On the Optimization Problem for the Quasi-Linear Neutral Functional-Differential Equation with the Discontinuous Initial Condition

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In the paper, necessary conditions of optimality of the delay parameter containing in the phase coordinates, the initial vector and initial function, the control function are obtained for the quasi-linear neutral optimization problem with the discontinuous initial condition.

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$; let $I=[t_0,t_1]$ be a fixed interval and let $\sigma>0$, $\tau_2>\tau_1>0$ be given numbers,with $t_0+\max\{\sigma,\tau_2\}<t_1$. Suppose that $O\subset\mathbb{R}^n$, $U\subset\mathbb{R}^r$ are compact and convex sets. Further, the $n\times n$-dimensional matrix function $A(t,x)$ is continuous on the set $I\times O$ and continuously differentiable with respect to $x^i, i=1,2,\dots,n$; the n-dimensional function $f(t,x,y,u)$ is continuous on the set $I\times O^2\times U$ and continuously differentiable with respect to x, y, u. We denote by Φ and Ω the sets of continuously differentiable initial functions $\varphi(t)\in O$, $t\in I_1=[\widehat{\tau},t_0]$, where $\widehat{\tau}=t_0-\max\{\sigma,\tau_2\}$ and measurable control functions $u(t)\in U$, $t\in I$, respectively.

To each element

$$w=(\tau,x_0,\varphi(t),u(t))\in W=(\tau_1,\tau_2)\times O\times\Phi\times\Omega$$

we assign the quasi-linear controlled neutral functional-differential equation

$$\dot{x}(t)=A(t,x(t))\dot{x}(t-\sigma)+f\big(t,x(t),x(t-\tau),u(t)\big),\quad t\in I \tag{1}$$

with the discontinuous initial condition

$$x(t)=\varphi(t),\quad t\in[\widehat{\tau},t_0),\qquad x(t_0)=x_0. \tag{2}$$

The condition (2) is called the discontinuous initial condition because in general $\varphi(t_0)\neq x_0$. Discontinuity at the initial moment may be related to the instant change in a dynamical process (changes of investment, environment and so on).

Definition 1. Let $w\in W$, a function $x(t)=x(t;w)\in O$, $t\in[\widehat{\tau},t_1]$ is called a solution of equation (1) with condition (2) or a solution corresponding to the element w if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Let the scalar-valued functions $q^i(\tau, x_0, x)$, $i = 0, 1, \dots, l$, be continuously differentiable on $[\tau_1, \tau_2] \times O^2$.

Definition 2. An element $w = (\tau, x_0, \varphi(t), u(t)) \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$, satisfying the conditions

$$q^i(\tau, x_0, x(t_1)) = 0, \ \ i = 1, 2, \dots, l. \tag{3}$$

By W_0 we denote the set of admissible elements.

Definition 3. An element $w_0 = (\tau_0, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$q^0(\tau_0, x_{00}, x_0(t_1)) \le q^0(\tau, x_0, x(t_1)) \tag{4}$$

holds, where $x_0(t) = x(t; w_0)$.

(1)–(4) is called the quasi-linear neutral optimization problem with the discontinuous initial condition.

Theorem 1. *Let $w_0 = (\tau_0, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ be an optimal element and let $x_0(t)$ be the corresponding solution, with*

$$t_0 + \tau_0 \notin \{t_1 - \sigma, t_1 - 2\sigma, \dots\}$$

and the function $u_0(t)$ is continuous at the point $t_0 + \tau_0$. Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, with $\pi_0 \le 0$, and a solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot{\chi}(t) = -\psi(t)\Big\{\dfrac{\partial}{\partial x}\big[A[t]\dot{x}_0(t-\sigma)\big] + f_x[t]\Big\} - \psi(t+\tau_0) f_y[t+\tau_0], \\ \psi(t) = \chi(t) + \psi(t+\sigma)A[t+\sigma], \ \ t \in I \end{cases}$$

with the initial condition

$$\chi(t_1) = \psi(t_1) = \pi Q_{0x}, \quad \chi(t) = \psi(t) = 0, \ \ t > t_1,$$

where

$$Q = (q^0, q^1, \dots, q^l)^T, \quad Q_{0x} = \frac{\partial Q(\tau_0, x_{00}, x_0(t_1))}{\partial x},$$

$$\frac{\partial}{\partial x}\Big[A[t]\dot{x}_0(t-\sigma)\Big] = \frac{\partial}{\partial x}\big[A(t,x)\dot{x}_0(t-\sigma)\big]_{x=x_0(t)}, \quad A[t] = A(t, x_0(t)),$$

$$f_y[t] = f_y\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big),$$

such that the following conditions hold:

– *the condition for the delay τ_0*

$$\pi Q_{0\tau} = \psi(t_0+\tau_0)\Big[f\big(t_0+\tau_0, x_0(t_0+\tau_0), x_{00}, u_0(t_0+\tau_0)\big) - f\big(t_0+\tau_0, x_0(t_0+\tau_0), \varphi_0(t_0), u_0(t_0+\tau_0)\big)\Big] + \int_{t_0}^{t_1} \psi(t) f_y[t]\dot{x}_0(t-\tau_0)\,dt;$$

– *the condition for the initial vector x_{00}*

$$\big(\pi Q_{0x_0} + \chi(t_0)\big)x_{00} = \max_{x_0 \in O}\big(\pi Q_{0x_0} + \chi(t_0)\big)x_0;$$

– *the condition for the initial function* $\varphi_0(t)$

$$\int\limits_{t_0-\sigma}^{t_0} \psi(t+\sigma)A[t+\sigma]\dot{\varphi}_0(t)\,dt + \int\limits_{t_0-\tau_0}^{t_0} \psi(t+\tau_0)f_y[t+\tau_0]\varphi_0(t)\,dt$$
$$= \max_{\varphi(t)\in\Phi} \int\limits_{t_0-\sigma}^{t_0} \psi(t+\sigma)A[t+\sigma]\dot{\varphi}(t)\,dt + \int\limits_{t_0-\tau_0}^{t_0} \psi(t+\tau_0)f_y[t+\tau_0]\varphi(t)\,dt;$$

– *the condition for the control function* $u_0(t)$

$$\int\limits_{t_0}^{t_1} \psi(t)f_u[t]u_0(t)\,dt = \max_{u(t)\in\Omega} \int\limits_{t_0}^{t_1} \psi(t)f_u[t]u(t)\,dt.$$

Theorem 1 is proved by the scheme given in [2] on the basis of the representation formula of a solution [1]. The case when $A(t,x)=A(t)$ and Q does not depend on the parameter τ is considered in [3]. Now we consider a particular case of the problem (1)–(4):

$$\dot{x}(t) = A(t)\dot{x}(t-\sigma) + B(t)x(t) + C(t)x(t-\tau) + D(t)u(t), \ \ t\in I, \tag{5}$$
$$x(t) = \varphi(t), \quad x(t_0) = x_0, \tag{6}$$
$$q^i(\tau, x(t_1)) = 0, \ \ i = 1,2,\dots,l, \tag{7}$$
$$q^0(\tau, x(t_1)) \to \min. \tag{8}$$

Here $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are the continuous matrix functions with dimensions $n\times n$ and $n\times r$, respectively; $\varphi(t)$ is a fixed initial function; x_0 is a fixed initial vector. In this case we have $w=(\tau,u(t))\in W=(\tau_1,\tau_2)\times\Omega$ and $w_0=(\tau_0,u_0(t))$;

$$Q(\tau,x) = (q^0(\tau,x),\dots,q^l(\tau,x))^T, \quad Q_{0x} = \frac{\partial Q(\tau_0,x_0(t_1))}{\partial x}.$$

Theorem 2. *Let* $w_0=(\tau_0,u_0(t))$ *be an optimal element for problem* (5)–(8) *and let* $x_0(t)$ *be the corresponding solution, with*

$$t_0+\tau_0 \notin \{t_1-\sigma, t_1-2\sigma,\dots\}$$

and the function $u_0(t)$ *is continuous at the point* $t_0+\tau_0$. *Then there exist a vector* $\pi=(\pi_0,\dots,\pi_l)\neq 0$ *with* $\pi_0\le 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system*

$$\begin{cases} \dot{\chi}(t) = -\psi(t)B(t) - \psi(t+\tau_0)C(t+\tau_0), \\ \psi(t) = \chi(t) + \psi(t+\sigma)A(t+\sigma), t\in I \end{cases}$$

with the initial condition

$$\chi(t_1) = \psi(t_1) = \pi Q_{0x}, \quad \chi(t) = \psi(t) = 0, \ \ t > t_1,$$

such that the following conditions hold:

– *the condition for the delay* τ_0

$$\pi Q_{0\tau} = \psi(t_0+\tau_0)C(t_0+\tau_0)[x_0-\varphi(t_0)] + \int\limits_{t_0}^{t_1} \psi(t)C(t)\dot{x}_0(t-\tau_0)\,dt;$$

– *the condition for the control function* $u_0(t)$

$$\int_{t_0}^{t_1} \psi(t)D(t)u_0(t)\,dt = \max_{u(t)\in\Omega} \int_{t_0}^{t_1} \psi(t)D(t)u(t)\,dt.$$

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