u}$ to equation (2.1).

Since the coefficients r, q in pendulum-like equation (0.1) are supposed to be T-periodic, the linearizations of (0.1) along its equilibria are second-order ODEs with periodic coefficients Therefore, it is not surprising that, in the proofs of stability/instability of the equilibria of (0.1), we apply Floquet theory for the linear equation

$$u'' + q(t)u' + p(t)u = 0 \tag{2.3}$$

in which a T-periodic coefficient $p : \mathbb{R} \to \mathbb{R}$ depends on the given equilibrium. It is well-known that stability criteria for equation (2.3) can be formulated in terms of Floquet multipliers but, unfortunately, the those criteria are not effective. In our results, we formulate stability criteria for the equilibria of (0.1) in terms of the classes $\mathbb{V}^+(T)$ and $\mathbb{V}^-(T)$ introduced for the the linear periodic problem

$$u'' + q(t)u' + p(t)u = 0; \quad u(0) = u(T), \ \ u'(0) = u'(T). \tag{2.4}$$

Definition 2.4. We say that a pair of functions $(p, q) \in L([0,T]) \times L([0,T])$ belongs to the set $\mathbb{V}^+(T)$ (resp. $\mathbb{V}^-(T)$) if for any function $v : [0,T] \to \mathbb{R}$, which is absolutely continuous together with its first derivative and satisfies

$$v''(t) + q(t)v'(t) + p(t)v(t) \geq 0 \ \text{ for a.e. } \ t \in [0,T], \quad v(0) = v(T), \quad v'(0) = v'(T),$$

the inequality $v(t) \geq 0$ $t \in [0,T]$ (resp. $v(t) \leq 0$ for $t \in [0,T]$).

In other words, $(p, q) \in \mathbb{V}^+(T)$ (resp. $(p, q) \in \mathbb{V}^-(T)$) if and only if Green's function of the periodic problem (2.4) exists and is positive (resp. negative). In another terminology, $(p, q) \in \mathbb{V}^+(T)$ (resp. $(p, q) \in \mathbb{V}^-(T)$) if and only if the anti-maximum principle (resp. maximum principle) holds for periodic problem (2.4).

3 Main results

We first formulate two general results for pendulum-like equation (0.1) in terms of Floquet multipliers of the linear equations

$$u'' + q(t)u' + r(t)u = 0 \tag{3.1_0}$$

and

$$u'' + q(t)u' - r(t)u = 0, \tag{3.1_π}$$

which are in fact linearizations of (0.1) at its equilibria $u_* = 0$ and $u^* = 0$, respectively.

Proposition 3.1. *Let $\varrho_1, \varrho_2 \in \mathbb{C}$ be Floquet multipliers of equation* (3.1_0) *(resp. equation* (3.1_π)*). Then:*

(1) *If $|\varrho_1| < 1$ and $|\varrho_2| < 1$, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(12) *If $|\varrho_k| > 1$ for some $k \in \{1, 2\}$, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation* (0.1) *is Krasovskii unstable.*

Remark 3.1. Since equations (3.1_0) and (3.1_π) are equations with periodic coefficients, Proposition 3.1 can be easily reformulated in terms of Lyapunov exponents as well as in terms of stability of linearized equations as follows:

(A) If $\int_0^T q(s)\,\mathrm{d}s > 0$ and the linear equation (3.1_0) (resp. equation (3.1_π)) is asymptotically Lyapunov stable, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation (0.1) is asymptotically Lyapunov stable.

(B) If $\int_0^T q(s)\,\mathrm{d}s > 0$ and the linear equation (3.1_0) (resp. equation (3.1_π)) is Lyapunov unstable, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation (0.1) is Krasovskii unstable.

One can see that asymptotic Krasovskii stability/instability of the equilibria of (0.1) is more or less completely described in terms of Floquet multipliers of their linearizations. However, it is a far more delicate question to guarantee Lyapunov instability of the equilibria of equation (0.1).

Proposition 3.2. *Let $q(t) \equiv Const.$, equation (3.1_π) be disconjugate on $\mathbb{R}$ and have a real Floquet multiplier ϱ satisfying $\varrho > 1$. Then, the equilibrium $u^* = \pi$ of equation* (0.1) *is Lyapunov unstable.*

Remark 3.2. Similarly as in Remark 3.1, Proposition 3.2 can be reformulated in terms of instability of the linearized equation:

(C) If $q(t) \equiv q_0 > 0$ and the linear equation (3.1_π) is disconjugate on $\mathbb{R}$ and Lyapunov unstable, then the equilibrium $u^* = \pi$ of equation (0.1) is Lyapunov unstable.

