On Non-Belonging to the Second Baire Class of the Topological Entropy of One Family of Non-Autonomous Dynamical Systems on an Interval Continuously Depending on a Real Parameter

Alexandr N. Vetokhin[1,2]

[1]*Lomonosov Moscow State University, Moscow, Russia*
[2]*Bauman Moscow State Technical University, Moscow, Russia*
E-mail: anveto27@yandex.ru

Let (X,d) be a compact metric space, $f \equiv (f_1, f_2, \dots)$ be a sequence of continuous mappings from X to X. Along with the original metric d, we define on X an additional system of metrics

$$d_n^f(x,y) = \max_{0 \leq i \leq n-1} d\big(f^{\circ i}(x), f^{\circ i}(y)\big), \quad f^{\circ i} \equiv f_i \circ \cdots \circ f_1 \circ \mathrm{id}_X, \quad x,y \in X, \quad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$ and $\varepsilon > 0$, let $N(f,\varepsilon,n)$ denote the maximum number of points in X whose pairwise d_n^f-distances are greater than ε. Such a set of points is called (f,ε,n)-separated. Then the *topological entropy* of a nonautonomous dynamical system (X,f) is the quantity

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln N(f,\varepsilon,n). \tag{1}$$

Note that the topological entropy does not depend on the choice of a metric generating the given topology on X, so definition (1) is correct.

Given a metric space $\mathcal{M}$ and a sequence of continuous mappings

$$f \equiv (f_1, f_2, \dots), \quad f_k : \mathcal{M} \times X \to X, \tag{2}$$

we form a function

$$\mu \mapsto h_{\mathrm{top}}(f(\mu, \cdot)). \tag{3}$$

For arbitrary $\mathcal{M}$, X and for any sequence of mappings (2) function (3) belongs to the third Baire class [4]. In the case when X is a Cantor perfect set [4] or a segment of the real line [5] and $\mathcal{M}$ is the set of irrational numbers on the segment $[0;1]$ with the metric induced by the standard metric of the real line, there is a sequence of mappings (2) for which function (3) is everywhere discontinuous and does not belong to the second Baire class.

A natural question arises on the smallest Baire class to which function (3) belongs in the case $\mathcal{M} = [0;1]$.

Theorem. *Let $\mathcal{M} = X = [0;1]$, then there exists a sequence of continuous mappings* (2) *such that the function* (3) *is everywhere discontinuous and does not belong to the second Baire class on the space $\mathcal{M}$.*

Proof. Given a continuous function $\alpha : \mathcal{M} \to \mathcal{M}$

$$\alpha(\mu) = \begin{cases} 0, & \text{if } \mu = 0, \\ \mu\Big(1 - \sin\dfrac{1}{\mu}\Big), & \text{if } 0 < \mu \leqslant 1, \end{cases}$$

we will construct a sequence of functions

$$\alpha_k(\,\cdot\,) = \max\left\{\frac{1}{k}\,, \alpha^{\circ[\log_2(k+4)]}(\,\cdot\,)\right\}, \quad k = 1, 2, \dots$$

($[\,\cdot\,]$ is the integer part of number) and a sequence of mappings from $[0;1]^2$ to $[0;1]$

$$f \equiv (f_1, f_2, \dots),$$
$$f_k(\mu, x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant 1 - \alpha_k(\mu), \\ 2x - 1 + \alpha_k(\mu), & \text{if } 1 - \alpha_k(\mu) < x \leqslant 1 - \dfrac{\alpha_k(\mu)}{2}\,, \\ -2x + 3 - \alpha_k(\mu), & \text{if } 1 - \dfrac{\alpha_k(\mu)}{2} < x \leqslant 1. \end{cases}$$

By definition, the function f_k is continuous on $[0;1]^2$.

Let $\mathbf{E}$ denote the set of those μ from $[0;1]$ for which the equality $\lim\limits_{k\to\infty} \alpha_k(\mu) = 0$ holds. It is not empty because it contains zero.

Lemma 1. *Let* $\mu \in \mathbf{E}$*, then*

$$h_{\text{top}}(f(\mu,\,\cdot\,)) = 0.$$

Proof. We recall another formula for calculating the topological entropy of a nonautonomous dynamical system [1]. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, denote by $B_f(x, \varepsilon, n)$ the open ball $\{y \in X : d_n^f(x, y) < \varepsilon\}$. A set $U \subset X$ is called an (f, ε, n)-covering if

$$X \subset \bigcup_{x \in U} B_f(x, \varepsilon, n).$$

Let $S(f, \varepsilon, n)$ denote the minimum number of elements of an (f, ε, n)-covering, then the topological entropy can be calculated by the formula

$$h_{\text{top}}(f) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}}\, \frac{1}{n}\, \ln S(f, \varepsilon, n).$$

We fix $\varepsilon > 0$ and $\mu \in \mathbf{E}$, then there is a number k_0 such that $\alpha_{k_0}(\mu) < \frac{1}{2}\,\varepsilon$ and for any $k \geqslant k_0$ the inequality $\alpha_k(\mu) \leqslant \alpha_{k_0}(\mu)$ holds. Let $U_{k_0} \subset [0;1]$ be a minimal $(f(\mu,\,\cdot\,), \frac{1}{2}\varepsilon, k_0)$-covering of the interval $[0;1]$. The set $U_{k_0} \cup \{x_0\}$, where $f^{\circ k_0}(\mu, x_0) = 1 - \alpha_{k_0}(\mu)$, due to the inclusion

$$f_k\big(\mu, [1 - \alpha_{k_0}(\mu), 1]\big) \subset \big[1 - \alpha_{k_0}(\mu), 1\big],$$

for $k > k_0$ is an $(f(\mu,\,\cdot\,), \frac{1}{2}\varepsilon, k)$-covering of the interval $[0;1]$, therefore

$$h_{\text{top}}(f(\mu,\,\cdot\,)) \leqslant \lim_{k\to\infty} \frac{1}{k}\, \ln\big(|U_{k_0}| + 1\big) = 0.$$

