Dirichlet Problem for Singular Fractional Differential Equations with Given Maximal Value for Positive Solutions

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1 Introduction

Let $J=[0,1]$, $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$, while $\|x\|_{L^1}=\int\limits_0^1|x(t)|\,\mathrm{d}t$ is the norm in $L^1(J)$.

We discuss the singular fractional differential equation

$$D^\alpha x(t)+\mu\,f\big(t,x(t),D^\beta x(t)\big)=0, \tag{1.1}$$

depending on the real parameter μ. Here $\alpha\in(1,2]$, $\beta\in(0,\alpha-1]$, f satisfies the local Carathéodory conditions on $J\times(0,\infty)\times\mathbb{R}$, $\lim\limits_{x\to0+}f(t,x,y)=\infty$ for a.e. $t\in J$ and $y\in\mathbb{R}$, and D^γ is the Riemann–Liouville fractional derivative of order γ.

Together with equation (1.1) the boundary conditions

$$x(0)=0,\quad x(1)=0, \tag{1.2}$$

$$\max\big\{x(t):\ t\in J\big\}=A \tag{1.3}$$

are considered, where $A>0$ is given.

We are looking for a value of the parameter μ in (1.1) for which problem (1.1)–(1.3) has a positive solution.

Definition. We say that $x:J\to\mathbb{R}$ is *a positive solution of problem* (1.1)–(1.3) if

(a) $x,D^\beta x\in C(J)$, $D^\alpha x\in L^1(J)$, $x>0$ on $(0,1)$,

(b) x satisfies the boundary conditions (1.2), (1.3),

(c) there exists $\mu_*>0$ such that (1.1) for $\mu=\mu_*$ holds for a.e. $t\in J$.

The special case of (1.1) (for $\alpha=2$, $\beta=1$) is the differential equation

$$x''(t)=\mu f(t,x(t),x'(t)).$$

The existence result for solutions of this equation satisfying the boundary conditions (1.2), (1.3) was given in [1].

The Riemann–Liouville fractional derivative $D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is defined as [3, 4]

$$D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\,x(s)\,\mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of γ. If $\gamma \in \mathbb{N}$, then $D^\gamma x(t) = x^{(\gamma)}(t)$. The Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is given as

$$I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)\, ds$$

and I^0 is the identity operator. Γ is the Euler gamma function.

We work with the following growth conditions for the function f in (1.1):

(H_1) There exists $m > 0$ such that

$$f(t,x,y) \ge m(1-t)^{2-\alpha} \quad \text{for a.e. } t \in J \text{ and all } (x,y) \in (0,\infty) \times \mathbb{R}.$$

(H_2) For a.e. $t \in J$ and all $(x,y) \in (0,\infty) \times \mathbb{R}$,

$$f(t,x,y) \le \phi(t)g(x) + \rho(t)\big(p(x) + w(|y|)\big),$$

where $\phi, \rho \in L^1(J)$, $g \in C(0,\infty)$, $p, w \in C[0,\infty)$ are positive, g is nonincreasing, p, w are nondecreasing and

$$\lim_{\kappa\to 0+} \kappa \int_0^1 \phi(t) g(\kappa t(1-t))\, dt = 0, \qquad \lim_{v\to\infty} \frac{w(v)}{v} = 0.$$

The existence results for problem (1.1)–(1.3) are proved by the combination of the regularization and sequential techniques with the Leray–Schauder degree method.

2 Preliminaries

Let α, β be from (1.1) and

$$G(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \le s \le t \le 1, \\[2ex] \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Then for $h \in L^1(J)$

$$\int_0^1 G(t,s)h(s)\, ds = -I^\alpha h(t) + I^\alpha h(t)\big|_{t=1} t^{\alpha-1},$$

and

$$D^\beta \int_0^1 G(t,s)h(s)\, ds = -I^{\alpha-\beta}h(t) + t^{\alpha-\beta-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\beta)} h(s)\, ds \tag{2.1}$$

since

$$D^\beta t^{\alpha-1} = \frac{t^{\alpha-\beta-1}\Gamma(\alpha)}{\Gamma(\alpha-\beta)}.$$

Let $X = \{x \in C(J) :\ D^\beta x \in C(J)\}$. X is a Banach space equipped with the norm

$$\|x\|_* = \max\big\{\|x\|, \|D^\beta x\|\big\}.$$

Lemma 2.1. *Let $h \in L^1(J)$. Then*

$$x(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s \tag{2.2}$$

is the unique solution in X of the equation

$$D^\alpha x(t) + h(t) = 0, \tag{2.3}$$

satisfying the Dirichlet condition (1.2).

