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Invariant Toroidal Manifolds of One Class of Discontinuous Dynamical Systems

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1 Introduction

The work [4] is devoted to the study of the main issues of the theory of differential equations with impulse action. In [2], conditions are established that guarantee the hyperbolicity of systems of differential equations with impulse action. The obtained hyperbolicity conditions allow us to study the existence of bounded solutions of inhomogeneous multidimensional systems of differential equations with impulse perturbation. In [1], sufficient conditions for the existence of an asymptotically stable invariant toroidal manifold of linear extensions of a dynamical system on a torus are obtained in the case of a matrix of the system that commutes with its integral. The proposed approach is applied to the study of the stability of invariant sets of a certain class of discontinuous dynamical systems. In [3], a review of the most modern methods for studying the stability of solutions of impulse differential equations and their application to impulse control problems is carried out. The exponential stability of a trivial torus is proved for one class of nonlinear extensions of dynamical systems on a torus. The obtained results are applied to the study of the stability of toroidal sets of impulsive dynamical systems. The concept of an impulsive non-autonomous dynamical system is introduced. For it, the existence and properties of an impulsive attracting set are investigated. The obtained results are applied to the study of the stability of a two-dimensional impulsive-perturbed Navier–Stokes system. In all the above works, the foundations of the qualitative theory of differential equations with impulsive action are outlined. In essence, the foundations of the qualitative theory of impulsive systems were laid, which are based on the qualitative theory of differential equations, methods of asymptotic integration of such equations, the theory of difference equations and generalized functions. At the same time, the issues of the existence of solutions of weakly nonlinear impulsive systems have not yet been fully investigated.

2 Setting of the problem and the main results

Consider a system of differential equations that is subject to impulse action when an angular variable passes through ϕ fixed set Γ on a torus $\mathfrak{T}^m$:

$$\begin{cases} \dfrac{d\phi}{dt} = \omega, \quad \dfrac{dx}{dt} = A(\phi)x + f(\phi, x), \quad \phi \notin \Gamma, \\ \triangle x\big|_{\phi\in\Gamma} = B(\phi)x + I(\phi, x), \quad \triangle x\big|_{\phi\in\Gamma} = a, \end{cases} \tag{2.1}$$

where $x \in R^n$, $\phi \in \mathfrak{T}^m$ and a – constant m-dimensional vectors, $\Gamma(m-1)$ is a dimensional manifold defined by the equation $\langle k, \phi\rangle = 0 \pmod{2\pi}$ and $k = (k_1, \dots, k_m)$ is an integer vector such that $\langle k, \phi\rangle = 0 \pmod{2\pi}$.

The last condition ensures that the angular phase variable belongs to the $\phi(t)$ set Γ at the moment of the pulse action both at $t - 0$, and at $t + 0$. $A(\phi)$ and $B(\phi)$ are continuous, 2π are square matrices that are periodic in each component ϕ_i, the functions $f(\phi,x)$ and $I(\phi,x)$, defined for all $\phi\in\mathfrak{T}^m$, are continuous (piecewise continuous with discontinuities of the first kind in ϕ), 2π-periodic in ϕ and satisfy the Lipschitz condition uniformly with respect to $\phi\in\mathfrak{T}^m$:

$$\big\|f(\phi,x')-f(\phi,x'')\big\|+\big\|I(\phi,x')-I(\phi,x'')\big\|\leq N\|x'-x''\| \tag{2.2}$$

for each $x'\in R^n$.

Let us denote by the $\phi_t(\phi)$ solution of the system

$$\frac{d\phi}{dt}=\omega,\quad \phi\notin\Gamma,\quad \triangle x\big|_{\phi\in\Gamma}=a. \tag{2.3}$$

Let us find out under what condition the trajectory of this solution densely fills the torus everywhere $\mathfrak{T}^m$. Any continuous trajectory $\phi=\omega t+\phi_0$ crosses the manifold Γ at equal time intervals $\beta=\frac{2\pi}{\langle k,\omega\rangle}$.

Consider the trajectory of system (2.3) passing through the point $\phi_t=\phi_t(0)$. The next point of intersection of it with Γ will be $\frac{2\pi}{\langle k,\omega\rangle}\omega$. After the first jump we get a point $\frac{2\pi}{\langle k,\omega\rangle}\omega+a$.

Let us call a section of motion consisting of one continuous arc and one jump one stroke of motion. The starting points of individual strokes of motion are, as we see, the points $S(\frac{2\pi}{\langle k,\omega\rangle}\omega+a)$ where S is an integer.

We obtain the following statement.

Lemma 2.1. *Any continuous trajectory $\phi_t(\phi)$ of system (2.3) is closed if and only if the coordinates of the vector $D=\frac{1}{\langle k,\omega\rangle}\,\omega+\frac{1}{2\pi}\,a$ are rational.*

Before formulating the main result related to system (2.1), consider the following system of equations:

$$\begin{cases}\dfrac{d\phi}{dt}=\omega,\quad \dfrac{dx}{dt}=A(\phi)x+f(\phi),\quad \phi\notin\Gamma,\\ \triangle x\big|_{\phi\in\Gamma}=B(\phi)x+I(\phi),\quad \triangle x\big|_{\phi\in\Gamma}=a\end{cases} \tag{2.4}$$

in which $A(\phi)$, $B(\phi)$, ω, a, Γ – the same as in system (2.1); $f(\phi)$ and $I(\phi)$ – continuous (piecewise continuous with discontinuities of the first kind along ϕ), 2π-periodic in ϕ function.

Let us denote by $I(t,\tau,\phi)$ the normalized fundamental matrix of the system

$$\begin{cases}\dfrac{dx}{dt}=A(\phi_t(\phi))x,\quad t\neq t_i(\phi),\\ \triangle x\big|_{t=t_i}=B(\phi_t(\phi)).\end{cases} \tag{2.5}$$

Note that, as shown in [4], $I(t,\tau,\phi)$ the system of differential equations $\frac{dx}{dt}=A(\phi_t(\phi))x$ associated with the matricant $\Omega^t_\tau(\phi)$ as follows:

$$I(t,\tau,\phi)=\Omega^t_{t_i(\phi)}\Pi\Big[(E+B_{i-k+1})\Omega^{t_{i-k+1}(\phi)}_{t_{i-k}(\phi)}\Big],\ \ t_i(\phi)\leq t\leq t_{i+1}(\phi).$$

In the following, we assume that $I(t,\tau,\phi)$ satisfies the inequality

$$|I(t,\tau,\phi)|\leq Ke^{-\gamma(t-\tau)},\ \ t\geq\tau. \tag{2.6}$$

Lemma 2.2. *Let in the system of equations (2.4) the functions $f(x)$ and $I(\phi)$ be periodic, continuous (piecewise continuous on τ^m). $A(\phi)$ and $B(\phi)$ are continuous on τ^m 2π- periodic matrices. If the matrix $I(t,\tau,\phi)$ satisfies estimate (2.6), then the system of equations (2.4) has an asymptotically stable invariant set*

$$x=u(\phi),\quad u(\phi+2\pi)=u(\phi),$$

where $u(\phi)$ is a piecewise-continuous function with discontinuities of the first kind on the set Γ such that for some constant C we obtain the inequality:

$$\|U(\phi)\| \le C \cdot \max\Big\{ \max_{\phi\in\tau^m} \|f(\phi)\|, \max_{\phi\in\tau^m} \|I(\phi)\| \Big\}.$$

Thus, the function $u(\phi)$ defines the invariant set of the system of equations (2.4). The asymptotic stability of this set is ensured by inequality (2.6).

Let us note some special cases of systems (2.5) for which the fundamental matrix $Y(t,\tau,\phi)$ implies estimate (2.6).

