On a Control-Volume Model for Fiber Coating

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1 Introduction

The study of liquid films flowing down vertical fibers is a topic of significant interest in fluid dynamics due to its wide range of industrial applications, such as in heat exchangers, desalination processes, and fiber coating technologies, one can find a review of industrial applications in [1]. The dynamics of such films are influenced by various factors including gravity, viscosity, surface tension, and the geometry of the fiber. This introduction aims to provide an overview of the current research in this area, highlighting key contributions from recent and well-cited studies.

Ruan et al. (2021) presented a comprehensive framework using control-volume methods to model liquid films on vertical fibers. Their work included both numerical simulations and experimental validations, demonstrating the formation of traveling wave solutions and droplet patterns on fibers [8]. Similarly, Kalliadasis and Chang (2020) focused on direct numerical simulations of thin film flows, employing domain mapping techniques to solve the Navier–Stokes equations [4].

Quere (2003) offered an extensive review of the fluid dynamics involved in coating fibers, highlighting various flow regimes observed in experiments [7]. Kalliadasis and Chang (1994) provided a detailed analysis of droplet formation processes during the coating of vertical fibers, using both experimental and theoretical approaches to elucidate the underlying mechanisms [4]. This foundational work is supported by later studies such as those by Ji and Witelski (2017), who examined the three-dimensional dynamics of thin liquid films under various conditions, revealing complex behaviours observed in experiments [3].

2 Setting of the problem and the main results

Following the framework of Ruan et al. [2, 8] which is based on a control-volume approach, we expresses the conservation of mass and axial momentum via a coupled system for the fluid film radius $h(x,t)$ and the mean axial velocity $u(x,t)$:

$$u_t + a\Big(\frac{u^2}{2}\Big)_x + b\kappa_x = c\,\frac{[(h^2-1)u_x]_x}{h^2-1} + 1 - \frac{u}{g(h)} \ \text{ in } \Omega_T, \tag{2.1}$$

$$2hh_t + a[u(h^2-1)]_x = 0 \ \text{ in } \Omega_T, \tag{2.2}$$

$$u = h_x = 0 \ \text{ on } \partial\Omega \times (0,T), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad h(x,0) = h_0(x), \tag{2.4}$$

where $\Omega \subset \mathbb{R}^1$ is an open interval, $\Omega_T := \Omega \times (0,T)$. Here, the dimensionless parameter a represents the square of the Froude number, b is the reciprocal of the Bond number, c represents the ratio of axial viscous to gravitational forces, and $g(h) = h^2 - 1$ represents the axial velocity profile. The film thickness is given by $h(x,t) - 1$, and κ represents the combined azimuthal and streamwise curvatures of the free surface:

$$\kappa = \Big(1 - \frac{1}{2}\,h_x^2\Big)h^{-1} - h_{xx}.$$

Time evolution of solutions for thin liquid film models with full and approximated curvature terms was compared for example in [6], where the authors showed that the qualitative behaviour of solutions (with periodic boundary conditions) is almost the same.

Let us denote by

$$v = h^2 - 1.$$

Then we can rewrite (2.1) and (2.2) in the following form:

$$u_t + a\Big(\frac{u^2}{2}\Big)_x + b\kappa_x = c\,\frac{(v\,u_x)_x}{v} + 1 - \frac{u}{v} \ \text{ in } \Omega_T, \tag{2.5}$$

$$v_t + a(uv)_x = 0 \ \text{ in } \Omega_T. \tag{2.6}$$

Integrating (2.6) in Ω_t, we find that $v(x,t)$ satisfies

$$\int\limits_{\Omega} v(x,t)\,dx = \int\limits_{\Omega} v_0(x)\,dx := M > 0 \ \ \forall\, t \geqslant 0. \tag{2.7}$$

Furthermore, we assume that the initial data (v_0, u_0) satisfy

$$\begin{gathered} h_0 \geqslant 1, \ \text{ i.e. } \ v_0 := h_0^2 - 1 \geqslant 0 \ \text{ for all } \ x \in \overline{\Omega}, \\ \sqrt{v_0} \in H^1(\Omega), \ \ h_0 h_{0,x}^2, v_0 u_0^2, -\log(v_0) \in L^1(\Omega). \end{gathered} \tag{2.8}$$

Definition 2.1. A pair (h,u) is a weak solution to (2.5), (2.6) with the boundary conditions (2.3) and the initial conditions (h_0,u_0) if $1 \leqslant h \in C(\overline{Q}_T)$, $v = h^2 - 1$, and u satisfy the regularity properties

$$\begin{gathered} \sqrt{v} \in L^\infty(0,T;H^1(\Omega)), \ \ -\log(v), vu^2 \in L^\infty(0,T;L^1(\Omega)), \\ hh_x^2 \in L^\infty(0,T;L^1(\Omega)), \ \ h^{-\frac{1}{4}}h_x \in L^4(\Omega_T), \\ \sqrt{h}\,h_{xx}, \chi_{\{v>0\}}\sqrt{v}\,u_x, u \in L^2(\Omega_T), \end{gathered}$$

and the following holds

$$\iint\limits_{\Omega_T} v\phi_t\;dx\,dt + \int\limits_{\Omega} v_0\phi(x,0)\,dx + a\iint\limits_{\Omega_T} uv\phi_x\;dx\,dt = 0,$$

$$\begin{aligned} \iint\limits_{\Omega_T} uv\psi_t\;dx\,dt + \int\limits_{\Omega} u_0v_0\psi(x,0)\,dx + \frac{a}{2}\iint\limits_{\Omega_T} \chi_{\{v>0\}}vu^2\psi_x\;dx\,dt \\ + 2b\iint\limits_{\Omega_T}\Big(1 - \frac{1}{2}\,h_x^2 - hh_{xx}\Big)h_x\psi\;dx\,dt + b\iint\limits_{\Omega_T}\Big(\Big(1 - \frac{1}{2}\,h_x^2\Big)h^{-1} - h_{xx}\Big)v\psi_x\;dx\,dt \\ - c\iint\limits_{\Omega_T}\chi_{\{v>0\}}vu_x\psi_x\;dx\,dt + \iint\limits_{\Omega_T}(v-u)\psi\;dx\,dt = 0 \end{aligned}$$

for all $\phi \in C_c^\infty(\overline{\Omega}_T)$ and $\psi \in C_c^\infty(\overline{\Omega}_T)$ such that $\phi(x,T) = \psi(x,T) = 0$.

We note that the set $v = 0$ coincides with the set $h = 1$. Based on Definition 2.1, we will establish the existence of weak solutions to the problem and prove the following theorem.

Theorem. *Let the initial data* (h_0, u_0) *satisfy* (2.7), (2.8) *and* $T > 0$. *Then there exists a weak solution* (h, u) *in the sense of Definition 2.1, where* $v = h^2 - 1$. *Moreover, the set* $\{v(\,\cdot\,, t) = 0\}$ *has Lebesgue measure zero for any* $t \in [0, T]$.

The proof of Theorem is based on the method of energy-entropy a priori estimates which was active developed in [5, 9].

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