Proof. By [2, Lemma 2.2], x is the unique solution of problem (2.3), (1.2) in $C(J)$. Since $\alpha-\beta \geq 1$, $I^{\alpha-\beta}h \in C(J)$. Hence (2.1) and (2.2) give $D^\beta x \in C(J)$. Consequently, $x \in X$. □

Lemma 2.2. *Let $m > 0$ be from (H_1), $h \in L^1(J)$, $h(t) \geq m(1-t)^{2-\alpha}$ for a.e. $t \in J$ and let*

$$K = \frac{m}{2\Gamma(\alpha-1)}\,.$$

Then

$$\int_0^1 G(t,s)h(s)\,\mathrm{d}s \geq Kt(1-t) \ \text{ for } \ t \in J.$$

3 Regular problems

For $n \in \mathbb{N}$, let

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \ x \geq \dfrac{1}{n}\,, \\ f\Big(t,\dfrac{1}{n},y\Big) & \text{if } \ x < \dfrac{1}{n} \end{cases}$$

for a.e. $t \in J$ and $y \in \mathbb{R}$. Then f_n satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2$.

We now discuss the regular fractional differential equation

$$D^\alpha x(t) + \mu f_n\big(t, x(t), D^\beta x(t)\big) = 0 \tag{3.1}$$

together with the boundary conditions (1.2) and

$$\max\big\{x(t):\ t \in J\big\} = \lambda A, \tag{3.2}$$

where $A > 0$ is from (1.3) and $\lambda \in (0,1]$.

Lemma 3.1. *Let (H_1) and (H_2) hold and let $K > 0$ be from Lemma 2.2. Then there exists a positive constant P independent of $\lambda \in (0,1]$ such that for solutions x of problem* (3.1), (1.2), (3.2) *with $\mu = \mu_x$ in* (3.1) *the estimates*

$$\|D^\beta x\| < P, \ \ 0 < \mu_x \leq \frac{4A}{K}$$

hold and $x > 0$ on $(0,1)$.

Let $Y = X \times \mathbb{R}$ and an operator $\mathcal{L}$ acting on $Y \times [0,1]$ be given by the formula

$$\mathcal{L}(x,\mu,\lambda)(t) = \mu \int_0^1 G(t,s)\Big(m(1-\lambda)(1-s)^{2-\alpha} + \lambda f_n\big(s, x(s), D^\beta x(s)\big)\Big)\, \mathrm{d}s,$$

where $m > 0$ is from (H_1).

Lemma 3.2. *Let (H_1) hold. Then $\mathcal{L} : Y \times [0,1] \to X$ and $\mathcal{L}$ is a completely continuous operator.*

Let $A > 0$ be from (1.3), $K > 0$ from Lemma 2.2 and $P > 0$ from Lemma 3.1. Let

$$\Omega = \Big\{(x,\mu) \in Y : \ \|x\| < A + 1, \ \ \|D^\beta x\| < P, \ \ |\mu| < \frac{4A}{K} + 1\Big\}$$

and "deg" stand for the Leray–Schauder degree, $\mathcal{I}$ be the identical operator on Y, $\theta = (0,0) \in Y$.

Lemma 3.3. *Let (H_1) and (H_2) hold. Then problem $(3.1), (1.2), (1.3)$ has at least one positive solution.*

Sketch of the proof.

Step 1. Let $\mathcal{K} : \overline{\Omega} \times [0,1] \to Y$,

$$\mathcal{K}(x,\mu,\lambda) = \big(\mathcal{L}(x,\mu,\lambda), \Lambda(x,\mu,\lambda)\big),$$

where $\Lambda : Y \times [0,1] \to \mathbb{R}$,

$$\Lambda(x,\mu,\lambda) = \lambda\Big(\max\big\{x(t) : \ t \in J\big\} + \min\big\{x(t) : \ t \in J\big\}\Big) + (1-\lambda)x\Big(\frac{1}{2}\Big) + \mu.$$

$\mathcal{K}$ is a compact operator. Since $\mathcal{K}(x,\mu,\lambda) \neq (x,\mu)$ for $(x,\mu) \in \partial\Omega$, $\lambda \in [0,1]$ and $\mathcal{K}(\,\cdot\,,\,\cdot\,,0)$ is an odd operator, we conclude from the Borsuk antipodal theorem and the homotopy property that $\deg(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,0), \Omega, \theta) \neq 0$ and

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1), \Omega, \theta\big) \neq 0. \tag{3.3}$$