Lemma 1 is proved. □

Lemma 2. *Let* $\mu \notin \mathbf{E}$*, then*

$$h_{\text{top}}(f(\mu,\,\cdot\,)) \geqslant \frac{1}{2}\, \ln 2.$$

Proof. Let $\mu \notin \mathbf{E}$, then there exists a subsequence $(\alpha_{k_j}(\mu))_{j=1}^{\infty} \subset (\alpha_k(\mu))_{k=1}^{\infty}$ and a number $q > 0$ such that $\inf\limits_{j\in\mathbb{N}} \alpha_{k_j}(\mu) = q$.

For all $j \in \mathbb{N}$, $k \in \{2^{k_j}, \dots, 2^{k_j+1} - 1\}$ and $x \in [0;1]$ the equality $f_k(\mu, x) = f_{2^{k_j}}(\mu, x)$ holds. Using the affine transformation φ we map the square $[1 - \alpha_{k_j}(\mu), 1]^2$ onto the square $[0,1]^2$, and the mapping $f_{2^{k_j}}(\mu, \cdot)\big|_{[1-\alpha_{k_j}(\mu),1]}$ becomes the mapping

$$g(x) = \begin{cases} 2x, & \text{if } \ 0 \leqslant x \leqslant \dfrac{1}{2}\,, \\ 2 - 2x, & \text{if } \ \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

For any $n \in \mathbb{N}$, consider the set of points of the form

$$\sum_{k=1}^{n} \frac{a_k}{2^k}\,, \ \text{ where } \ a_k \in \{0, 1\}.$$

Using mathematical induction, we prove the equality

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = \begin{cases} 0, & \text{if } \ a_n = 0, \\ 1, & \text{if } \ a_n = 1. \end{cases} \tag{4}$$

Indeed, for $n = 1$ we have $g(0) = 0$ and $g(\frac{1}{2}) = 1$.

Let

$$\sum_{k=1}^{n} \frac{a_k}{2^k} < \frac{1}{2}\,,$$

then

$$\sum_{k=1}^{n} \frac{a_k}{2^k} = \sum_{k=2}^{n} \frac{a_k}{2^k}$$

and

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(2\sum_{k=2}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(\sum_{k=1}^{n-1} \frac{a_{k+1}}{2^k}\Big) = \begin{cases} 0, & \text{if } \ a_n = 0, \\ 1, & \text{if } \ a_n = 1. \end{cases}$$

Let

$$\sum_{k=1}^{n} \frac{a_k}{2^k} \geqslant \frac{1}{2}\,,$$

then $a_1 = 1$ and

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(2 - 2\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(1 - \sum_{k=1}^{n-1} \frac{a_{k+1}}{2^k}\Big) = \begin{cases} 0, & \text{if } \ a_n = 0, \\ 1, & \text{if } \ a_n = 1. \end{cases}$$

Thus, equality (4) is proved.

By (4), for $\varepsilon < \frac{1}{q}$ and $n \in \{1, \dots, 2^{k_j+1} - 2^{k_j} - 1\}$, we have that $d^{f(\mu,\cdot)}_{2^{k_j}+n}$-distance between any preimages of two points

$$\varphi^{(-1)}\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}, 0\Big) \ \text{ and } \ \varphi^{(-1)}\Big(\sum_{k=1}^{n} \frac{a_k}{2^k} + \frac{1}{2^n}, 0\Big)$$

under the mapping $f^{\circ(2^{k_j}-2)}(\mu,\,\cdot\,)$ is greater than ε, and therefore

$$N(f(\mu,\,\cdot\,),\varepsilon,2^{k_j+1}) \geqslant 2^{k_j+1}-2^{k_j},$$

whence we get

$$h_{\mathrm{top}}(f(\mu,\,\cdot\,)) \geqslant \lim_{\varepsilon\to 0}\varlimsup_{j\to\infty}\frac{1}{2^{k_j+1}}\ln(2^{2^{k_j+1}-2^{k_j}}) = \frac{1}{2}\ln 2.$$

Lemma 2 is proved. □

Completion of the proof of the theorem. In the paper [2] it was established that the set $\mathbf{E}$ is an $F_{\sigma\delta}$-set and is not a $G_{\delta\sigma}$-set. We use the following statement from [3]: if a functional $h:\mathcal{M}\to\mathbb{R}$ belongs to the second Baire class, then the intersection of the closures of the sets $h(\mathbf{E})$ and $h(\mathcal{M}\setminus\mathbf{E})$ is nonempty. By Lemmas 1 and 2, we have

$$h_{\mathrm{top}}(f(\mathbf{E},\,\cdot\,)) \leqslant 0 < \frac{1}{2}\ln 2 \leqslant h_{\mathrm{top}}(f(\mathcal{B}\setminus\mathbf{E},\,\cdot\,)),$$

therefore, the function $\mu\mapsto h_{\mathrm{top}}(f(\mu,\,\cdot\,))$ does not belong to the second Baire class. Theorem is proved. □

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