Inequality (2.6) is also satisfied if A and B are constant matrices that commute with each other, and are non-degenerate and the real parts of all eigenvalues $E+B$ of the matrix $\wedge = A + \frac{\langle k,\omega\rangle}{2\pi}\,\ln(E+B)$ are negative.

So, based on the results obtained, we obtain the following theorem.

Theorem. *Let the system of equations* (2.1) *be such that inequalities* (2.2) *and* (2.6) *are satisfied. Then we can specify a positive number* N_0 *such that for all* $0 \le N \le N_0$ *the system of equations* (2.1) *has an asymptotically stable invariant set* $x = u(\phi), u(\phi+2\pi) = u(\phi)$*, where* $u(\phi)$ *is a piecewise-continuous function with discontinuities of the first kind on the set* Γ *such that*

$$\triangle U\Big|_{\phi\in\Gamma} = B(\phi)u(\phi) + I(\phi, u(\phi)).$$

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Once More on Typicality and Atypicality of Power-Law Asymptotic Behavior of Solutions to Emden–Fowler Type Differential Equations

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Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \tag{1}$$

where $n \geq 2$, $k > 1$, and p is a positive continuous function that is Lipschitz-continuous in its last n variables. Also consider a special case of (1), namely,

$$y^{(n)} = p_0|y|^k \operatorname{sign} y \tag{2}$$

with $p_0 > 0$.

Immediate calculations show that equation (2) has positive solutions with exact power-law behavior, namely,

$$y(x) = C(x^* - x)^{-\alpha} \tag{3}$$

defined on $(-\infty,\, x^*)$ with

$$\alpha = \frac{n}{k-1}\,, \quad C = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}}, \tag{4}$$

and arbitrary $x^* \in \mathbb{R}$.

We discuss the problem posed by I. Kiguradze (see [9, Problem 16.4]) on asymptotic behavior of all positive non-extensible (so-called “blow-up”) solutions to equations (2) and (1).

For $n = 2$ (see [9]), $n = 3, 4$ (see [1,2], [3, **5.1**]), it appears that if $p(x, y_1, y_2, \dots, y_{n-1})$ tends to p_0 as $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$, then all such solutions to equation (1) (and equation (2)) have the following power-law asymptotic behavior:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \ \ x \to x^* - 0, \tag{5}$$

with α and C defined by (4).

For equation (1) with any n and some additional assumptions on the function p, the existence of solutions with power-law asymptotic behavior is proved, for $5 \leq n \leq 11$, the existence of an $(n-1)$-parametric family of such solutions is obtained (see [3, **5.1**]).

It is also proved that for weakly super-linear equations (2) (see [5]) and (1) (see [6]) Kiguradze's conjecture on the power-law asymptotic behavior of all blow-up solutions is true.

Theorem 1 ([6]). *Suppose that $p \in C(\mathbb{R}^{n+1}) \cap Lip_{y_0,\dots,y_{n-1}}(\mathbb{R}^n)$ and $p \to p_0 > 0$ as $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. Then for any integer $n > 4$ there exists $K_n > 1$ such that for any real $k \in (1, K_n)$, any solution to equation* (1) *tending to $+\infty$ as $x \to x^* - 0$ has the power-law asymptotic behavior* (5).

In the case $n \geq 12$, even if we deal with equation (2), another type of asymptotic behavior of singular solutions appears (see [4, 6, 8, 10]).

Theorem 2 ([8])**.** *For any $n \geq 12$, there exists $k_n > 1$ such that equation* (2) *has a solution $y(x)$ with*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha - j} h_j(\log(x^* - x)), \quad j = 0, 1, \dots, n-1, \tag{6}$$

where all h_j are periodic positive non-constant functions on $\mathbb{R}$.

If we have stronger nonlinearity, then the power-law asymptotic behavior becomes atypical. The following theorem generalizes the results of [7].

Theorem 3. *If $12 \leq n \leq 100000$, then there exists $k_n > 1$ such that at any point $x_0 \in \mathbb{R}$ the set of initial data of asymptotically power-law solutions to equation* (2) *has zero Lebesgue measure whenever $k > k_n$.*

In order to study the blow-up solutions to equation (2) having the vertical asymptote $x = x^*$, we use the substitutions

$$x^* - x = e^{-t}, \quad y = (C + v)e^{\alpha t} \tag{7}$$

with C defined by (4) to transform equation (2) with $p_0 = 1$ to another one, which can be reduced to the first-order system

$$\frac{dV}{dt} = A_\alpha V + F_\alpha(V), \tag{8}$$

where A_α is a constant $n \times n$ matrix with eigenvalues satisfying the equation

$$\prod_{j=0}^{n-1} (\lambda + \alpha + j) = \prod_{j=0}^{n-1} (1 + \alpha + j), \tag{9}$$

and F_α is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying

$$\|F_\alpha(V)\| = O(\|V\|^2) \ \text{ and } \ \|F'_{\alpha,V}(V)\| = O(\|V\|) \ \text{ as } \ V \to 0.$$

In order to study equation (1), the same substitution as (7) of variables is used, and a more complicated system than (8) with an additional term $G(t, V)$ appears (see [3]).

The proof of Theorem 3 is based on the following statement.

Lemma. *If there is no purely imaginary root to equation* (9)*, but there exists at least one root not equal to 1 and having positive real part, then for any $x_0 \in \mathbb{R}$, the set of initial data of asymptotically power-law solutions to equation* (2) *has zero Lebesgue measure whenever $k > k_n$.*

Remark. The occurrence of the order 12 for equation (2) in Theorems 2 and 3 is explained by the fact that all roots but one ($\lambda = 1$) to equation (9) with $n < 12$ have negative real parts, which implies the existence of an $(n-1)$-parametric family of solutions with power-law asymptotic behavior (5) of solutions to equation (1). Equation (9) with $n = 12$ and some α has a pair of complex-conjugate purely imaginary roots, which implies the appearance of a solution of the form (6) to equation (2). The order 100000 appearing in Theorem 3 is not final. It is possible to continue the calculations and obtain the same result for equations of order higher than 100000. The previous result (see [7]) was obtained for $12 \leq n \leq 203$.

Open problems

- Is there any blow-up solution to equation (2) with asymptotic behavior other than (5) and (6) ?
- Is there any blow-up solution with non-power-law asymptotic behavior to equation (2) with strong power-law nonlinearity when $5 \le n \le 11$?
- Is it possible to find exactly a constant $K_n^* > 1$ such that for any $k \in (1, K_n^*)$ all blow-up solutions to (2) have power-law asymptotic behavior (5), while other blow-up solutions appear whenever $k > K_n^*$?

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On Smooth Controllability in Parabolic Control Problem with a Pointwise Observation

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1 Introduction

Consider an extremum problem for the parabolic mixed problem

$$u_t = (a(x)u_x)_x + b(x)u_x + h(x)u, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ 0 < t < T, \tag{1.2}$$

$$u(x,0) = 0, \ \ 0 < x < 1, \tag{1.3}$$

where the real functions a, b and h are smooth in $\overline{Q}_T$, $0 < a_0 \le a(x) \le a_1 < \infty$, $\varphi \in W_2^2(0,T)$, $\psi \in W_2^2(0,T)$. Here $W_2^2(0,T)$ is the Sobolev space of weakly differentiable functions with the norm

$$\|y\|^2_{W_2^k(0,T)} = \int\limits_0^T \Big(\sum_{j=0}^k \big(y^{(k)}(t)\big)^2 \Big)\, dt.$$

We study the control problem with a pointwise observation: by controlling the temperature φ at the left end of the segment (the function ψ is assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z \in W_2^1(0,T)$ over the entire time interval $(0,T)$. This problem arises in the model of climate control in industrial greenhouses [4,5]. Note that extremal problems for parabolic equations were considered in [11,13–16] (as usual, problems with final or distributed observation). But the results and methods of investigation are not similar to our methods.

