Regularly Varying Solutions of Differential Equations of the Second Order with Nonlinearities of Exponential Types

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The differential equation

$$y'' = \alpha_0 p(t) f(t, y, y'), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$ is a continuous function, $f : [a, \omega[\times \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ is continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$[1] or $]Y_i, y_i^0]$, is considered. We also suppose that the function f satisfies the conditions

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \cdot \frac{\partial f}{\partial t}(t, v_0, v_1)}{f(t, v_0, v_1)} = \gamma \ \text{ uniformly by } \ v_0 \in \Delta_{Y_0}, \ \ v_1 \in \Delta_{Y_1}, \tag{2}$$

$$\lim_{\substack{y_k \to Y_k \\ y_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(t, v_0, v_1)}{f(t, v_0, v_1)} = \sigma_k \ \text{ uniformly by } \ t \in [a, \omega[, v_j \in \Delta_{Y_j}, \ \ j \neq k, \ \ k \in \{0, 1\}. \tag{3}$$

By conditions (2), (3) the function f is in some sense close to regularly varying function by every variable.

We call the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ a regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma. \tag{4}$$

Here $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [5] that regularly varying functions have the next properties.

R_1: The function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if the next representation takes place

$$\varphi(z) = z^\sigma \theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

R_2: If the function $L : \Delta_{Y^0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

R_3: If the function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then φ is regularly varying as $z \to Y$ of index σ.

[1] As $Y_i = +\infty$ $(Y_i = -\infty)$ assume $y_i^0 > 0$ $(y_i^0 < 0)$.

R_4: For every regularly varying as $z \to Y$ function φ the property (4) takes place uniformly as $\lambda \in [c,d]$ for every segment $[c,d] \subset]0,+\infty[$.

Definition. Solution y of the equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ if it is defined on $[t_0,\omega[\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \;\; (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

For different values of parameter λ_0 the class of such solutions contains regularly slowly and rapidly varying as $t \uparrow \omega$ functions. $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) are regularly varying functions as $t\uparrow\omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R\setminus\{0,1\}$.

A lot of works (see, for example, [2,3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $f(t,y,y') \equiv \varphi_0(y)\varphi_1(y')$, where φ_0 and φ_1 are regularly varying functions. For more general case as f depends only on y and y' asymptotic properties and necessary and sufficient conditions of existence of such solutions of the equation (1) have been obtained in [1].

We need the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t-\omega & \text{as } \omega < +\infty, \end{cases}$$

$$\Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \;\; (i=0,1),$$

$$J_1(t) = \int_{A_\omega^1}^{t} \left(\alpha_0 p(\tau)|\pi_\omega(\tau)|^{\gamma+\sigma_0}\left|\frac{\lambda_0-1}{\lambda_0}\right|^{\sigma_0}\right) d\tau,$$

$$A_\omega^1 = \begin{cases} a, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma+\sigma_0}\,d\tau = +\infty, \\ \omega, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma+\sigma_0}\,d\tau < +\infty, \end{cases}$$

$$J_2(t) = \left|(1-\sigma_0-\sigma_1)\right|^{\frac{1}{1-\sigma_0-\sigma_1}} \operatorname{sign} y_1^0 \int_{B_\omega^2}^{t} |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\,d\tau,$$

$$B_\omega^2 = \begin{cases} b, & \text{if } \int_b^\omega |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\,d\tau = +\infty, \\ \omega, & \text{if } \int_b^\omega |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\,d\tau < +\infty. \end{cases}$$

The following theorem is obtained for the equation (1).

Theorem 1. *Let in the equation* (1) $\sigma_1 \neq 1$. *Then for the existence of* $P_\omega(Y_0,Y_1,\lambda_0)$*-solutions to the equation* (1) *in cases* $\lambda_0 \in R\setminus\{0,1\}$, *it is necessary and if*

$$\lambda_0 \neq \sigma_1 - 1 \;\; \textit{or} \;\; (\sigma_1-1)(\sigma_0+\sigma_1-1) > 0,$$

then also sufficient

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1) > 0, \quad \pi_\omega(t)\alpha_0 y_1^0\lambda_0(\lambda_0-1) > 0 \;\; \textit{as} \;\; t \in [a,\omega[,$$

$$\lim_{t\uparrow\omega} y_0^0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}} = Y_1,$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_2'(t)}{J_2(t)} = \frac{\lambda_0}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_1'(t)}{J_1(t)} = \frac{1-\sigma_0-\sigma_1}{\lambda_0-1}\,.$$

The next result is devoted to research $P_\omega(Y_0,Y_1,\lambda_0)$-solutions in special and most complex case $\lambda_0=0$. In this case, such decisions or their first order derivatives will be slowly changing functions as $t\uparrow\omega$, which significantly complicates the study. Therefore, consider the differential equation

$$y'' = \alpha_0 p(t)|y|^{\sigma_0}y'|^{\sigma_1}\exp\big(R\big(\big|\ln|\pi_\omega(t)yy'|\big|\big)\big), \tag{5}$$

where α_0, p are the same as in a general equation, and R is continuously differentiable, with a monotone derivative, regularly variable at infinity function return order μ, $0<\mu<1$.

