Stability of Solutions of One-Dimensional Stochastic Differential Equations Controlled by Rough Paths

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Abstract

We prove theorems on the stability of solutions of one-dimensional stochastic differential equations controlled by rough paths with arbitrary positive Holder exponent.

1 Introduction

Consider the one-dimensional stochastic differential equation

$$dY_t = f(Y_t)dX_t, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where X_t is a random process whose paths are a.s. Holder continuous of order $\alpha \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function with bounded continuous derivatives of any order $m \in \{0, \ldots, [1/\alpha] + 1\}$.

In the present paper, we prove that the conditions ensuring the existence and uniqueness of solutions of Eq. (1.1) also guarantee the continuous dependence of the solutions on the initial data on any finite interval; the Lyapunov stability of the zero solution of Eq. (1.1) is studied on the basis of the stability of the zero solution of the corresponding ordinary differential equation (ODE) $dZ_t = f(Z_t)dt$. Here a solution of Eq. (1.1) is understood as a solution of a stochastic differential equation weakly controlled by the corresponding rough path [1]. To define solutions, we need a number of notions introduced in the papers [1] and [2].

2 Definition of rough paths

Fix some $T > 0$ and $\alpha \in (0,1]$. Let V be a finite-dimensional Euclidean space. By $C^\alpha([0,T],V)$ and $C_2^\alpha([0,T],V)$ we denote the sets of functions $f : [0,T] \to V$ and $g : [0,T]^2 \to V$, respectively, with finite norms

$$\|f\|_\alpha := \sup_{s,t\in[0,T],\, s\neq t} \frac{|f_t - f_s|}{|t-s|^\alpha},$$

$$\|g\|_{\alpha,2} := \sup_{s,t\in[0,T], s\neq t} \frac{|g_{s,t}|}{|t-s|^\alpha}.$$

Further, for a function of two variables $g_{s,t}$ we write $\|g\|_\alpha$ instead of $\|g\|_{\alpha,2}$. For a function f_t of one variable, by $f_{s,t}$ we denote the increment $f_t - f_s$.

For an integer non-negative k and finite-dimensional Euclidean spaces V and W, by $C_b^k(V,W)$ we denote the set of functions $h : V \to W$ with finite norm

$$\|h\|_{C_b^k} := \sum_{i=0}^{k} \|D^i h\|_\infty,$$

where
$$\|D^i h\|_\infty = \sup_{t\in[0,T]} |D^i h_t|.$$

Set $n = [1/\alpha]$. By $\mathscr{C}^\alpha([0,T],V)$ we denote the set of Holder α-continuous rough paths, i.e., the set of elements $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ such that $\mathbf{X}^i \in C_2^{i\alpha}([0,T], V^{\otimes i})$ for any $i = 1, \dots, n$, and any $s, u, t \in [0,T]$ there holds the Cheng identity
$$\mathbf{X}_{s,t} = \mathbf{X}_{s,u} \boxplus \mathbf{X}_{u,t},$$
where
$$\big(\mathbf{X}_{s,u} \boxplus \mathbf{X}_{u,t}\big)^i = \sum_{j=0}^{i} \mathbf{X}_{s,u}^j \otimes \mathbf{X}_{u,t}^{i-j}.$$
Note that the operation $\boxplus$ defines multiplication on the tensor algebra $T^{(n)}(V) = \bigoplus\limits_{i=0}^{n} V^{\otimes i}$, where $V^{\otimes 0} = \mathbb{R}$. Thus, an element $\mathbf{X} : [0,T]^2 \to T^{(n)}(V)$ is uniquely determined by the values $\mathbf{X}_{0,t}$, $t \in [0,T]$, because $\mathbf{X}_{s,t} = (\mathbf{X}_{0,s})^{-1} \boxplus \mathbf{X}_{0,t}$. In what follows, we write $\mathbf{X}_t$ instead of $\mathbf{X}_{0,t}$.

A rough path $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ is said to be geometric if
$$\operatorname{Sym}(\mathbf{X}_{s,t}^i) = \frac{1}{i!}\,(\mathbf{X}_{s,t}^1)^{\otimes i} \ \ \forall\, i = 1, \dots, n.$$
The set of geometric rough paths will be denoted by $\mathscr{C}_g^\alpha([0,T],V)$.

We say that an element $\mathbf{X} \in \mathscr{C}^\alpha([0,T],V)$ is a rough path over $X \in C^\alpha([0,T],V)$ if $\mathbf{X}_{0,t}^1 = X_t$ for any $t \in [0,T]$.

