On the Solvability of a Family of Nonlinear Two-Point Boundary Value Problems for Systems of Integro-Differential Equations and its Application

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1 Introduction

Conditions for the existence of solutions of systems of integro-differential equations and boundary-value problems for these systems are considered in [8, 9]. A general theory for these systems is developed and effective methods for solving them are proposed. In this paper, we study the x-parametric family of nonlinear boundary-value problems for integro-differential equations

$$\frac{\partial V}{\partial t}=f\Big(x,t,\psi(t)+\int\limits_0^x V(\xi,t)\,d\xi,V\Big),\ \ V\in\mathbb{R}^n,\ \ (x,t)\in[0,\omega]\times[0,T], \tag{1.1}$$

$$g\big(x,V(x,0),V(x,T)\big)=0,\ \ x\in[0,\omega], \tag{1.2}$$

where the functions $f:[0,\omega]\times[0,T]\times\mathbb{R}^{2n}\to\mathbb{R}^n$ and $g:[0,\omega]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous.

Our study is based on the parametrization method [10, 12] proposed by Dzhumabaev. The parametrization method was developed to various boundary-value problems for some types of differential equations [2, 13, 15], such that Fredholm integro-differential equations [3–7, 17], delay differential equations [16], hyperbolic equations [1, 14] etc. The application of this method made it possible to derive the solvability conditions for the above problems.

This approach can also be applied to study the nonlinear nonlocal boundary value problem for the system of partial differential equations ($m=1,2,\dots$)

$$\frac{\partial^{m+1}u}{\partial t\partial x^m}=f\Big(x,t,\frac{\partial^{m-1}u}{\partial x^{m-1}},\frac{\partial^m u}{\partial x^m}\Big),\ \ u\in\mathbb{R}^n,\ \ (x,t)\in[0,\omega]\times(0,T),$$

$$\frac{\partial^k u}{\partial x^k}\Big|_{x=0}=\psi_k(t),\ \ t\in[0,T],\ \ k=0,1,\dots,m-1,$$

$$g\Big(x,\frac{\partial^m u(x,t)}{\partial x^m}\Big|_{t=0},\frac{\partial^m u(x,t)}{\partial x^m}\Big|_{t=T}\Big)=0.$$

Therefore, the study of problem (1.1), (1.2) is of interest from the point of view of its application to nonlocal boundary value problems for a class of partial differential equations.

In the present paper, we propose a modified algorithm of the parametrization method for finding an isolated solution of problem (1.1), (1.2) and derive sufficient conditions for the existence of such a solution.

2 Setting of the problem and the main results

We consider the nonlinear nonlocal boundary value problem (1.1), (1.2) for the x-parametric family of integro-differential equations.

A solution of problem (1.1), (1.2) is a function $V(x,t) \in C([0,\omega] \times [0,T]\mathbb{R}^n)$, which is continuously differentiable on $[0,T]$ (at fixed $x \in [0,\omega]$) and satisfies the system of integro-differential equations (1.1) and the boundary conditions (1.2).

We take $h > 0$, $Nh = T$ $(N \in \mathbb{N})$, and make the partition $[0,\omega] \times [0,T) = \bigcup\limits_{r=1}^{N} \Omega_r$, where $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$.

We will use the following notations.

- $C([0,\omega], \mathbb{R}^{n(N+1)})$ is the space of systems of functions $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_{N+1}(x))$ with the norm
$$\|\lambda\|_1 = \max_{x\in[0,\omega]} \max_{r=\overline{1,(N+1)}} \|\lambda_r(x)\|,$$
here the functions $\lambda_r : [0,\omega] \to \mathbb{R}^n$ are continuous, $r = \overline{1,(N+1)}$;