Now, we provide stability criteria for the equilibria of (0.1) in terms of the classes $\mathbb{V}^+(T)$ and $\mathbb{V}^-(T)$.

Theorem 3.1. *The following conclusions hold:*

(1) *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and $(r,q) \in \operatorname{Int}\mathbb{V}^+(T)$, then the equilibrium $u_* = 0$ of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(2) *If $q(t) \equiv q_0 > 0$ and $(r,q_0) \in \mathbb{V}^-(T)$, then the equilibrium $u_* = 0$ of equation* (0.1) *is Lyapunov unstable and consequently, Krasovskii unstable.*

(3) *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and $(-r,q) \in \operatorname{Int}\mathbb{V}^+(T)$, then the equilibrium $u^* = \pi$ of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(4) *If $q(t) \equiv q_0 > 0$ and $(-r,q_0) \in \mathbb{V}^-(T)$, then the equilibrium $u^* = \pi$ of equation* (0.1) *is Lyapunov unstable and consequently, Krasovskii unstable.*

Let us mention here that some effective conditions for the inclusions $(p,q) \in \mathbb{V}^+(T)$ and $(p,q) \in \mathbb{V}^-(T)$ are derived, e.g., in [1, 6, 13] (see also [4, 11] for the case of $q(t) \equiv 0$).

Theorem 3.2. *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and*

$$(r,q) \in \operatorname{Int}\mathbb{V}^+(T), \quad (-r,q) \in \operatorname{Int}\mathbb{V}^+(T),$$

then both equilibria $u_* = 0$ and $u^* = \pi$ of equation (0.1) *are asymptotically Lyapunov stable and there exists a T-periodic solution $u_{\rm per}$ to equation* (0.1) *such that*

$$0 < u_{\rm per}(t) < \pi \ \ \textit{for} \ \ t \in \mathbb{R}. \tag{3.2}$$

Remark 3.3. It seems that under the hypotheses of the previous theorem, the solution $u_{\rm per}$ should be unique. Unfortunately, we cannot prove this fact.

However, it follows from the proof of Theorem 3.2 and [5, Proposition 3.1] that if $u_{\rm per}$ in the previous theorem is a unique T-periodic solution satisfying (3.2), then it is Lyapunov unstable.

Applying the results of [9, 11, 13], we can derive from Theorem 3.1(1,3,4) the following effective criteria for equation (1.1), i.e., for pendulum equation (0.2) with $\psi(t) := A\sin(\Omega t)$.

Corollary 3.1. *Let $b > 0$ and either*

$$\Big(\frac{A}{\ell}\Big)^2\Omega^2\Big[1-\Big(\frac{\pi}{e^{\frac{2\pi A}{\ell}}-1}\Big)^2\Big] \le -\frac{g}{\ell}+\Big(\frac{b}{2m}\Big)^2 \le 0 \tag{3.3}$$

or

$$\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi), \quad 0 \le -\frac{g}{\ell}+\Big(\frac{b}{2m}\Big)^2 \le \frac{4\ln(1+\pi)}{\pi^3}\Big(\frac{A}{\ell}\Big)^2\Omega^2.$$

Then, the equilibrium $\varphi_* = 0$ of equation (1.1) *is asymptotically Lyapunov stable.*

Corollary 3.2. *Let $b > 0$ and*

$$\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi), \quad \frac{g}{\ell}+\Big(\frac{b}{2m}\Big)^2 \le \frac{4\ln(1+\pi)}{\pi^3}\Big(\frac{A}{\ell}\Big)^2\Omega^2. \tag{3.4}$$

Then, the equilibrium $\varphi^* = \pi$ of equation (1.1) *is asymptotically Lyapunov stable.*

Remark 3.4. The second inequality in (3.4) can be rewritten into the form

$$\Omega \ge \sqrt{\frac{\pi^3}{4\ln(1+\pi)}}\,\frac{\ell}{A}\sqrt{\frac{g}{\ell}+\Big(\frac{b}{2m}\Big)^2}.$$

It is clear that the requirement on Ω in Corollary 3.2 is stronger than condition (1.2) derived by Bogolyubov and Kapitza under the assumption $\frac{A}{\ell} \ll 1$. On the other hand, in their approaches, the assumption $\frac{A}{\ell} \ll 1$ is essential (Kapitza chose $\frac{A}{\ell} = 0.05$ in his example), but we require quite weaker and the explicit assumption $\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi) \approx 0.22$.