Continuing the research in [1–3,6–10], we consider some special quality functional, which is in demand in applications, providing, among other things, uniform proximity of the solution and the objective function, implemented by the norm in the space $W_2^1(0,T)$. Since in applied problems the control and observation time T is sufficiently large, the influence of the initial function is relatively small and can be neglected, setting the initial function equal to zero.

As in [12, p. 6], we denote by $V_2^{1,0}(Q_T)$ Banach space of functions $u \in W_2^{1,0}(Q_T)$ (Sobolev space of functions with the norm $\|u\|^2_{W_2^{1,0}(Q_T)} = \int\limits_{Q_T} (u_x^2 + u^2)\, dx\, dt$) with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0 \le t \le T} \|u(\,\cdot\,, t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,, t)$ is a continuous mapping from $[0, T]$ to $L_2(0, 1)$. Let $\widetilde{W}_2^1(Q_T)$ be the set of all functions $\eta \in W_2^1(Q_T)$, satisfying the conditions $\eta(\,\cdot\,, T) = 0$, $\eta(0, \,\cdot\,) = 0$.

Definition 1.1. A function $u \in V_2^{1,0}(Q_T)$, satisfying the condition $u(0, t) = \varphi(t)$ and the equality

$$\int\limits_{Q_T} \big(a(x)u_x\eta_x - b(x)u_x\eta - h(x)u\eta - u\eta_t\big)\, dx\, dt = a(1)\int\limits_0^T \psi(t)\eta(1, t)\, dt$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$, is called a weak solution to problem (1.1)–(1.3).

2 Main results

Theorem 2.1. *If $\varphi, \psi \in W_2^2(0, T)$ and $\varphi(0) = \psi(0) = 0$, then problem (1.1)–(1.3) has a unique weak solution $u \in V_2^{1,0}(Q_T)$ with $u_t \in V_2^{1,0}(Q_T)$, and the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} + \|u_t\|_{V_2^{1,0}(Q_T)} \le C_1\big(\|\varphi\|_{W_2^2(0,T)} + \|\psi\|_{W_2^2(0,T)}\big) \tag{2.1}$$

holds with some constant C_1, independent of φ and ψ.

Denote by $\Phi \subset W_2^2(0, T)$ nonempty set of control functions φ satisfying the condition $\varphi(0) = 0$, and let $Z \subset W_2^1(0, T)$ be nonempty set of objective functions z satisfying the condition $z(0) = 0$. Consider the functional

$$J[z, \varphi] = \big\|u_\varphi(x_0, t) - z(t)\big\|^2_{W_2^1(0,T)}, \quad \varphi \in \Phi, \quad z \in Z, \tag{2.2}$$

where u_φ is the solution to problem (1.1)–(1.3) with the given control function φ. Considering the function z to be fixed, we have the following minimization problem

$$m[z, \Phi] = \inf_{\varphi \in \Phi} J[z, \varphi]. \tag{2.3}$$

Theorem 2.2. *If the set Φ is closed, convex and bounded in $W_2^2(0, T)$, then for any $z \in Z$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$m[z, \Phi] = J[z, \varphi_0]. \tag{2.4}$$

Definition 2.1. We will say that problem (1.1)–(1.3), (2.3) is densely controllable from the set Φ to the set Z (see [8, 16]), if for all $z \in Z$ the equality

$$m[z, \Phi] = 0 \tag{2.5}$$

holds.

Theorem 2.3. *Problem (1.1)–(2.2) is densely controllable from the set $\Phi = \{\varphi \in W_2^2(0, T) : \varphi(0) = 0\}$ to the set $Z = \{z \in W_2^1(0, T) :\ z(0) = 0\}$.*

3 Proofs

Proof of Theorem 2.1. By results of [8], we can prove that under assumptions of $\varphi \in W_1^2(0,T)$, $\psi \in W_1^2(0,T)$ there exists a unique solution $u \in V_2^{1,0}(Q_T)$ of problem (1.1)–(1.3). This solution satisfies the estimate

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}\big). \tag{3.1}$$

The function $v = u_t$ is a solution to the problem

$$v_t = (a(x)v_x)_x + b(x)v_x + h(x)v, \ \ (x,t) \in Q_T, \tag{3.2}$$

$$v(0,t) = \varphi'(t), \quad v_x(1,t) = \psi'(t), \ \ 0 < x < 1, \ \ t > 0, \tag{3.3}$$

$$v(x,0) = 0, \ \ 0 < x < 1. \tag{3.4}$$

Using the results of [8], under assumptions of $\varphi' \in W_1^2(0,T)$, $\psi' \in W_1^2(0,T)$ there exists a solution $v \in V_2^{1,0}(Q_T)$ of problem (3.2)–(3.4). This solution satisfies the estimate

$$\|v\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi'\|_{W_2^1(0,T)} + \|\psi'\|_{W_2^1(0,T)}\big).$$

Therefore,

$$\|u_t\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi\|_{W_2^2(0,T)} + \|\psi\|_{W_2^2(0,T)}\big). \tag{3.5}$$

Combining estimates (3.1) and (3.5), we obtain the required inequality (2.1). □

The proof of Theorem 2.2 is based on the following lemma concerning the best approximation in Hilbert spaces.

Lemma 3.1 ([4])**.** *Let A be a convex closed set in a Hilbert space H. Then for any $x \in H$ there exists a unique element $y \in A$ such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Proof of Theorem 2.2. Denote

$$B = \big\{y = u_\varphi(x_0,\,\cdot\,) : \ \ \varphi \in \Phi\big\} \subset W_2^1(0,T).$$

By the convexity of Φ the set B is a convex subset in $W_2^1(0,T)$. The set Φ is bounded and closed in $W_2^1(0,T)$ and by estimate (2.1) we obtain that B is a bounded and closed set in $W_2^1(0,T)$. Now we apply Lemma 3.1 to the case $H = W_2^1(0,T)$, $A = B$, $x = z \in Z \subset H$. By Lemma 3.1 there exists a unique function $y \in B$ such that

$$m[z,\Phi] = \|y - z\|^2_{W_2^1(0,T)}.$$

So, $y = u_{\varphi_0}(x_0,\,\cdot\,)$ for some $\varphi_0 \in \Phi$ such that

$$m[z,\Phi] = J[z,\varphi_0].$$

Now we can prove that such $\varphi_0 \in \Phi$ is unique by the same technique of maximum principle and unique continuation theorems as in [8]. □

Proof of Theorem 2.3. For $u_\varphi(x_0,0) = z(0) = 0$ we have the representation

$$u_\varphi(x_0,t) - z(t) = \int_0^t (u_{\varphi t}(x_0,\tau) - z'(\tau))\,d\tau, \ \ 0 \le t \le T. \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{L_2(0,T)}\\&=\int_0^T\Big(\int_0^t\big(u_{\varphi t}(x_0,\tau)-z'(\tau)\big)\,d\tau\Big)^2\,dt\le\int_0^T t\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,t)}\,dt\\&\le\frac{T^2}{2}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}.\end{aligned}\tag{3.7}$$

So, from (3.6) and (3.7) we have

$$\begin{aligned}J[z,\varphi]&=\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{W_2^1(0,T)}\\&=\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{L_2(0,T)}+\big\|u_{\varphi t}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}\\&\le\Big(1+\frac{T^2}{2}\Big)\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}.\end{aligned}\tag{3.8}$$

Now, by the results of [8] and [9], problem (3.2)–(3.4) is densely controllable from $W_2^1(0,T)$ to $L_2(0,T)$. Therefore, for an arbitrary $z'\in L_2(0,T)$ we have

$$\inf_{\varphi'\in W_2^1(0,T)}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}=0.\tag{3.9}$$

Now, by (3.8), (3.9),

$$\inf_{\varphi\in W_2^2(0,T)}J[z,\varphi]\le\Big(1+\frac{T^2}{2}\Big)\inf_{\varphi'\in W_2^1(0,T)}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}=0.\qquad\square$$

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On Oscillation of Solutions to One Neutral Type Differential Equation

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Consider a second-order differential equation of neutral type with constant delays

$$(y - py_\tau)'' + q(t)f(y_\sigma) = 0, \quad y_\rho(t) \equiv y(t-\rho), \ \ t \in [t_0, +\infty), \tag{1}$$

where $0 < p < 1$, $\tau, \sigma > 0$, $q \in C[t_0, +\infty)$, $q \geqslant 0$.