Definition of weakly controlled rough paths

Let $X \in C^\alpha([0,T],V)$ and let $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ be a rough path over X. Let W be a finite-dimensional Euclidean space. We say that a function $Y_t \in C^\alpha([0,T],W)$ is weakly controlled by the rough path $\mathbf{X} \in \mathscr{C}^\alpha([0,T],V)$ if there exist functions $Y^{(1)} : [0,T] \to \mathcal{L}(V,W), \dots, Y^{(n-1)} : [0,T] \to \mathcal{L}(V^{\otimes(n-1)}, W)$ such that
$$\begin{aligned}
Y_{s,t} &= Y_s^{(1)} \mathbf{X}_{s,t}^1 + \dots + Y_s^{(n-1)} \mathbf{X}_{s,t}^{n-1} + R_{s,t}^{Y,n},\\
Y_{s,t}^{(1)} &= Y_s^{(2)} \mathbf{X}_{s,t}^1 + \dots + Y_s^{(n-1)} \mathbf{X}_{s,t}^{n-2} + R_{s,t}^{Y,n-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
Y_{s,t}^{(n-2)} &= Y_s^{(n-1)} \mathbf{X}_{s,t}^1 + R_{s,t}^{Y,2},\\
Y_{s,t}^{(n-1)} &= R_{s,t}^{Y,1};
\end{aligned}$$
and the norm $\|R^{Y,i}\|_{i\alpha}$ is finite for each of the remainder terms $R^{Y,i}$, $i = 1, \dots, n$. The function $Y^{(i)}$ will be called the i-th rough derivative of Y.

Define the Banach space
$$\mathcal{D}_{\mathbf{X}}^\alpha([0,T],W) = \Big\{(Y, Y^{(1)}, \dots, Y^{(n-1)}) : \ \ Y \in C^\alpha([0,T],W), \ \ \sum_{i=1}^{n} \|R^{Y,i}\|_{i\alpha} < \infty\Big\}$$
with the seminorm
$$\big\|(Y, Y^{(1)}, \dots, Y^{(n-1)})\big\|_{\mathcal{D}_{\mathbf{X}}^\alpha} = \sum_{i=1}^{n} \|R^{Y,i}\|_{i\alpha}.$$

The norm of an element $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)}) \in \mathcal{D}^\alpha_{\mathbf{X}}([0,T], W)$ is defined by the formula

$$\|\mathbf{Y}\|_{\mathcal{D}^\alpha_{\mathbf{X}}} := \sum_{i=0}^{n-1} |Y_0^{(i)}| + \big\|(Y, Y^{(1)}, \dots, Y^{(n-1)})\big\|_{\mathcal{D}^\alpha_{\mathbf{X}}},$$

where $Y_t^{(0)} = Y_t$.

3 Definition of the integral over rough paths

Let V and W be some finite-dimensional Euclidean spaces,

$$\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n) \in \mathscr{C}^\alpha([0,T], V), \quad Y \in C^\alpha([0,T], \mathcal{L}(V,W)),$$
$$(Y, Y^{(1)}, \dots, Y^{(n-1)}) \in \mathcal{D}^\alpha_{\mathbf{X}}([0,T], \mathcal{L}(V,W)).$$

Take some $s, t \in [0,T]$, $s < t$, and let $\mathcal{P}$ be an arbitrary finite partition of the interval $[s,t]$ by points.

The rough path integral $\int_s^t Y_r\, d\mathbf{X}_r$ is defined as the following limit of integral sums (if the limit exists, then it is finite and does not depend on the choice of partitions of the interval $[s,t]$ by points):

$$\int_s^t Y_r\, d\mathbf{X}_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \sum_{i=0}^{n-1} Y_u^{(i)} \mathbf{X}^{i+1}_{u,v}.$$

4 Definition of rough paths on a half-line

Let $X : \mathbb{R}_+ \to \mathbb{R}$; i.e., assume that for each $T > 0$ the restriction $X|_{[0,T]}$ belongs to the space $C^\beta([0,T], \mathbb{R})$, $\beta \in (\frac{1}{n+1}, \frac{1}{n}]$. For each $i \in \{1, \dots, n\}$ we define $\mathbf{X}^i_{s,t} = \frac{(X_{s,t})^i}{i!}$, $s, t \in \mathbb{R}_+$. The element $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)\mathbb{R}^2_+ \to T^{(n)}(\mathbb{R})$ is called a geometric rough path over X. The set of geometric rough paths is denoted by $\mathscr{C}^\beta_g(\mathbb{R}_+, \mathbb{R})$.