Corollary 3.3. *If $b \ge 0$ and $\Omega^2 \le \frac{g}{A}$, then the equilibrium $\varphi^* = \pi$ of equation* (1.1) *is Lyapunov unstable.*

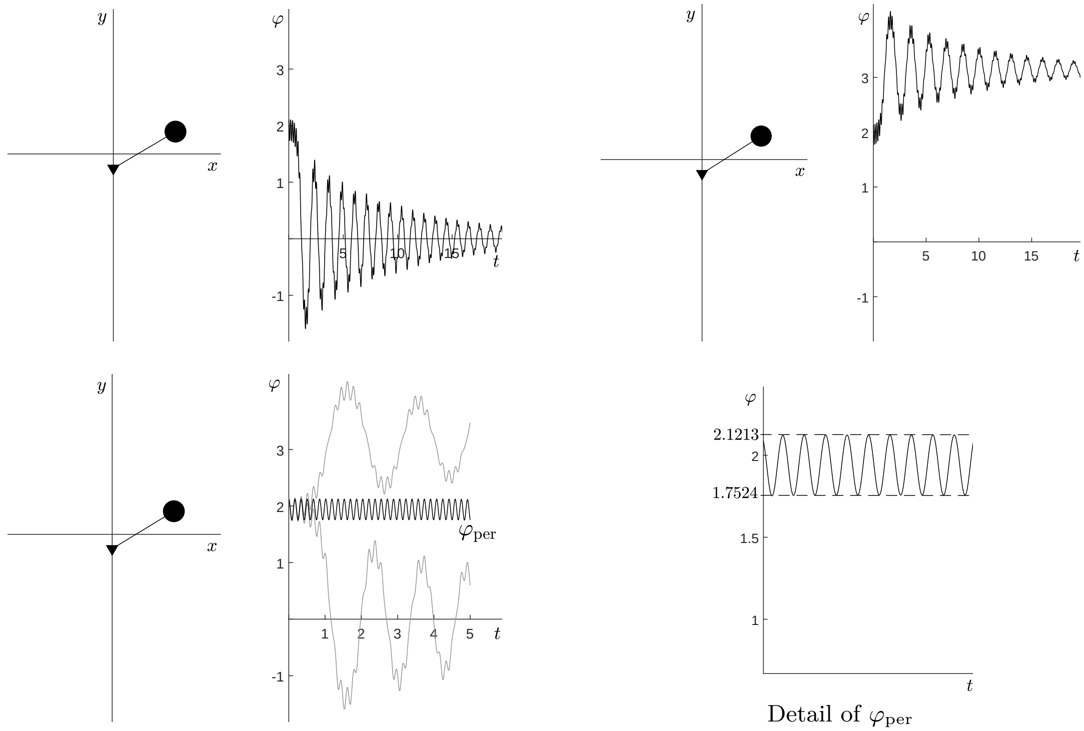
Corollary 3.4. *If $\frac{A}{\ell} < 1$ and $(\frac{A}{\ell})^2\Omega^2 \le \frac{g}{\ell}$, then there exists $b_0 > 0$ such that for any $b \in [0, b_0[$, the equilibrium $\varphi^* = \pi$ of equation* (1.1) *is Lyapunov unstable.*

Remark 3.5. Instability criterion provided in Theorem 3.13.1 cannot be applied to equation (1.1). If $q(t) := \frac{b}{m} > 0$ and $r(t) := \frac{g}{\ell} - \frac{A}{\ell}\,\Omega^2\sin(\Omega t)$, then (q, r) cannot belong to the class $\mathbb{V}^-(T)$, because the inequality $\int_0^T r(s)\,\mathrm{d}s < 0$ is a necessary condition for the validity of the inclusion $(q, r) \in \mathbb{V}^-(T)$ with $q(t) := Const$.

Corollary 3.5. *Let $b > 0$ and conditions* (3.3) *and* (3.4) *hold. Then, both equilibria $\varphi_* = 0$ and $\varphi^* = \pi$ of equation* (1.1) *are asymptotically Lyapunov stable and, moreover, there exists a $\frac{2\pi}{\Omega}$-periodic solution φ_{per} to equation* (1.1) *satisfying*

$$0 < \varphi_{\mathrm{per}}(t) < \pi \ \ \text{for} \ \ t \in \mathbb{R}.$$

The solution φ_{per} in the previous corollary corresponds to the "quasistatic solution" of Bogolyubov as well as to the "quasistatic balance" of Kapitza described in Section 1. On pictures below, there are the results of some numerical simulations showing that free damped pendulum with periodically oscillating suspension point can actually move periodically if its both lower and upper equilibria are asymptotically stable.



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