On Investigation of Non-Linear Differential Systems with Mixed Boundary Conditions

A. Rontó

Faculty of Business and Management, Brno University of Technology
Brno, Czech Republic
E-mail: andras.ronto@vut.cz

M. Rontó

Institute of Mathematics, University of Miskolc, Miskolc, Hungary
E-mail: matronto@uni-miskolc.hu

N. Rontóová

Faculty of Business and Management, Brno University of Technology
Brno, Czech Republic
E-mail: Natalie.Rontoova@vut.cz

I. Varga

Faculty of Mathematics, Uzhhorod National University, Uzhhorod, Ukraine
E-mail: iana.varga@uzhnu.edu.ua

We study a non-linear differential system with mixed boundary conditions on a compact interval $[a,b]$

$$\begin{aligned} x'(t) &= f_1(t,x(t),y(t)), \quad t\in[a,b],\\ y'(t) &= f_2(t,x(t),y(t)), \quad t\in[a,b], \end{aligned} \tag{1}$$

$$x(a)=x(b), \tag{2}$$

$$\phi(y)=d, \tag{3}$$

where $x:[a,b]\to\mathbb{R}^p$, $y:[a,b]\to\mathbb{R}^q$, $d\in\mathbb{R}^q$. It is supposed that f_1, f_2 are continuous as functions $f_1:[a,b]\times U\times V\to\mathbb{R}^p$, $f_2:[a,b]\times U\times V\to\mathbb{R}^q$, where bounded sets $U\subset\mathbb{R}^p$, $V\subset\mathbb{R}^q$ are specified later (see (4)). We also assume the continuity of $\phi:V\to\mathbb{R}^q$. Continuously differentiable solutions of problem (1)–(3) are considered. For problem (1)–(3) we will use an approach similar to that of [2,3].

For vectors $x=\operatorname{col}(x_1,\dots,x_n)\in\mathbb{R}^n$ the notation $|x|=\operatorname{col}(|x_1|,\dots,|x_n|)$ is used and the inequalities between vectors are understood componentwise; the operations max and min for vectors are understood similarly. I denotes the identity matrix. For a non-negative vector ϱ, we define the componentwise ϱ-neighbourhood of a point z by putting

$$\mathscr{O}_\varrho(z)=\big\{\xi\in\mathbb{R}^n:\ \ |\xi-z|\le\varrho\big\}.$$

The ϱ-neighbourhood of a set $\Omega\subset\mathbb{R}^n$ is then defined as $\mathscr{O}_\varrho(\Omega)=\bigcup\limits_{z\in\Omega}\mathscr{O}_\varrho(z)$. The particular sets Ω and values of ϱ used in the assumptions are specified below in (4), (5).

We will use a reduction of the given problem to a family of simpler auxiliary boundary value problems [2]. Let us fix certain compact convex sets $\mathscr{V}_a \subset \mathbb{R}^q$, $\mathscr{V}_b \subset \mathbb{R}^q$ and $\mathscr{U} \subset \mathbb{R}^p$, take some positive vectors $\varrho_U \in \mathbb{R}^p$, $\varrho_V \in \mathbb{R}^q$ and put

$$U = \mathscr{O}_{\varrho_U}(\mathscr{U}), \quad V = \mathscr{O}_{\varrho_V}(\mathscr{C}(\mathscr{V}_a, \mathscr{V}_b)), \tag{4}$$

where

$$\mathscr{C}(X,Y) = \big\{(1-\theta)x + \theta y : \ \ x \in X, \ \ y \in Y, \ \ 0 \le \theta \le 1\big\}.$$

It is convenient to choose the sets $\mathscr{U}$, $\mathscr{V}_a$, $\mathscr{V}_b$ as some parallelepipeds. We will consider solutions (x,y) of problem (1)–(3) with values $x(a) = x(b) \in \mathscr{U}$, $y(a) \in \mathscr{V}_a$, $y(b) \in \mathscr{V}_b$ and range in $U \times V$.

Introduce the notation

$$\delta_{U\times V}(f_k) = \frac{1}{2}\Big(\max_{[a,b]\times U\times V} f_k - \min_{[a,b]\times U\times V} f_k\Big), \ \ k = 1,2,$$

and assume that the positive vectors ϱ_U, ϱ_V can be chosen so that

$$\varrho_U \ge \frac{b-a}{2}\,\delta_{U\times V}(f_1), \quad \varrho_V \ge \frac{b-a}{2}\,\delta_{U\times V}(f_2). \tag{5}$$

Let f_1, f_2 satisfy the Lipchitz condition on U, V:

$$\big|f_k(t,x,y) - f_k(t,\widetilde{x},\widetilde{y})\big| \le K_{k1}|x-\widetilde{x}| + K_{k2}|y-\widetilde{y}|, \ \ k = 1,2, \tag{6}$$

for $t \in [a,b]$, $\{x,\widetilde{x}\} \subset U$, $\{y,\widetilde{y}\} \subset V$, where K_{11}, K_{12}, K_{21}, K_{22} are positive matrices of dimensions $p\times p$, $p\times q$, $q\times p$, $q\times q$. We assume that the maximal in modulus eigenvalue of the matrix $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is small enough:

$$r(Q) < 1, \tag{7}$$

where $Q = \frac{3}{10}\,(b-a)K$.