We say that a function $Y \in C^\alpha(\mathbb{R}_+, \mathbb{R})$, $\frac{1}{n+1} < \alpha < \beta$, is weakly controlled by a geometric rough path $\mathbf{X} \in \mathscr{C}^\beta_g(\mathbb{R}_+, \mathbb{R})$ if there exist $Y^{(i)} : \mathbb{R}_+ \to \mathbb{R}$, $i \in \{1, \dots, n-1\}$, such that the $i\alpha$-Holder norms of $R^{Y,i}$, $i \in \{1, \dots, n\}$, are finite on each bounded segment $\mathbb{R}_+$. We say that a vector function $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ belongs to the set $\mathcal{D}^\alpha_{\mathbf{X}}(\mathbb{R}_+, \mathbb{R})$ if for each $T > 0$ its restriction $\mathbf{Y}|_{[0,T]}$ belongs to the space $\mathcal{D}^\alpha_{\mathbf{X}}([0,T], \mathbb{R})$.

5 Stochastic differential equations weakly controlled by rough paths with arbitrary positive Holder exponent

Suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ with a flow $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras are given an $\mathcal{F}_t$-adapted random process X_t, $t \in \mathbb{R}_+$, such that almost all trajectories of X_t belong to the space $C^\beta(\mathbb{R}_+, \mathbb{R})$, $\beta \in (\frac{1}{n+1}, \frac{1}{n}]$. Define a process $\mathbf{X}_\cdot = (1, \mathbf{X}^1_{0,\cdot}, \dots, \mathbf{X}^n_{0,\cdot})$ as a random variable a.s. taking values in $\mathscr{C}^\beta_g(\mathbb{R}_+, \mathbb{R})$ a.s., where $\mathbf{X}^i_{s,t} = \frac{(X_{s,t})^i}{i!}$.

Let $Y \in C^\alpha([0,T], \mathbb{R})$, $(Y, Y^{(1)}, Y^{(2)}, \dots, Y^{(n-1)}) \in \mathcal{D}^\alpha_{\mathbf{X}}([0,T], \mathbb{R})$; $f \in C^n_b(\mathbb{R}, \mathbb{R})$. Define $Z_t = f(Y_t)$. By analogy with the Faà di Bruno's formula, we set

$$Z^{(k)} = \sum_{j=1}^{k} D^j f(Y) B_{k,j}(Y^{(1)}, \dots, Y^{(k-j+1)}), \quad k = 1, \dots, n-1, \tag{5.1}$$

where the $B_{k,j}(x_1,\dots,x_{k-j+1})$ – are Bell polynomials.

Consider the stochastic differential equation

$$dY_t = f(Y_t)d\mathbf{X}_t, \ \ t\in\mathbb{R}_+. \tag{5.2}$$

Definition 5.1. Let $\xi:\Omega\to\mathbb{R}$ be an $\mathcal{F}_0$-measurable random variable. A solution of Eq. (5.2) with the initial condition $Y_0=\xi$ is an $\mathcal{F}$-measurable random variable $\mathbf{Y}=(Y,Y^{(1)},\dots,Y^{(n-1)})$ with values in $\mathcal{D}^\alpha_{\mathbf{X}}(\mathbb{R}_+,\mathbb{R})$ a.s., $\frac{1}{n+1}<\alpha<\beta$, such that the random process $\mathbf{Y}_t$ is $\mathcal{F}_t$-adapted and a.s. the equality

$$Y_t=\xi+\int\limits_0^t f(Y_s)\,d\mathbf{X}_s$$

holds for all $t\in\mathbb{R}_+$, where the rough derivatives of the function $f(Y)$, occurring in the definition of the integral on the right-hand side, are determined by formulas (5.1). A solution of Eq. (5.2) with the initial condition $Y_0=\xi$ is said to be unique if for two arbitrary solutions $\mathbf{Y}$, $\overline{\mathbf{Y}}$ of Eq. (5.2) with the initial condition $Y_0=\xi$ one has the equality $P(\mathbf{Y}=\overline{\mathbf{Y}})=1$.