Denote $\rho \equiv \max\{\tau, \sigma\}$.

Definition 1. The solution to equation (1) is the function $y \in C[t_0 - \rho, +\infty)$, satisfying this equation, such that $y - py_\tau \in C^2[t_0, +\infty)$.

Definition 2. The solution y of equation (1) is called oscillatory if for any $t_1 \geqslant t_0$ there exists $t_2 > t_1$ such that $y(t_2) = 0$.

Definition 3. We will say that a function f such that $f'(y) \geqslant 0$, $y \in \mathbb{R}$, and $yf(y) > 0$, $y \neq 0$, satisfies:

- the superlinear condition, if for any $\varepsilon > 0$ the inequalities hold:

$$0 < \int\limits_{\varepsilon}^{+\infty} \frac{dy}{f(y)} < +\infty, \quad 0 < -\int\limits_{-\infty}^{-\varepsilon} \frac{dy}{f(y)} < +\infty;$$

- the sublinear condition, if for any $\varepsilon > 0$ the inequalities hold:

$$0 < \int\limits_{0}^{\varepsilon} \frac{dy}{f(y)} < +\infty, \quad 0 < -\int\limits_{-\varepsilon}^{0} \frac{dy}{f(y)} < +\infty.$$

In the case $p = \tau = \sigma = 0$ and $f(y) = |y|^\gamma \operatorname{sgn} y$, equation (1) is an Emden–Fowler type equation

$$y'' + q(t)|y|^\gamma \operatorname{sgn} y = 0. \tag{2}$$

The following criteria for the oscillation of all its solutions are known.

Theorem A (Atkinson [2])**.** *If $q \in C[0, +\infty)$, $q \geqslant 0$ and $\gamma = 2n - 1$, $n \in \mathbb{N}$, $n > 1$, then all solutions to equation* (2) *are oscillatory iff*

$$\int\limits_{0}^{+\infty} tq(t)\, dt = +\infty.$$

Theorem B (Belohorec [3]). *If $q_j \in C[0,+\infty)$, $q_j \geqslant 0$ and $\gamma_j = p_j/r_j \in (0,1)$, where p_j, r_j – natural, odd and $j \in \mathbb{N}$, then all solutions of the equation $y'' + \sum\limits_{j=1}^{n} q_j(t)y^{\gamma_j} = 0$ are oscillatory iff*

$$\int\limits_0^{+\infty} \sum_{j=1}^{n} t^{\gamma_j} q_j(t)\,dt = +\infty.$$

A strengthening of Atkinson's theorem for all real $\gamma > 1$ was proven in [4], the oscillation of solutions of high-order Emden–Fowler type equations was studied in [5]. A more general case of equation (2) was considered in [1].

In [6] criteria for the oscillation of all solutions of equation (1) in the cases of superlinearity and sublinearity of the function f are proved. The following results complement and clarify these criteria.

Lemma 1. *Let y be the solution of equation (1) such that $y > 0$ for every $t \geq t_0 \geq 0$ and $z = y - py_\tau$. Then for every $t \geq t_1$, where $t_1 \geq t_0 + \rho$ is sufficiently large, one of the conditions holds:*

1) $z'' \leq 0$, $z' > 0$, $z < 0$;

2) $z'' \leq 0$, $z' > 0$, $z > 0$.

Moreover, the first condition is satisfied when $\lim\limits_{t\to+\infty} y(t) = 0$. Otherwise, the second condition is true.

Lemma 2. *For every continuous function φ, defined on the segment $[t_0-\rho, t_0]$, equation (1) has a solution y, extendable to the interval $[t_0,+\infty)$ and satisfying the initial conditions $y(t) = \varphi(t)$ for $t \in [t_0-\rho, t_0]$.*

Theorem 1. *Let the function $f \in C^1(\mathbb{R})$ be superlinear. Then:*

1) *if $\int\limits_{t_0}^{+\infty} tq(t)\,dt = +\infty$, then all not vanishing at infinity solutions to equation (1) are oscillatory;*

2) *if all solutions to equation (1) are oscillatory, then $\int\limits_{t_0}^{+\infty} tq(t)\,dt = +\infty$.*

Proof. 1) Let y be a non-vanishing non-oscillatory solution to equation (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \geq t_0 \geq 0$. By Lemma 1 for $z = y - py_\tau \geq y$ we have $z'' \leq 0$, $z' > 0$, $z > 0$ for all $t \geq t_1$.

Then

$$0 = z''(t) + q(t)f(y_\sigma(t)) \geq z''(t) + q(t)f(z_\sigma(t)).$$

Let

$$w(t) = \frac{tz'(t)}{f(z_\sigma(t))} \geq 0.$$

We obtain

$$w'(t) + tq(t) \leq \frac{z'(t)}{f(z_\sigma(t))} - \frac{tf'(z_\sigma(t))z'(t)}{[f(z_\sigma(t))]^2}\, z'_\sigma(t) \leq \frac{z'(t)}{f(z_\sigma(t))}\,.$$

Let's integrate the inequality

$$w(t) - w(t_1) + \int\limits_{t_1}^{t} sq(s)\,ds \le \int\limits_{t_1}^{t} \frac{z'(s)}{f(z_\sigma(s))}\,ds \le \int\limits_{t_1}^{t} \frac{z'_\sigma(s)}{f(z_\sigma(s))}\,ds,$$

$$w(t) - w(t_1) + \int\limits_{t_1}^{t} sq(s)\,ds \le \int\limits_{z_\sigma(t_1)}^{z_\sigma(t)} \frac{dv}{f(v)}\,,$$

$$\int\limits_{t_1}^{t} sq(s)\,ds \le w(t_1) + \int\limits_{z_\sigma(t_1)}^{\infty} \frac{dv}{f(v)} = const < +\infty.$$

Tending t to infinity, we arrive at a contradiction.

2) See [6]. □

Remark. The divergence of the integral $\int\limits_0^{+\infty} tq(t)\,dt$ does not guarantee (contrary to the statement from [6]) the oscillation of all solutions to equation (1). For example, the function $y(t) = e^{-t}$ is a particular solution to the equation

$$\Big(y - \frac{1}{2}\,y_1\Big)'' + \Big(\frac{e}{2} - 1\Big)e^{2t-3}y_1^3 = 0,$$

and $\lim\limits_{t\to+\infty} y(t) = 0$ and $\int\limits_0^{+\infty} tq(t)\,dt = +\infty$, where $q(t) \equiv t(e/2-1)e^{2t-3}$.

Theorem 2. *Let the function $f \in C(\mathbb{R})$ be sublinear and $f(uv) \geqslant f(u)f(v)$ for $uv \geqslant 0$. Then:*

1) *if $\int\limits_{t_0}^{+\infty} f(t)q(t)\,dt = +\infty$, then all not vanishing at infinity solutions to equation (1) are oscillatory;*

2) *if all solutions to equation (1) are oscillatory, then $\int\limits_{t_0}^{+\infty} f(t)q(t)\,dt = +\infty$.*

Proof. 1) Let y be a non-vanishing non-oscillatory solution to equation (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \ge t_0 \ge 0$. By Lemma 1 for $z = y - py_\tau \ge y$ we have $z'' \le 0$, $z' > 0$, $z > 0$ for all $t \ge t_1$.