We introduce the vectors of parameters $z \in \mathbb{R}^p$, $\gamma \in \mathbb{R}^q$, $\lambda \in \mathbb{R}^q$ by formally putting

$$z = x(a) = x(b), \quad \gamma = y(a), \quad \lambda = y(b)$$

and, instead of problem (1)–(3), consider the following two auxiliary boundary value problems with periodic and two-point linear separated conditions at a and b:

$$x'(t) = f_1(t,x,y), \ \ t \in [a,b], \tag{8}$$

$$x(a) = z, \ \ x(b) = z \tag{9}$$

and

$$y'(t) = f_2(t,x,y), \ \ t \in [a,b], \tag{10}$$

$$y(a) = \gamma, \ \ y(b) = \lambda. \tag{11}$$

As will be seen from statements below, there is a certain relation to the original problem depending on the choice of the values of z, γ and λ. Let us relate problems (8), (9) and (10), (11) to the sequences of functions

$$x_{m+1}(t,z,\gamma,\lambda) = z + \int_a^t f_1\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds - \frac{t-a}{b-a}\int_a^b f_1\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds \tag{12}$$

and

$$y_{m+1}(t,z,\gamma,\lambda) = \gamma + \int_a^t f_2\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\, ds$$
$$- \frac{t-a}{b-a}\int_a^b f_2\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\, ds + \frac{t-a}{b-a}(\lambda-\gamma), \quad (13)$$

where $t \in [a,b]$, $m = 0,1,\dots,$

$$x_0(t,z) = z, \quad y_0(t,\gamma,\lambda) = \gamma + \frac{t-a}{b-a}(\lambda-\gamma).$$

Theorem 1. *Let conditions* (5), (6), (7) *be fulfilled. Then, for all fixed* $z \in \mathscr{U}$, $\gamma \in \mathscr{V}_a$, $\lambda \in \mathscr{V}_b$:

1. *Each of the functions of sequence* (12) *has range in* U, *is continuosly differentiable on* $[a,b]$, *and satisfies conditions* (9). *The limit*
$$x_\infty(t,z,\gamma,\lambda) = \lim_{m\to\infty} x_m(t,z,\gamma,\lambda) \quad (14)$$
exists uniformly in $(t,z,\gamma,\lambda) \in [a,b] \times \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b$. *Function* (14) *satisfies the boundary condition* (9).

2. *Each of the functions of sequence* (13) *has range in* V, *is continuosly differentiable on* $[a,b]$, *and satisfies conditions* (11). *The limit*
$$y_\infty(t,z,\gamma,\lambda) = \lim_{m\to\infty} y_m(t,z,\gamma,\lambda) \quad (15)$$
exists uniformly in $(t,z,\gamma,\lambda) \in [a,b] \times \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b$. *Function* (15) *satisfies the boundary condition* (11).

3. *The functions* $x_\infty(\,\cdot\,,z,\gamma,\lambda)$, $y_\infty(\,\cdot\,,z,\gamma,\lambda)$ *form the unique continuously differentiable solution of the system of integral equations*
$$x(t) = z + \int_a^t f_1(s,x(s),y(s))\, ds - \frac{t-a}{b-a}\int_a^b f_1(s,x(s),y(s))\, ds,$$
$$y(t) = \gamma + \frac{t-a}{b-a}(\lambda-\gamma) + \int_a^t f_2(s,x(s),y(s))\, ds - \frac{t-a}{b-a}\int_a^b f_2(s,x(s),y(s))\, ds.$$

4. *The following error estimate holds:*
$$\big|x_\infty(t,z,\gamma,\lambda) - x_m(t,z,\gamma,\lambda)\big| \le \frac{10}{9}\,\alpha_1(t)\left\{Q^m(I_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{U\times V}(f_1)\\ \delta_{U\times V}(f_2)\end{pmatrix}\right\}_1^p,$$
$$\big|y_\infty(t,z,\gamma,\lambda) - y_m(t,z,\gamma,\lambda)\big| \le \frac{10}{9}\,\alpha_1(t)\left\{Q^m(I_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{U\times V}(f_1)\\ \delta_{U\times V}(f_2)\end{pmatrix}\right\}_{p+1}^{p+q},$$
where
$$\alpha_1(t) = 2(t-a)\Big(1-\frac{t-a}{b-a}\Big), \quad t\in[a,b],$$
and $\{u\}_1^p = \operatorname{col}(u_1,u_2,\dots,u_p)$, $\{u\}_{p+1}^{p+q} = \operatorname{col}(u_{p+1},u_{p+2},\dots,u_{p+q})$ *for a vector* $u \in \mathbb{R}^n$.