Consider the ODE

$$dZ_t=f(Z_t)dt, \ \ t\in\mathbb{R}. \tag{5.3}$$

Let $S_t=e^{tV_f}$, $t\in\mathbb{R}$, be the flow generated by Eq. (5.3), i.e., $Z_t=S_tZ_0$, where the operator $V_f:C(\mathbb{R},\mathbb{R})\to C(\mathbb{R},\mathbb{R})$ acts according to the rule $(V_fg)_t=f(g_t)$.

The following assertion was proved in [1].

Proposition 5.1 ([1])**.** *Let $\alpha,\beta\in(\frac{1}{n+1},\frac{1}{n}]$, $\alpha<\beta$, $\mathbf{X}=(1,\mathbf{X}^1,\dots,\mathbf{X}^n)\in\mathscr{C}^\beta_g(\mathbb{R}_+,\mathbb{R})$ a.s. If $f\in C^{n+1}_b(\mathbb{R},\mathbb{R})$, then for any $\mathcal{F}_0$-measurable random variable $\xi:\Omega\to\mathbb{R}$ there exists a unique solution $\mathbf{Y}=(Y,Y^{(1)},\dots,Y^{(n-1)})$ of Eq. (1.1) with the initial condition $Y_0=\xi$, and a.s. one has*

$$Y_t=S_{X_{0,t}}\xi, \ \ Y^{(i)}_t=D^{i-1}_f f(Y_t), \ \ i\in\{1,\dots,n-1\}, \ \ t\in\mathbb{R}_+.$$

6 Continuous dependence of solutions on the initial data

Along with Eq. (5.2), consider the perturbed equation

$$dY_t=\widetilde{f}(Y_t)d\mathbf{X}_t, \ \ t\in\mathbb{R}_+. \tag{6.1}$$

Theorem 6.1. *Let $\alpha,\beta\in(\frac{1}{n+1},\frac{1}{n}]$, $\alpha<\beta$, $p\ge 1$, $T>0$, $\mathbf{X}=(1,\mathbf{X}^1,\dots,\mathbf{X}^n)\in\mathscr{C}^\beta_g(\mathbb{R}_+,\mathbb{R})$ a.s., $\xi:\Omega\to\mathbb{R}$ be a $\mathcal{F}_0$-measurable random variable; $f\in C^{n+1}_b(\mathbb{R},\mathbb{R})$. If $\mathbb{E}\|X\|^p_{\alpha,[0,T]}<\infty$, then for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,T)$ such that for any $\widetilde{f}\in C^{n+1}_b(\mathbb{R},\mathbb{R})$ and $\mathcal{F}_0$-measurable random variable $\widetilde{\xi}:\Omega\to\mathbb{R}$ such that*

$$\|\widetilde{f}-f\|_{C^{n+1}_b}+\mathbb{E}|\widetilde{\xi}-\xi|^p\le\delta,$$

there holds the inequality

$$\sum_{i=0}^{n-1}\mathbb{E}\|\widetilde{Y}^{(i)}-Y^{(i)}\|^p_{\alpha,[0,T]}\le\varepsilon,$$

where $\mathbf{Y}=(Y,Y^{(1)},\dots,Y^{(n-1)})$ is a solution of Eq. (5.2) with the initial condition $Y_0=\xi$ and $\widetilde{\mathbf{Y}}=(\widetilde{Y},\widetilde{Y}^{(1)},\dots,\widetilde{Y}^{(n-1)})$ is a solution of Eq. (6.1) with the initial condition $Y_0=\widetilde{\xi}$.

Proof. Assume that the assertion in the theorem does not hold; i.e., there exists an $\varepsilon_0 > 0$ such that for any $\delta_k = \frac{1}{k}$, $k \in \mathbb{N}$, there exist $f_k \in C_b^{n+1}(\mathbb{R},\mathbb{R})$ and $\mathcal{F}_0$-measurable random variables $\xi_k : \Omega \to \mathbb{R}$ such that

$$\|f_k - f\|_{C_b^{n+1}} + \mathbb{E}|\xi_k - \xi|^p \le \delta_k,$$
$$\sum_{i=0}^{n-1} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|^p_{\alpha,[0,T]} \ge \varepsilon_0,$$

where $\mathbf{Y}_k = (Y_k, Y_k^{(1)}, \dots, Y_k^{(n-1)})$ is a solution of equation

$$dY_t = f_k(Y_t)d\mathbf{X}_t, \ \ t \in \mathbb{R}_+,$$

with the initial condition

$$\mathbf{Y}_0 = \big(\xi_k, f_k(\xi_k), Df_k(\xi_k)f_k(\xi_k), \dots\big).$$

Let $S_{k,t} = e^{tV_{f_k}}$ be the flow corresponding to the equation $dZ_t = f_k(Z_t)dt$. By Proposition 1 we get

$$Y_t = S_{X_t - X_0}\xi, \quad Y_t^{(i)} = D_f^{i-1} f(Y_t),$$
$$Y_{k,t} = S_{k,X_t - X_0}\xi_k, \quad Y_{k,t}^{(i)} = D_{f_k}^{i-1} f_k(Y_{k,t}).$$