Theorem 2. *Let*
$$\lim_{t\uparrow\omega} \frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)} = 0.$$
Then, for the existence of $P_\omega(Y_0,Y_1,0)$-solutions to the equation (5) for which there is a finite or infinite boundary
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)}\,,$$
the following conditions and inequalities are sufficient and sufficient
$$\lim_{t\uparrow\omega} y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0|I(t)|^{\frac{1}{1-\sigma_1}} = Y_1,$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \sigma_1-1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = 0,$$
$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0, \quad \frac{y_0^0y_1^0(1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1} > 0 \ \ as \ \ t\in\,]a,\omega[\,.$$
In addition, for each such solution, the following asymptotic representations hold as $t\uparrow\omega$
$$\frac{y(t)}{|\exp(R(|\ln|\pi_\omega(t)y(t)y'(t)||))|y(t)|^{\sigma_0}|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}}J(t)[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)J(t)}{(1-\sigma_1)J'(t)}\,[1+o(1)],$$
where
$$I(t) = \alpha_0\int_{A_\omega}^{t} p(\tau)\,d\tau, \quad J(t) = \int_{B_\omega}^{t} |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau,$$
the integration limits A_ω, B_ω are chosen so that the corresponding integrals are either 0*, or* ∞*.*

For differential equations of more specific type, one can get more detailed information about $P_\omega(Y_0,Y_1,0)$-solutions to the equation (3).

In [4] it was considered the differential equation

$$y'' = mt^{\sigma_1-2}\exp(k\ln^\gamma t)|y|^{\sigma_0}|y'|^{\sigma_1}\exp\big((|\ln|yy'|)^\mu\big) \tag{6}$$

on the interval $[t_0;+\infty[$ $(t_0>0)$, where $m\in\,]-\infty,0[\,$, $k\in\,]0,+\infty[\,$, $\gamma,\mu\in\,]0;1[\,$, $\sigma_0,\sigma_1\in\mathbb{R}$, $\sigma_0+\sigma_1\neq 1$, $\sigma_1\neq 1$ is the equation of the form (1), where $\alpha_0=\operatorname{sign} m=-1$, $p(t)=mt^{\sigma_1-2}\exp(k\ln^\gamma t)$,

$\varphi_0 = |y|^{\sigma_0}$, $\varphi_1 = |y|^{\sigma_1}$, $R(z) = z^\mu$. This function φ_1 satisfies the condition S. Let us consider the case, when $\omega = Y_0 = Y_1 = +\infty$.

If $\mu - \gamma < 0$, then for the existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions of the equation (6) the following condition

$$1 - \sigma_0 - \sigma_1 > 0 \tag{7}$$

is necessary and sufficient.

Moreover, for each such solution the following asymptotic representations take place as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^\mu}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\gamma k} \exp\Big(\frac{k\ln^\gamma t}{1-\sigma_1}\Big) \ln^{1-\gamma} t[1+o(1)],$$

$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)\gamma k}{(1-\sigma_1)^2} \frac{\ln^{\gamma-1} t}{t} [1+o(1)].$$

Let us now consider the case $\mu-\gamma > 0$. In this case for $\mu-\gamma > 0$ for existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions to the equation (6) the condition (7) is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^\mu}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1)} \exp\Big(\frac{k\ln^\gamma t}{1-\sigma_1}\Big) \ln^{1-\mu} t[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\mu}{\sigma_0+\sigma_1-1} t^{\sigma_1-2} \ln^{\gamma-1} t[1+o(1)].$$

In case $\mu = \gamma$ we obtain that for the existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions to the equation (6) the condition (7) together with the condition

$$(1-\sigma_1-k)(1-\sigma_1) > 0$$

is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^\mu}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1-k)} \exp\Big(\frac{k\ln^\gamma t}{1-\sigma_1}\Big) \ln^{1-\mu} t[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\mu(1-\sigma_1-k)}{(\sigma_0+\sigma_1-1)(1-\sigma_1)} t^{\sigma_1-2} \ln^{\gamma-1} t[1+o(1)].$$

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