We have

$$0 = z''(t) + q(t)f(y_\sigma(t)) \ge z''(t) + q(t)f(z_\sigma(t)).$$

Since

$$z(t) = z(t_1) + \int\limits_{t_1}^{t} z'(s)\,ds \ge z'(t)(t - t_1),$$

then

$$f(z_\sigma(t)) \ge f(z'_\sigma(t)(t - \sigma - t_1)).$$

For any $\lambda \in (0;1)$, if $t_2 \ge t_1$ is sufficiently large, $t - \sigma - t_2 \ge \lambda t$ for all $t \ge t_2$. Therefore,

$$f\big(z'_\sigma(t)(t - \sigma - t_1)\big) \ge f(\lambda z'_\sigma(t)t) \ge f(\lambda z'_\sigma(t))f(t)$$

and
$$\frac{z''(t)}{f(\lambda z'_\sigma(t))} + q(t)f(t) \le 0.$$

Integrating the resulting inequality, we obtain
$$\int_{t_2}^{t} \frac{z''(s)}{f(\lambda z'_\sigma(s))}\,ds + \int_{t_2}^{t} q(s)f(s)\,ds \le 0,$$
$$\int_{t_1}^{t} q(s)f(s)\,ds \le -\int_{t_2}^{t} \frac{z''(s)}{f(\lambda z'_\sigma(s))}\,ds \le -\int_{t_2}^{t} \frac{z''(s)}{f(\lambda z'(s))}\,ds,$$
$$\int_{t_2}^{t} q(s)f(s)\,ds \le \int_{\lambda z'(t)}^{\lambda z'(t_2)} \frac{dv}{\lambda f(v)} = \int_{0}^{\lambda z'(t_2)} \frac{dv}{\lambda f(v)} - \int_{0}^{\lambda z'(t)} \frac{dv}{\lambda f(v)}\,.$$

Then, by the property of sublinearity of the function f we have
$$\int_{t_2}^{t} q(s)f(s)\,ds \le const < +\infty.$$

Tending t to infinity, we arrive at a contradiction.

2) See [6]. □

Theorem 3. *If the function $f \in C(\mathbb{R})$ is sublinear, $\sigma > \tau$ and $\int\limits_{t_0}^{+\infty} q(t)\,dt = +\infty$, then all solutions to equation* (1) *are oscillatory.*

Proof. Let y be a non-oscillating solution to (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \ge t_0 \ge 0$.

Let us show that both cases described in Lemma 1 are impossible.

1) If $z > 0$ for all $t \ge t_1$, where $t_1 \ge t_0 + \rho$, we have
$$z = y - py_\tau \ge y.$$

Due to $f' \ge 0$ and equation (1), we obtain
$$z''(t) + q(t)f(z_\sigma(t)) \le 0.$$

Integrating this inequality on the interval $[t_1, t]$, we get
$$\int_{t_1}^{t} q(s)f(z_\sigma(s))\,ds \le z'(t_1),$$
$$\int_{t_1}^{t} q(s)\,ds \le \frac{z'(t_1)}{f(z_\sigma(t_1))} \le \frac{z'(t_1)}{f(z(t_1))} = const < +\infty.$$

Tending t to infinity, we come to a contradiction.

2) If $z < 0$ for all $t \geq t_1 \geq t_0 + \rho$, then

$$z(t) = y(t) - py_\tau(t) < -py_\tau(t),$$
$$y_\sigma(t) < -\frac{z_{\sigma-\tau}(t)}{p}\,.$$

Then, since f is increasing, from equation (1) we have

$$z''(t) + q(t)f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big) \leq 0.$$

Let us integrate this inequality on the interval $[t-\sigma+\tau, t]$.

$$z'_{\sigma-\tau}(t) - z'(t) + \int\limits_{t-\sigma+\tau}^{t} q(s)f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big)p\,ds \leq 0.$$

Taking into account the fact that z is positive and increasing, we have

$$-z'_{\sigma-\tau}(t)\Big/ f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big) + \int\limits_{t-\sigma+\tau}^{t} q(s)\,ds \leq 0.$$

Let $w(t) \equiv -z_{\sigma-\tau}(t)/p$. Integrating the inequality on $[t_2, t_3]$, we obtain

$$p\int\limits_{w(t_2)}^{w(t_3)} \frac{dw}{f(w)} + \int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\,ds\,dt \leq 0,$$
$$\int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\,ds\,dt \leq p\int\limits_{0}^{w(t_2)} \frac{dt}{w(t)} - p\int\limits_{0}^{w(t_3)} \frac{dt}{w(t)}\,,$$
$$\int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\,ds\,dt \leq p\int\limits_{0}^{w(t_2)} \frac{dt}{w(t)}\,.$$

Due to the sublinearity of the function f, we get

$$\int\limits_{t_2}^{\infty}\int\limits_{t-\sigma+\tau}^{t} q(s)\,ds\,dt < +\infty,$$

which contradicts the condition $\int\limits_{t_0}^{\infty} q(t)dt = +\infty$. □

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Autonomous Seminonlinear Boundary Value Problems with Switchings at Non-Fixed Times

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We study the problem of constructing solutions [3,4,7]

$$z(\,\cdot\,,\varepsilon)\in\mathbb{C}^1\big\{[0,T]\setminus\{\tau(\varepsilon)\}_I\big\}\cap\mathbb{C}[0,T],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary value problem for the equation

$$z'(t,\varepsilon)=A\,z(t,\varepsilon)+\varepsilon\,Z(z(t,\varepsilon),\varepsilon),\quad \ell z(\,\cdot\,,\varepsilon)=0, \tag{1}$$

which are continuous at $t=\tau(\varepsilon)$. At the point $t=\tau(\varepsilon)$: $0<\tau(\varepsilon)<T$, its $\tau(0):=\tau_0$ the solution of the boundary value problem (1) might have a limited discontinuity of first derivative [3,7]. The solution of the boundary value problem (1) is found in a small neighbourhood of the solution

$$z_0(t)\in\mathbb{C}\big\{[0,T]\setminus\{\tau_0\}_I\big\}\cap\mathbb{C}[0,T]$$

of the generating boundary value problem

$$z_0'(t)=A\,z_0(t),\quad \ell z_0(\,\cdot\,)=0. \tag{2}$$

At the point $t=\tau_0$, the solution of the boundary value problem (2) might have a limited discontinuity of the derivative. Here, $A\in\mathbb{R}^{n\times n}$ is a constant matrix, $Z(z,\varepsilon)$ is a nonlinear vector function, piecewise analytic in the unknown z in a small neighbourhood of the solution of the generating problem (2) and piecewise analytic in a small parameter ε on the interval $[0,\varepsilon_0]$. In addition,

$$\ell z(\,\cdot\,,\varepsilon):=\begin{pmatrix} z(0,\varepsilon)-z(T,\varepsilon)\\ z(\tau(\varepsilon)+0,\varepsilon)-z(\tau(\varepsilon)-0,\varepsilon)\end{pmatrix}=0,\quad \ell z_0(\,\cdot\,):=\begin{pmatrix} z_0(0)-z_0(T)\\ Z_0(\tau_0+0)-z_0(\tau_0-0)\end{pmatrix}=0$$

are linear bounded vector functionals. The condition for the solvability of the autonomous nonlinear boundary value problem (1) with switchings leads to the equation

$$F_0(c_0,\tau_0):=P_{Q_r^*}\,\ell K\big[Z(z_0(s,c_0),0);\tau_0\big](\,\cdot\,)=0. \tag{3}$$

The necessary conditions for the existence of a solution to the autonomous nonlinear boundary value problem (1) with switchings in the critical case are given by the following lemma.