Without loss of generality we may assume that $X_0 = 0$. Set $g(\tau) = S_\tau \xi$, $g_k(\tau) = S_{k,\tau}\xi_k$, $\psi_k(\tau) = g_k(\tau) - g(\tau)$, $\tau \in \mathbb{R}$. Thus,

$$\|Y_k - Y\|_{\alpha,[0,T]} = \sup_{s\neq t} \frac{|\psi_k(X_t) - \psi_k(X_s)|}{|t-s|^\alpha}$$
$$= \sup_{s\neq t} \frac{|(X_t - X_s)D\psi_k(X_s + \theta_k(X_t - X_s))|}{|t-s|^\alpha} \le \|X\|_{\alpha,[0,T]}\|D\psi_k\|_\infty.$$

Since $\mathbb{E}\|X\|^p_{\alpha,[0,T]} < \infty$, we have

$$\lim_{k\to\infty} \mathbb{E}\|Y_k - Y\|^p_{\alpha,[0,T]} = 0.$$

Take arbitrary $i \in \{1, \dots, n-1\}$. Denote

$$h(y) = D_f^{i-1} f(y), \ \ h_k(y) = D_{f_k}^{i-1} f_k(y), \ \ \varphi_k(y) = h_k(y) - h(y), \ \ y \in \mathbb{R}.$$

Then

$$\|Y_k^{(i)} - Y^{(i)}\|_{\alpha,[0,T]} = \sup_{s\neq t} \frac{|h_k(Y_{k,t}) - h_k(Y_{k,s}) - h(Y_t) + h(Y_s)|}{|t-s|^\alpha}$$
$$= \sup_{s\neq t} \frac{|(Y_{k,t} - Y_{k,s})D\varphi_k(Y_{k,s} + \theta_k(Y_{k,t} - Y_{k,s}))|}{|t-s|^\alpha} + \sup_{s\neq t} \frac{|h(Y_{k,t}) - h(Y_{k,s}) - h(Y_t) + h(Y_s)|}{|t-s|^\alpha}$$
$$\le \|Y_k - Y\|_{\alpha,[0,T]}\|D\varphi_k\|_\infty + \|Dh\|_\infty\|Y_k - Y\|_{\alpha,[0,T]} + C\|D^2h\|_\infty\big(\|Y_k - Y\|_{\alpha,[0,T]} + |\xi_k - \xi|\big).$$

Hence,

$$\lim_{k\to\infty} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|^p_{\alpha,[0,T]} = 0.$$

Therefore,

$$\lim_{k\to\infty} \sum_{i=0}^{n-1} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|_{\alpha,[0,T]} = 0.$$

The resulting contradiction completes the proof of the theorem. □

7 Lyapunov stability of solutions on the half-line

Let us proceed to the stability analysis of the zero solution of Eq. (5.2) under the assumption that $f(0) = 0$. Additionally, we assume that the function $f \in C^{n+1}(\mathbb{R}, \mathbb{R})$ is such that no solution Z_t, $t \geq 0$, of Eq. (5.3) has blow-ups. In what follows, the zero solution of Eq. (5.2) is understood as the solution $\mathbf{Y} \equiv 0$ of Eq. (5.2) with the zero initial condition $Y_0 = 0$.

Definition 7.1. We say that the zero solution of Eq. (5.2) is stable in probability if for any ε_1, $\varepsilon_2 > 0$ there exists $\delta = \delta(\varepsilon_1, \varepsilon_2) > 0$ such that for each $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \delta$ a.s., there holds the inequality

$$P\Big(\sup_{t\geq 0} |Y_t| \geq \varepsilon_1\Big) \leq \varepsilon_2,$$

where $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ is the solution of Eq. (5.2) with the initial condition $Y_0 = \xi$. We say that the zero solution of Eq. (5.2) is asymptotically stable in probability if it is stable in probability and there exists a $\Delta > 0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \Delta$ a.s., one has the convergence in probability $Y_t \xrightarrow[t\to+\infty]{P} 0$. Let $p \geq 1$; we say that the zero solution of Eq. (5.2) is p-stable if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \delta$ a.s., there holds the inequality $\sup\limits_{t\geq 0} \mathbb{E}|Y_t|^p \leq \varepsilon$.