- $C([0,\omega] \times [0,T], \Omega_r, R^{nN})$ is the space of systems of functions
$$V[x,t] = \big(V_1(x,t), V_2(x,t), \ldots, V_N(x,t)\big)$$
with the norm
$$\big\|V[\,\cdot\,]\big\|_2 = \max_{r=\overline{1,N}} \max_{x\in[0,\omega]} \sup_{t\in[t_{r-1},t_r)} \|V_r(x,t)\|,$$
where the functions $V_r(x,t) \in C(\Omega_r)$ have finite limits $\lim\limits_{t\to t_r-0} V_r(x,t)$ uniform in x, $x \in [0,\omega]$ $(r = \overline{1,N})$;

The restriction of the function $V(x,t)$ into Ω_r is denoted by $V_r(x,t)$, i.e. $V_r(x,t) = V(x,t)$, $(x,t) \in \Omega_r$, $r = \overline{1,N}$.

Let us set additional parameters
$$\lambda_r(x) = V_r(x,(r-1)h), \quad r = \overline{1,N}$$
and
$$\lambda_{N+1}(x) = \lim_{t\to T-0} V_N(x,t), \quad x \in [0,\omega],$$
and introduce the functions
$$\widetilde{V}_r(x,t) = V_r(x,t) - \lambda_r(x) \text{ on } \Omega_r, \quad r = \overline{1,N}.$$

We then obtain the family of multipoint nonlinear boundary value problems for integro-differential equations with parameters
$$\frac{\partial \widetilde{V}_r}{\partial t} = f\bigg(x,t,\psi(t) + \int_0^x \lambda_r(\xi)\,d\xi + \int_0^x \widetilde{V}_r(\xi,t)\,d\xi, \lambda_r(x) + \widetilde{V}_r\bigg), \quad (x,t) \in \overline{\Omega}_r, \quad r = \overline{1,N}, \tag{2.1}$$
$$\widetilde{V}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \tag{2.2}$$
$$g\big(x, \lambda_1(x), \lambda_{N+1}(x)\big) = 0, \quad x \in [0,\omega], \tag{2.3}$$
$$\lambda_r(x) + \lim_{t\to rh-0} \widetilde{V}_r(x,t) - \lambda_{r+1}(x) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}. \tag{2.4}$$

It can be easily shown that the families of problems (1.1), (1.2) and (2.1)–(2.4) are equivalent.

Suppose that for all $r=\overline{1,N+1}$ and $x\in[0,\omega]$ the family of parameters $\lambda_r(x)$ is known. Then the functions $\widetilde{V}_r(x,t)$, $(x,t)\in\Omega_r$ $(r=\overline{1,N})$, can be determined from the Cauchy problem (2.1), (2.2). For a fixed $x\in[0,\omega]$, this problem is equivalent to the family of mixed type systems of integral equations

$$\widetilde{V}_r(x,t)=\int\limits_{(r-1)h}^{t} f\bigg(x,\tau,\psi(t)+\int\limits_0^x \lambda_r(\xi)\,d\xi+\int\limits_0^x \widetilde{V}_r(\xi,\tau)\,d\xi,\lambda_r(x)+\widetilde{V}_r(x,\tau)\bigg)\,d\tau, \qquad (2.5)$$
$$t\in[(r-1)h,rh],\ \ r=\overline{1,N}.$$

By substituting the values of $\lim\limits_{t\to rh-0}\widetilde{V}_r(x,t)$, found from (2.5), into (2.3) and (2.4), we obtain

$$g\big(x,\lambda_1(x),\lambda_{N+1}(x)\big)=0,$$
$$\lambda_r(x)+\int\limits_{(r-1)h}^{rh} f\bigg(x,t,\psi(t)+\int\limits_0^x \big(\lambda_r(\xi)+\widetilde{V}_r(\xi,\tau)\big)\,d\xi,\lambda_r(x)+\widetilde{V}_r(x,t)\bigg)\,dt-\lambda_{r+1}(x)=0.$$

This is a system of nonlinear functional equations in parameters $\lambda_r(x)$, $x\in[0,\omega]$, $r=\overline{1,N+1}$. We rewrite this system in the form

$$Q_{1,h}\bigg(x,\lambda(x),\int\limits_0^x \lambda(\xi)\,d\xi,\widetilde{V}\bigg)=0,\ \ \lambda(x)\in\mathbb{R}^{n(N+1)},\ \ x\in[0,\omega]. \qquad (2.6)$$

Condition 2.1. There exists $h>0:Nh=T$ $(N\in\mathbb{N})$, such that the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V}=0$, has a solution $\lambda^{(0)}(x)=(\lambda_1^{(0)}(x),\lambda_2^{(0)}(x),\dots,\lambda_{N+1}^{(0)}(x))\in C([0,\omega],\mathbb{R}^{n(N+1)})$.