The idea of proof is to show that (12), (13) are Cauchy sequences in the Banach spaces $C([a,b],\mathbb{R}^p)$ and $C([a,b],\mathbb{R}^q)$, respectively.

Under conditions of Theorem 1, functions (14), (15) are solutions of the Cauchy problems for the forced systems

$$x'(t)=f_1(t,x(t),y(t))+\Delta_U(z,\gamma,\lambda),\ \ x(a)=z,$$
$$y'(t)=f_2(t,x(t),y(t))+\Delta_V(z,\gamma,\lambda),\ \ x(a)=\gamma,$$

where $\Delta_U:\mathscr{U}\times\mathscr{V}_a\times\mathscr{V}_b\to\mathbb{R}^p$ and $\Delta_V:\mathscr{U}\times\mathscr{V}_a\times\mathscr{V}_b\to\mathbb{R}^p$ are the mappings given by the formulas

$$\Delta_U(z,\gamma,\lambda)=-\frac{1}{b-a}\int_a^b f_1\big(s,x_\infty(s,z,\gamma,\lambda),y_\infty(s,z,\gamma,\lambda)\big)\,ds,$$

$$\Delta_V(z,\gamma,\lambda)=\frac{1}{b-a}\,(\lambda-\gamma)-\frac{1}{b-a}\int_a^b f_2\big(s,x_\infty(s,z,\gamma,\lambda),y_\infty(s,z,\gamma,\lambda)\big)\,ds.$$

Theorem 2. *Under the assumptions of Theorem 1, the limit functions* (14), (15) *of sequences* (12), (13) *form a solution of the boundary value problem* (1)–(3) *if and only if the parameters* (z,γ,λ) *satisfy the system of* $p+2q$ *equations*

$$\Delta_U(z,\gamma,\lambda)=0,\quad \Delta_V(z,\gamma,\lambda)=0,\quad \Lambda(z,\gamma,\lambda)=0, \tag{16}$$

where

$$\Lambda(z,\gamma,\lambda)=\phi(y_\infty(\,\cdot\,,\gamma,\lambda))-d. \tag{17}$$

The proof can be carried out similarly to [1,2]. The next statement shows that the system of determining equations (16) determines all possible solutions of the original non-linear boundary value problem (1)–(3) having range in $U\times V$.

Theorem 3. *Let the assumptions of Theorem 1 hold.*

1. *If there exist some* $(z_*,\gamma_*,\lambda_*)\in\mathscr{U}\times\mathscr{V}_a\times\mathscr{V}_b$ *satisfying the system of determining equations* (16)*, then problem* (1)–(3) *has a solution* (x_*,y_*) *such that*

   $$x_*(a)=x_*(b)=z_*,\quad y_*(a)=\gamma_*,\quad y_*(b)=\lambda_*$$

   and, moreover,

   $$x_*(\,\cdot\,)=x_\infty(\,\cdot\,,z_*,\gamma_*,\lambda_*),\quad y_*(\,\cdot\,)=y_\infty(\,\cdot\,,z_*,\gamma_*,\lambda_*).$$

2. *If the boundary value problem* (1)–(3) *has a solution* (x_*,y_*) *with the range in* $U\times V$*, then the system of determining equations* (16) *is satisfied with*

   $$z=x_*(a),\quad \gamma=y_*(a),\quad \lambda=y_*(b).$$

The proof can be carried out by analogy to [1,2].

The solvability of system (16), under additional conditions, can be proved if a solution of an approximate determining system

$$\Delta_{U,m}(z,\gamma,\lambda)=0,\quad \Delta_{V,m}(z,\gamma,\lambda)=0,\quad \Lambda_m(z,\gamma,\lambda)=0,$$

has been found, where m is fixed and $\Delta_{U,m}$, $\Delta_{V,m}$, Λ_m are defined similarly to (17) with x_∞, y_∞ replaced by x_m, y_m. Practical calculations using Maple confirm the constructiveness of the proposed approach.

References

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