Lemma. *Suppose that there is the critical case* (2) *for the generating boundary value problem. In this case, the generating problem* (2) *has a one-parameter family of solutions* $z_0(t,c_0)$. *Suppose that an autonomous nonlinear boundary value problem* (1) *with switchings at non-fixed times in the neighbourhood of the generating solution* $z_0(t,c_0)$ *has the solution*

$$z(\,\cdot\,,\varepsilon)\in\mathbb{C}\big\{[0,T]\setminus\{\tau(\varepsilon)\big\}_I\big\}\cap\mathbb{C}[0,T],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0].$$

Under these conditions, the equality (3) *holds.*

The equation (3), will be further called the equation for the generating constants of the boundary value problem (1) with switchings in the critical case. Let us assume that the equation for the generating constants (3) of the boundary value problem (1) with switchings has real roots. Fixing one of the real solutions

$$c_0^*\in\mathbb{R}^r,\ \ \tau_0^*\in\mathbb{R}$$

of the equation (3) we get the problem of constructing a solution of the nonlinear boundary value problem (1) in a small neighbourhood of the solution $z_0(t,c_0^*)=X_r(t)\,c_0^*$, $c_0^*\in\mathbb{R}^r$, of the generating boundary value problem (2). The traditional condition for the solvability of a boundary value problem (1) with switchings in a small neighbourhood of the solution of the generating problem is the requirement [3]

$$P_{B_0^*}P_{Q_r^*}\neq 0,\quad B_0:=F'_{\check c_0}(c_0^*,\tau_0^*)\in\mathbb{R}^{r\times(r+1)},\quad \check c_0:=\begin{pmatrix}c_0 & \tau_0\end{pmatrix}^*, \tag{4}$$

where $P_{B_0^*}:\mathbb{R}^r\to\mathbb{N}(B_0^*)$ is an orthoprojector matrix [3]. The solution of the boundary value problem (1) with switchings is given by

$$z(t,\varepsilon):=z_0(t,c_0^*)+u_1(t,\varepsilon)+\cdots+u_k(t,\varepsilon)+\cdots,\ \ \tau(\varepsilon)=\tau_0^*+\xi_1(\varepsilon)+\xi_2(\varepsilon)+\cdots+\xi_k(\varepsilon)+\cdots.$$

The nonlinear vector function $Z(z(t,\varepsilon),\varepsilon)$ is analytical with respect to the unknown $z(t,\varepsilon)$ in a small neighbourhood of the solution of the generating boundary value problem (2) and the constant τ_0^*, therefore in the given neighbourhood there exist an expansion

$$Z(z(t,\varepsilon),\varepsilon)=Z_0(z_0(t,c_0^*),\varepsilon)+Z_1\big(z_0(t,c_0^*),u_1(s,\varepsilon),\varepsilon\big)+Z_2\big(z_0(t,c_0^*),u_1(s,\varepsilon),u_2(s,\varepsilon),\varepsilon\big)+\cdots.$$

The first approximation to the solution of the nonlinear periodic boundary value problem (1) in the critical case

$$z_1(t,\varepsilon)=z_0(t,c_0^*)+u_1(t,\varepsilon),\quad \tau_1(\varepsilon)=\tau_0^*+\xi_1(\varepsilon),$$
$$u_1(t,\varepsilon)=X_r(t)c_1(\varepsilon)+\varepsilon\,G\big[Z_0(z_0(s,c_0^*),z_0'(s,c_0^*),\varepsilon);\tau_0^*\big](t)$$

determines the solution of the nonlinear periodic boundary value problem of the first approximation

$$u_1'(t,\varepsilon)=A\,u_1(t,\varepsilon)+\varepsilon\,Z_0(z_0(t,c_0^*),\varepsilon),\ \ \ell u_1(\,\cdot\,,\varepsilon)=0.$$

The matrix B_0, which is the key matrix in the study of the boundary value problem (1), takes the form

$$B_0=P_{Q_r^*}\,\ell K\big[\mathcal{A}_0(s)X_r(s);1\big](\,\cdot\,);\quad \mathcal{A}_0(t)=\frac{\partial Z(z(t,\varepsilon),\varepsilon)}{\partial z(t,\varepsilon)}\bigg|_{\substack{z(t,\varepsilon)=z_0(t,c_0^*)\\ \varepsilon=0}}.$$

The second approximation to the solution of the nonlinear periodic boundary value problem (1), in the critical case

$$z_2(t,\varepsilon):=z_0(t,c_0^*)+u_1(t,\varepsilon)+u_2(t,\varepsilon),\quad \tau_2(\varepsilon)=\tau_0^*+\xi_1(\varepsilon)+\xi_2(\varepsilon),$$

determines the solution of the nonlinear periodic boundary value problem of the second approximation

$$u_2'(t,\varepsilon) = A\, u_2(t,\varepsilon) + \varepsilon\, Z_1(z_0(t,c_0^*), u_1(t,\varepsilon),\varepsilon), \quad \ell u_2(\,\cdot\,,\varepsilon) = 0.$$

The condition of solvability of the boundary value problem of the second approximation

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) := P_{Q_d^*}\ell K\big[Z_1\big(z_0(s,c_0^*), u_1(s,\varepsilon), z_0'(s,c_0^*), u_1(s,\varepsilon),\varepsilon); \xi_1(\varepsilon)\big](\,\cdot\,) = 0$$

is the linear equation

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) = B_0\, \check{c}_1(\varepsilon) + \gamma_1(\varepsilon) = 0, \quad \check{c}_1(\varepsilon) := \begin{pmatrix} c_1(\varepsilon) & \xi_1(\varepsilon)\end{pmatrix}^*,$$

which has solutions in case (4), where

$$\gamma_1(\varepsilon) := F_1(\check{c}_1(\varepsilon)) - B_0\, \check{c}_1(\varepsilon).$$

Indeed, let us denote the vector-functions

$$v(t,\varepsilon,\mu) := z_0(t,c_0^*) + \mu\, u_1(t,\varepsilon) + \cdots + \mu^k\, u_k(t,\varepsilon) + \cdots,$$
$$g(\varepsilon,\mu) := \tau_0^* + \mu\, \xi_1(\varepsilon) + \mu^2\, \xi_2(\varepsilon) + \cdots + \mu^k\, \xi_k(\varepsilon) + \cdots,$$

while

$$F_1(c_1(\varepsilon),\xi_1(\varepsilon)) = P_{Q_d^*}\ell K\big[Z_1(z_0(s,c_0^*), u_1(s,\varepsilon),\varepsilon); \xi_1(\varepsilon)\big](\,\cdot\,)$$
$$= P_{Q_d^*}\,\ell K\big[Z_\mu'(v(t,\varepsilon,\mu),\varepsilon); g_\mu'(\varepsilon,\mu)\big](\,\cdot\,)\Big|_{\mu=0} = P_{Q_d^*}\,\ell K\big[\mathcal{A}_0(s) u_1(s,\varepsilon); \xi_1(\varepsilon)\big](\,\cdot\,),$$

therefore

$$B_0 := F'_{\check{c}_1(\varepsilon)}(\check{c}_1(\varepsilon)) \in \mathbb{R}^{r\times(r+1)}.$$

Thus, under the condition (4), we obtain at least one solution to the first approximation boundary value problem

$$z_1(t,\varepsilon) := z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\, \gamma_1(\varepsilon),$$
$$u_1(t,\varepsilon) = X_r(t) c_1(\varepsilon) + \varepsilon\, G\big[Z_1(z_0(s,c_0^*), u_1'(s,\varepsilon),\varepsilon); \xi_1(\varepsilon)\big](t).$$