Let Condition 2.1 be met. We denote the solution of the Cauchy problem (2.1), (2.2), corresponding to $\lambda_r(x)=\lambda_r^{(0)}(x)$, by $\widetilde{V}_r^{(0)}(x,t)$. Let us define the function

$$V^{(0)}(x,t)=\begin{cases}\lambda_r^{(0)}(x)+\widetilde{V}_r^{(0)}(x,t) & \text{for } (x,t)\in\Omega_r,\ \ r=\overline{1,N},\\ \lambda_{N+1}^{(0)}(x) & \text{for } (x,t)\in[0,\omega]\cup\{T\}.\end{cases}$$

We choose some numbers $\rho_\lambda>0$, $\rho_{\widetilde{V}}>0$, $\rho_V>0$ and define the following sets:

$$S(\lambda^{(0)}(x),\rho_\lambda)=\Big\{\lambda(x)\in C([0,\omega],\mathbb{R}^{n(N+1)}):\ \ \|\lambda-\lambda^{(0)}\|_1<\rho_\lambda\Big\},$$
$$S(\widetilde{V}^{(0)}(x,[t]),\rho_{\widetilde{V}})=\Big\{\widetilde{V}(x,[t])\in C(\overline{\Omega},\Omega_r,\mathbb{R}^{nN}):\ \ \big\|(\widetilde{V}-\widetilde{V}^{(0)})[\,\cdot\,]\big\|_2<\rho_{\widetilde{V}}\Big\},$$
$$S(V^{(0)}(x,t),\rho_V)=\Big\{V(x,t)\in C(\overline{\Omega},\mathbb{R}^n):\ \ \max_{(x,t)\in\overline{\Omega}}\|V(x,t)-V^{(0)}(x,t)\|<\rho_V\Big\},$$
$$G_f(x,\rho_V)=\Big\{(x,t,u,v)\in\overline{\Omega}\times\mathbb{R}^{2n}:\ \ (x,t)\in\overline{\Omega},\ \ \|u-u^{(0)}(x,t)\|<\omega\cdot\rho_V,\ \ \|v-v^{(0)}(x,t)\|<\rho_V\Big\},$$
$$G_g(x,\rho_\lambda)=\Big\{(x,w_1,w_2)\in[0,\omega]\times\mathbb{R}^{2n}:\ \ \|w_1-V^{(0)}(x,0)\|<\rho_\lambda,\ \ \|w_2-V^{(0)}(x,T)\|<\rho_\lambda\Big\}.$$

Condition 2.2. The function $f(x,t,u,v)$ has uniformly continuous partial derivatives f'_u, f'_v in $G_f(x,\rho_u,\rho_v)$ and the following inequalities hold:

$$\|f'_u(x,t,u,v)\|\le L_1,\quad \|f'_v(x,t,u,v)\|\le L_2\ \ \forall\,(x,t,u,v)\in G_f(x,\rho_u,\rho_v).$$

The function $g(x, w_1, w_2)$ has uniformly continuous partial derivatives g'_{w_1}, g'_{w_2} in $G_g(x, \rho_\lambda)$ and the following inequalities hold:

$$\|g'_{w_1}(x, w_1, w_2)\| \le L_3, \quad \|g'_{w_2}(x, w_1, w_2)\| \le L_4, \ \ (x, w_1, w_2) \in G_g(x, \rho_\lambda).$$

Here L_i $(i = \overline{1, 4})$ are some constants.