Theorem 7.1. *Let* $X_t \xrightarrow[t\to+\infty]{P} +\infty$ *and let the expectation* $\mathbb{E}\big(\sup\limits_{t\in[0,T]} |X_t|\big)$ *is finite for each* $T > 0$. *If the zero solution of Eq.* (5.3) *is Lyapunov stable (respectively, asymptotically stable) for* $t \geq 0$, *then the zero solution of Eq.* (5.2) *is stable in probability (respectively, asymptotically stable in probability).*

Proof. Without loss of generality, we can assume that $X_0 = 0$. Let Z_t be the solution of Eq. (5.3) with the initial condition $Z_0 = \xi$, then $Y_t = Z_{X_t}$. Fix arbitrary ε_1, $\varepsilon_2 > 0$.

Since $X_t \xrightarrow[t\to+\infty]{P} +\infty$, for any $\varepsilon_2 > 0$ there exists $\tau = \tau(\varepsilon_2) > 0$ such that

$$P(X_t \geq 0 \ \ \forall\, t > \tau) \geq 1 - \frac{\varepsilon_2}{2}\,.$$

Since $\mathbb{E}\big(\sup\limits_{t\in[0,\tau]} |X_t|\big)$ is finite, it follows by the Chebyshev inequality that there exists a constant $M = M(\tau, \varepsilon_2) > 0$ such that

$$P\big(|X_t| \leq M \ \forall\, t \in [0,\tau]\big) \geq 1 - \frac{\varepsilon_2}{2}\,.$$

Assume that the zero solution of Eq. (5.3) is Lyapunov stable for $t \geq 0$. Then there exists a $\delta = \delta(\varepsilon_1, M) > 0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \delta$ a.s., one has the inequality $\sup\limits_{t\geq -M} |Z_t| \leq \varepsilon_1$.

Thus, we have

$$\begin{aligned} P\Big(\sup_{t\geq 0} |Y_t| > \varepsilon_1\Big) &= P\Big(\sup_{t\geq 0} |Z_{X_t}| > \varepsilon_1\Big) \leq P\big(\exists\, t \geq 0 :\ X_t < -M\big) \\ &\leq P\big(\exists\, t \in [0,\tau] :\ X_t < -M\big) + P\big(\exists\, t > \tau : X_t < 0\big) \leq \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2. \end{aligned}$$

Thus, the zero solution of Eq. (5.2) is stable in probability.

Consequently, the zero solution of Eq. (5.3) is asymptotically stable for $t \geq 0$. Then there exists a $\Delta > 0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \Delta$ a.s., the solution Z_t of Eq. (5.3) with the initial condition $Z_0 = \xi$ has the following property: the convergence

$$Z_t \xrightarrow[t\to+\infty]{} 0$$

holds with probability 1. Take arbitrary ε_1, $\varepsilon_2 > 0$. There exists $\delta = \delta(\varepsilon_1)$ such that

$$P\big(|Z_t| \leq \varepsilon_1 \;\forall\, t \geq \delta\big) = 1.$$

Since $X_t \xrightarrow[t\to+\infty]{P} +\infty$, there exists $\delta_1 > 0$ such that

$$P\big(\exists\, t \geq \delta_1 :\; X_t < \delta\big) \leq \varepsilon_2.$$

Thus,

$$\begin{aligned} P\big(|Y_t| \leq \varepsilon_1 \;\forall\, t \geq \delta_1\big) &= P\big(|Z_{X_t}| \leq \varepsilon_1 \;\forall\, t \geq \delta_1\big) \\ &= 1 - P\big(\exists\, t \geq \delta_1 :\; |Z_{X_t}| > \varepsilon_1\big) \geq 1 - P\big(\exists\, t \geq \delta_1 :\; X_t < \delta\big) \geq 1 - \varepsilon_2. \end{aligned}$$

Hence, $Y_t \xrightarrow[t\to+\infty]{P} 0$, therefore, the zero solution of Eq. (5.2) is asymptotically stable in probability.

The proof of the theorem is complete. □

References

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