The conditions for solvability of boundary value problems of the following approximations

$$F_j(\check{c}_j(\varepsilon)) := P_{Q_d^*}\,\ell K\big[Z_j(z_0(s,c_0^*), u_1(t,\varepsilon), \ldots,\ u_j(s,\varepsilon), \xi_j(\varepsilon),\varepsilon)\big](\,\cdot\,) = 0$$

are linear equations

$$F_j(\check{c}_j(\varepsilon)) = B_0\, \check{c}_j(\varepsilon) + \gamma_j(\varepsilon) = 0, \quad j = 1,2,\ldots,k,$$

where

$$B_0 = F'(\check{c}_j(\varepsilon)), \quad \gamma_j(\varepsilon) := F(\check{c}_j(\varepsilon)) - B_0\, \check{c}_j(\varepsilon), \quad j = 1,2,\ldots,k.$$

In the case (4), the last equation has solutions. The sequence of approximations to the solution of the nonlinear periodic boundary value problem (1) in the critical case is determined by the iterative scheme

$$\begin{gathered}
z_1(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\, \gamma_1(\varepsilon),\\
u_1(t,\varepsilon) = X_r(t) c_1(\varepsilon) + \varepsilon\, G\big[Z_1(z_0(s,c_0^*), u_1'(s,\varepsilon),\varepsilon); \xi_1(\varepsilon)\big](t);\\
z_2(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \tau_2(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon),\\
u_2(t,\varepsilon) = X_r(t) c_2(\varepsilon) + \varepsilon\, G\big[Z_2(z_0(s,c_0^*), u_1(s,\varepsilon), u_2(s,\varepsilon),\varepsilon); \xi_2(\varepsilon)\big](t);\\
z_{k+1}(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + \cdots + u_{k+1}(t,\varepsilon),\\
\tau_{k+1}(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \cdots + \xi_{k+1}(\varepsilon),\\
u_{k+1}(t,\varepsilon) = X_r(t)\, c_{k+1}(\varepsilon) + \varepsilon\, G\big[Z_k(z_0(s,c_0^*), u_1(s,\varepsilon), \ldots, u_k(s,\varepsilon),\varepsilon); \xi_k(\varepsilon)\big](t),\\
k = 0,1,2,\ldots.
\end{gathered} \tag{5}$$

Theorem. *Suppose that there is the critical case of the generating boundary value problem* (2)*. In this case, the generating problem* (2) *has a family of solutions*

$$z_0(t,c_0)=X_r(t)\,c_0,\ \ c_0\in\mathbb{R}^r.$$

In the case of (4) *in the small neighbourhood of the generating solution* $z_0(t,c_0^*)$ *and the constant* τ_0^* *the problem* (1) *with switchings has at least one solution. The sequence of approximations to the solution*

$$z(\,\cdot\,,\varepsilon)\in\mathbb{C}\big\{[0,T]\setminus\{\tau(\varepsilon)\}_I\big\}\cap\mathbb{C}[0,T],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary value problem (1) *with switchings is determined by an iterative scheme* (5)*. If there exist constants* $0<\gamma<1$*, and* $0<\delta<1$ *such that inequalities hold*

$$\begin{gathered}\|u_1(t,\varepsilon)\|\le\gamma\|z_0(t,c_0^*)\|,\quad \|u_{k+1}(t,\varepsilon)\|\le\gamma\|u_k(t,\varepsilon)\|,\\ |\xi_1(\varepsilon)|\le\delta\,|\tau_0^*|,\quad |\xi_{k+1}(\varepsilon)|\le\delta\,|xi_k(\varepsilon)|,\ \ k=1,2,\dots,\end{gathered}\tag{6}$$

then the iterative scheme (5) *converges to the solution of the autonomous boundary value problem* (1) *with switchings.*

The obtained iterative scheme is applied to find approximations to the periodic solution of the equation with switchings at non-fixed moments of time, which models a nonisothermal chemical reaction [1, 2].

The obtained convergence condition (6) of the iterative scheme (5) allows us to estimate the interval of values of the small parameter $\varepsilon\in[0,\varepsilon_0]$, $0\le\varepsilon_*\le\varepsilon_0$, for which the convergence of the iterative scheme (5) is preserved, different from similar estimates [5, 6].

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Regularly Varying Solutions of Differential Equations of the Second Order with Nonlinearities of Exponential Types

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We consider the following differential equation

$$y'' = \alpha_0 p(t) \exp\big(R_0(y, y') + \exp(R_1(y, y'))\big), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, the functions $R_k : \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ $(k \in \{0, 1\})$ are continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$ [1] or $]Y_i, y_i^0]$. We also suppose the functions R_k satisfy the conditions

$$\lim_{\substack{(y_0,y_1)\to(Y_0,Y_1)\\ (y_0,y_1)\in\Delta_{Y_0}\times\Delta_{Y_1}}} R_k(y_0, y_1) = +\infty, \tag{2}$$

$$\lim_{\substack{y_i\to Y_i\\ y_i\in\Delta_{Y_i}}} \frac{y_i \frac{\partial R_k(y_0,y_1)}{\partial y_i}}{R_k(y_0, y_1)} = \gamma_{ki} \ \text{ uniformly by } \ y_j \ne y_i \ \ (k, i, j \in \{0, 1\}). \tag{3}$$

Here functions R_k $(k \in \{0, 1\})$ are in some sense near to regularly varying functions, that are useful for investigations of equations of such a type. Theory of such a functions and their properties are described in the book [4]. Functions that satisfy conditions (2), (3) can be written, for example, as $|y_0|^{\gamma_{k0}} |y_1|^{\gamma_{k1}} \exp(\ln^{\mu} |y_0 y_1|)$, $|y_0|^{\gamma_{k0}} |y_1|^{\gamma_{k1}} \ln^{\mu_1} |y_0 y_1| \ln\ln |y_0 y_1|$, $0 < \mu < 1$, $\mu_1 \in \mathbb{R}$. Differential equations of the second order, containing both power and exponential nonlinearities in the right-hand side, play an important role in the development of the qualitative theory of differential equations. Such equations also have many applications in practice. This happens, for example, when studying the distribution of the electrostatic potential in the cylindrical volume of the plasma of combustion products. The corresponding equation can be reduced to the following

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

In the works by Evtukhov V. M. and Drik N. G. (see, for example, [3]) under certain conditions for the p function, results were obtained about the asymptotic behavior of all correct solutions of this equation. Partial case of the equation (1) was studied in [2].

The solution y to the equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, if

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \ \ [t_0, \omega[\subset [a, \omega[, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t) y(t)} = \lambda_0.$$

[1] As $Y_i = +\infty$ $(Y_i = -\infty)$ assume $y_i^0 > 0$ $(y_i^0 < 0)$.