Step 2. Let $\mathcal{H} : \overline{\Omega} \times [0,1] \to Y$,

$$\mathcal{H}(x,\mu,\lambda) = \big(\mathcal{L}(x,\mu,1), \ \Phi(x,\mu,\lambda)\big),$$

where $\Phi : \overline{\Omega} \times [0,1] \to \mathbb{R}$,

$$\Phi(x,\mu,\lambda) = \max\big\{x(t) : \ t \in J\big\} + \min\big\{x(t) : \ t \in J\big\} - \lambda A + \mu,$$

and $A > 0$ is from (1.3). $\mathcal{H}$ is a compact operator and if $\mathcal{H}(x_*,\mu_*,1) = (x_*,\mu_*)$ for some $(x_*,\mu_*) \in \overline{\Omega}$, then x_* is a positive solution of problem $(3.1), (1.2), (1.3)$ for $\mu = \mu_*$ in (3.1). Since $\mathcal{H}(x,\mu,\lambda) \neq (x,\mu)$ for $(x,\mu) \in \partial\Omega$ and $\lambda \in [0,1]$, we conclude from $\mathcal{H}(\,\cdot\,,\,\cdot\,,0) = \mathcal{K}(\,\cdot\,,\,\cdot\,,1)$, the homotopy property and (3.3) that

$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1), \Omega, \theta\big) \neq 0.$$

Hence there exists a fixed point $(x_0,\mu_0) \in \Omega$ of $\mathcal{H}(\,\cdot\,,\,\cdot\,,1)$. Therefore x_0 is a positive solution of problem $(3.1), (1.2), (1.3)$ for $\mu = \mu_0$ in (3.1). □

4 Problem (1.1)–(1.3)

Theorem 4.1. *Let* (H_1) *and* (H_2) *hold. Then problem* (1.1)–(1.3) *has at least one positive solution.*

Sketch of the proof. By Lemmas 3.1 and 3.3, for each $n \in \mathbb{N}$ there exists a positive solution $x_n \in X$ of problem $(3.1), (1.2), (1.3)$ for $\mu = \mu_n$ in (3.1), $x_n > 0$ on $(0,1)$, $\|x_n\| = A$, $\|D^\beta x_n\| < P$ and $0 < \mu_n \leq 4A/K$. Hence the sequence $\{(x_n, \mu_n)\}$ is bounded in $X \times \mathbb{R}$. We begin by proving that $\{\mu_n\}$ has a positive lower bound $\Delta > 0$ and the sequences $\{x_n\}$ $\{D^\beta x_n\}$ are equicontinuous on J. Consequently, $\{(x_n, \mu_n)\}$ is relatively compact in Y. Without loss of generality we can assume that $\{(x_n, \mu_n)\}$ is convergent in Y and let $(x, \rho) \in Y$ be its limit. Then $\rho \geq \Delta$, $x(t) \geq \Delta K t(1-t)$, $\|D^\beta x\| \leq P$, x satisfies the boundary condition (1.2), (1.3) and

$$\lim_{n\to\infty} f_n\big(t, x_n(t), D^\beta x_n(t)\big) = f\big(t, x(t), D^\beta x(t)\big) \text{ for a.e. } t \in J.$$

Letting $n \to \infty$ in the equality

$$x_n(t) = \mu_n \int_0^1 G(t,s) f_n\big(s, x_n(s), D^\beta x_n(s)\big)\, \mathrm{d}s,$$

we get

$$x(t) = \rho \int_0^1 G(t,s) f\big(s, x(s), D^\beta x(t)\big)\, \mathrm{d}s, \quad t \in J,$$

by the Lebesgue dominated convergence theorem. Consequently, x is a positive solution of problem (1.1)–(1.3) for $\mu = \rho$ in (1.1). ☐

Example. Let $r_1, r_2 \in L^1(J)$ and $q_1 \in C(J)$ be nonnegative, $q_2 \in C(J)$ be positive, $\nu, \tau \in (0,1)$ and

$$f(t,x,y) = r_1(t) + \frac{q_1(t)}{x^\nu} + q_2(t)e^x + r_2(t)|y|^\tau \text{ for a.e. } t \in J, \ \ x > 0, \ \ y \in \mathbb{R}.$$

Then f satisfies the local Carathéodory conditions on $J \times (0,\infty) \times \mathbb{R}$, and the conditions (H_1), (H_2). Hence, by Theorem 4.1, there exists a positive solution of the problem

$$\left.\begin{aligned} D^\alpha x(t) + \mu\Big(r_1(t) + \frac{q_1(t)}{(x(t))^\nu} + q_2(t)e^{x(t)} + r_2(t)|D^\beta x(t)|^\tau\Big) = 0, \\ x(0) = 0, \ \ x(1) = 0, \ \ \max\big\{x(t) : \ t \in J\big\} = A, \ \ A > 0. \end{aligned}\right\}$$

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