Let Condition 2.2 be met. We take the pair $(\lambda^{(0)}(x), \widetilde{V}^{(0)}(x, [t]))$ and determine the sequence $(\lambda^{(k)}(x), \widetilde{V}^{(k)}(x, [t]))$, $k = 1, 2, \dots$, by the following algorithm.

Step 1.

(i) Find $\lambda^{(1)}(x) = (\lambda_1^{(1)}(x), \lambda_2^{(1)}(x), \dots, \lambda_{N+1}^{(1)}(x)) \in C([0, \omega], \mathbb{R}^{n(N+1)})$ by solving the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V} = \widetilde{V}^{(0)}$.

(ii) By solving the family of Cauchy problems (2.1), (2.2), where $\lambda(x) = \lambda^1(x)$, find the system of functions $\widetilde{V}^{(1)}(x, [t])$.

(iii) Define the function

$$V^{(1)}(x, t) = \begin{cases} \lambda_r^{(1)}(x) + \widetilde{V}_r^{(1)}(x, t) & \text{for } \ (x, t) \in \Omega_r, \ \ r = \overline{1, N}, \\ \lambda_{N+1}^{(1)}(x) & \text{for } \ (x, t) \in [0, \omega] \cup \{T\}. \end{cases}$$

Step k.

(i) Find $\lambda^{(k)}(x) = (\lambda_1^{(k)}(x), \lambda_2^{(k)}(x), \dots, \lambda_{N+1}^{(k)}(x)) \in C([0, \omega], \mathbb{R}^{n(N+1)})$ by solving the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V} = \widetilde{V}^{(k-1)}$.

(ii) By solving the family of Cauchy problems (2.1), (2.2), where $\lambda(x) = \lambda^2(x)$, find the system of functions $\widetilde{V}^{(2)}(x, [t])$.

(iii) Define the function

$$V^{(k)}(x, t) = \begin{cases} \lambda_r^{(k)}(x) + \widetilde{V}_r^{(k)}(x, t) & \text{for } \ (x, t) \in \Omega_r, \ \ r = \overline{1, N}, \\ \lambda_{N+1}^{(k)}(x) & \text{for } \ (x, t) \in [0, \omega] \cup \{T\}. \end{cases}$$

The following statement represents sufficient conditions for the existence of an isolated solution of the family of boundary value problems with parameters (2.1)–(2.4).

Theorem 2.1. *Let for some $h > 0 : Nh = T$ $(N = 1, 2, \dots)$, $\rho_\lambda > 0$, $\rho_{\widetilde{V}} > 0$, $\rho_V > 0$ fulfill Condition 2.1 and Condition 2.2 are met, the Jacobi matrix $\frac{\partial Q_{1,h}(x, \widetilde{w}_1, \widetilde{w}_2, \widetilde{V})}{\partial \widetilde{w}_1}$ has an inverse for $x \in [0, \omega]$ $(\widetilde{w}_1 = \lambda(x)$, $\widetilde{w}_2 = \int_0^x \lambda(\xi)\, d\xi)$ and for all $(\lambda(x), \widetilde{V}(x, [t])) \in S(\lambda^{(0)}(x), \rho_\lambda) \times S(\widetilde{V}^{(0)}(x, [t]), \rho_{\widetilde{V}})$, and let the following inequalities hold:*

(1) $\left\| \left(\dfrac{\partial}{\partial \widetilde{w}_1} Q_{1,h}\left(x, \lambda(x), \displaystyle\int_0^x \lambda(\xi)\, d\xi, \widetilde{V}\right)\right)^{-1} \right\| \le \gamma_1(h)$, *$x \in [0, \omega]$, $\gamma_1(h)$ – const;*

(2) $q_1(h) = \gamma_1(h) e^{h \cdot \gamma_1(h) L_1 \omega} \dfrac{(L_1\omega + L_2)^2 h^2}{1 - (L_1 \omega + L_2) h} < 1;$

(3) $$\frac{\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\max_{x\in[0,\omega]}\Big\|Q_{1,h}\Big(x,\lambda^{(0)}(x),\int\limits_0^x\lambda^{(0)}(\xi)\,d\xi,\widetilde{V}^{(0)}\Big)\Big\|<\rho_\lambda;$$

(4) $$\frac{\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\cdot\frac{(L_1\omega+L_2)h}{1-(L_1\omega+L_2)h}\max_{x\in[0,\omega]}\Big\|Q_{1,h}\Big(x,\lambda^{(0)}(x),\int\limits_0^x\lambda^{(0)}(\xi)\,d\xi,\widetilde{V}^{(0)}\Big)\Big\|<\rho_{\widetilde{V}};$$

(5) $\rho_\lambda+\rho_{\widetilde{V}}<\rho_V$.

Then for any $x\in[0,\omega]$ the sequence of pairs

$$\big(\lambda^{(k)}(x),\widetilde{V}^{(k)}(x,[t])\big)\in S(\lambda^{(0)}(x),\rho_\lambda)\times S(\widetilde{V}^{(0)}(x,[t]),\rho_{\widetilde{V}})$$

converges to $(\lambda^*(x),\widetilde{V}^*(x,[t]))$; an isolated solution of problem (2.1)–(2.4) *in $S(\lambda^{(0)}(x),\rho_\lambda)\times S(\widetilde{V}^{(0)}(x,[t]),\rho_{\widetilde{V}})$. Moreover, the following estimates hold:*

$$\|\lambda^*-\lambda^{(0)}\|_1\le\frac{h\cdot\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\,\frac{(L_1\omega+L_2)h}{1-h(L_1\omega+L_2)}\max_{r=\overline{1,N}}\widetilde{K}_r,$$
$$\|\widetilde{V}^*-\widetilde{V}^{(0)}\|_2\le\frac{(L_1\omega+L_2)h}{1-(L_1\omega+L_2)h}\,\|\lambda^*-\lambda^{(0)}\|_1,$$

where

$$\widetilde{K}_r=\sup_{(x,t)\in\Omega_r}\Big\|f\Big(x,t,\int\limits_0^x\lambda_r^{(0)}(\xi)\,d\xi,\lambda_r^{(0)}(x)\Big)\Big\|,\ \ r=\overline{1,N}.$$

The proof of Theorem 2.1 is based on the sequential implementation of the steps of the proposed algorithm. To find the solution of the nonlinear operator equation with respect to the family of parameters for each fixed $x\in[0,\omega]$, a sharper version of the local Hadamard theorem [12, p. 41] is used.

Remark. The conditions of Theorem 2.1 are sufficient for the feasibility and convergence of the proposed algorithm.

Due to the equivalence of problems (2.1)–(2.4) and problems (1.1), (1.2), the following statement is true.

Theorem 2.2. *Let for some $h>0$: $Nh=T$ $(N=1,2,\dots)$, $\rho_\lambda>0$, $\rho_{\widetilde{V}}>0$ and $\rho_V>0$ all conditions of Theorem 2.1 are met. Then for any $x\in[0,\omega]$ the sequence of functions $V^{(k)}(x,t)\in S(V^{(0)}(x,t),\rho_V)$ converges to $V^*(x,t)$, an isolated solution of problem* (1.1), (1.2) *in $S(V^{(0)}(x,t),\rho_V)$ and the following estimate holds:*

$$\max_{(x,t)\in\overline{\Omega}}\big\|V^*(x,t)-V^{(0)}(x,t)\big\|\le\frac{h\cdot\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\,\frac{h\cdot(L_1\omega+L_2)}{(1-h(L_1\omega+L_2))^2}\cdot K,$$

where

$$K=\max_{r=\overline{1,N}}\sup_{(x,t)\in\Omega_r}\Big\|f\Big(x,t,\int\limits_0^x(V^{(0)}(\xi,t)-V^{(0)}(\xi,(r-1)h))\,d\xi,V^{(0)}(x,t)-V^{(0)}(x,(r-1)h)\Big)\Big\|.$$

Theorem 2.2 is a corollary of Theorem 2.1.

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