The aim of the work is to establish the necessary and sufficient conditions for the existence to the equation (1) $P_\omega(Y_0, Y_1, \lambda_0)$-solutions and asymptotic representation as $t \uparrow \omega$ for such solutions and its first order derivatives in cases $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

To present the results, we introduce the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t - \omega & \text{as } \omega < +\infty, \end{cases}$$

and for every monotone continuously differentiable function $y : [t_0, \omega[\to \Delta_{Y_0}$ such that

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = Y_1,$$

$$\Phi_0(y(t)) = \int_{Y_0}^{y(t)} \exp\Big(- R_0\big(\tau, y'(t(\tau))\big) - \exp\big(R_1(\tau, y'(t(\tau)))\big)\Big)\, d\tau,$$

where $t(y)$ is the inverse function for $y(t)$,

$$\Phi_1(y) = \int_{Y_0}^{y} \frac{\Phi_0(\tau)}{\tau}\, d\tau, \quad Z_1 = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi_1(y),$$

$$I(t) = \alpha_0(\lambda_0 - 1) \int_{B_\omega^0}^{t} \pi_\omega(\tau) p(\tau)\, d\tau, \quad B_\omega^0 = \begin{cases} a & \text{as } \displaystyle\int_a^\omega \pi_\omega(\tau) p(\tau)\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_a^\omega \pi_\omega(\tau) p(\tau)\, d\tau < +\infty, \end{cases}$$

$$I_1(t) = \int_{B_\omega^1}^{t} \frac{\lambda_0 I(\tau)}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau, \quad B_\omega^1 = \begin{cases} a & \text{as } \displaystyle\int_a^\omega \frac{\lambda_0 I(\tau)}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_a^\omega \frac{\lambda_0 |I(\tau)|}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau < +\infty. \end{cases}$$

Remark. It follows from the conditions (2), (3), that functions Φ_0 and Φ_1 are rapidly varying as $y \to Y_0$ ($Y_0 \in \Delta_{Y_0}$) and

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\Phi_0''(y) \cdot \Phi_0(y)}{(\Phi_0'(y))^2} = 1, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\Phi_1''(y) \cdot \Phi_1(y)}{(\Phi_1'(y))^2} = 1.$$

The following theorem is obtained.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\gamma_{10}, \gamma_{11} \neq 0$. Then the conditions*

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0 \ \text{ as } \ t \in [a; \omega[\,,$$

$$y_1^0 \cdot \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t \uparrow \omega} I_1(t) = Z_1,$$

$$\lim_{t \uparrow \omega} \frac{I(t)}{\Phi_0(\Phi_1^{-1}(I_1(t)))} = 1, \quad \lim_{t \uparrow \omega} \frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,,$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_1'(t)}{I_1(t)} = \infty, \quad \lim_{t \uparrow \omega} \frac{I'(t)\pi_\omega(t)\Phi_0(\Phi_1^{-1}(I_1(t)))}{\Phi_0'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t)) I(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,,$$

are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions to the equation (1). Moreover, the equation (1) has a one-parametric family of solutions in case $(\lambda_0 - 1)\beta < 0$ and two-parametric family of solutions in cases

$$\big((1 < \lambda_0 < 3) \wedge (\beta > 0)\big) \vee (\lambda_0 > 3) \wedge (\beta < 0).$$

For every such solution the next asymptotic representations take place as $t \uparrow \omega$

$$y(t) = \Phi_1^{-1}(I_1(t))[1 + o(1)], \quad y'(t) = \frac{\lambda_0}{(\lambda_0 - 1)} \cdot \frac{\Phi_1^{-1}(I_1(t))}{\pi_\omega(t)}\,[1 + o(1)].$$

The differential equation

$$y'' = \alpha_0 p(t) \exp(R_0(y, y')), \tag{4}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, the function $R : \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ is continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$ [1] or $]Y_i, y_i^0]$, was considered in [1] and it is the equation of type (1), where $\gamma_{10} = \gamma_{11} = 0$.

In this case we have

$$\Phi_0(y) = \int\limits_{Y_0}^{y} \exp\big(-R(\tau, y'(t^{-1}(\tau)))\big)\, d\tau,$$

where $t^{-1}(y)$ is the inverse function for $y(t)$,

$$\Phi_1(y) = \int\limits_{Y_0}^{y} \frac{\Phi_0(\tau)}{\tau}\, d\tau, \quad Z_1 = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi_1(y).$$

For the equation (4) the next result is valid.

Theorem 2. *Let $\gamma_0\lambda_0 + \gamma_1 \in \mathbb{R} \setminus \{0, \lambda_0\}$. Then the conditions*

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0, \ \ t \in [a; \omega[\,,$$

$$y_1^0 \cdot \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t \uparrow \omega} I_1(t) = Z_1,$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t)\big(\frac{I_1(t)}{I_1'(t)}\big)'}{\frac{I_1(t)}{I_1'(t)}} = \frac{\lambda_0\gamma_0 + \gamma_1 + 1}{\lambda_0 - 1}\,, \quad \lim_{t \uparrow \omega} \frac{I_1''(t) I_1(t)}{(I_1'(t))^2} = 1,$$

$$\lim_{t \uparrow \omega} \frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_1''(t)}{I_1'(t)} = \infty$$

are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions to the equation (4) in cases $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$. Moreover, for every such solution the next asymptotic representations take place as $t \uparrow \omega$

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\,[1 + o(1)].$$

For equations of more concrete type we can find more precise representations. For $t \in [2, +\infty[$ let us consider the differential equation

$$y'' = \frac{1}{4}\, t^{-3} L(t) e^{|y|^4 - t^8} |y'|^3, \tag{5}$$

where $L : [2, +\infty[\to]0, +\infty[$ is slowly varying on infinity function.

In this case

$$
\begin{aligned}
I_1(t) = \frac{\lambda_0}{8\sqrt{|\lambda_0 - 1|}(\lambda_0 - 1)}\, t^{-14}(L(t))^{-\frac{1}{2}} e^{\frac{1}{2}t^8}[1 + o(1)],\\
\Phi_0(y) = \frac{1}{2y^3}\, e^{\frac{1}{2}|y|^4} \operatorname{sign} y[1 + o(1)] \ \text{ as } \ y \to +\infty,\\
\Phi_1(y) = \frac{1}{4y^7}\, e^{\frac{1}{2}|y|^4}[1 + o(1)] \ \text{ as } \ y \to +\infty,\\
\frac{\Phi_1'(y)}{\Phi_1(y)} = 2y^3[1 + o(1)] \ \text{ as } \ y \to +\infty.
\end{aligned}
$$

We have that $P_{+\infty}(Y_0, Y_1, \lambda_0)$-solutions of the equation (5) can be only $P_{+\infty}(+\infty, +\infty, 2)$-solutions. Moreover, for every such solution the next asymptotic representations take place as $t \to \infty$,

$$
\begin{aligned}
\frac{1}{y^7(t)} e^{\frac{1}{2}y^4(t)} = t^{-14}(L(t))^{-\frac{1}{2}} e^{\frac{1}{2}t^8}[1 + o(1)],\\
y'(t)y^3(t) = 2t^7[1 + o(1)] \ \text{ as } \ t \uparrow \omega.
\end{aligned}
$$

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Asymptotic Representations of Solutions to Differential Equations of the Fourth Order with Nonlinearities, Close to Regularly Varying

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The differential equation

$$y^{(4)} = \alpha_0 p(t) \prod_{i=0}^{3} \varphi_i(y^{(i)}) \exp\Big(\gamma \Big| \sum_{i=0}^{3} \ln |y^{(i)}| \Big|^{\mu}\Big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $\gamma \in R$, $\mu \in]0;1[$, $p : [a, \omega[$ [1] $\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, 1, 2, 3)$ are the continuous functions, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ [2] or the interval $]Y_i, y_i^0]$, is considered.

We suppose also that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum\limits_{i=0}^{3} \sigma_i \ne 1$.

According to properties of regularly varying functions (see, for example, the monograph [7]) it is clear that for every defined on $[t_0, \omega[\subset [a, \omega[$ solution y of the equation (1) such that

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1, 2, 3), \tag{2}$$

the representations $\varphi_i(y^{(i)}(t)) = |y^{(i)}(t)|^{\sigma_i + o(1)}$ take place as $t \uparrow \omega$. Therefore the equation (1) is in some sense similar to the well known differential equation of Emden–Fowler type.

The first results on the asymptotics of solutions of differential equations with regularly varying nonlinearities have been obtained in the works by V. Marić, M. Tomić [6], S. D. Taliaferro [8], V. M. Evtukhov, L. O. Kirillova [4] and some other authors for the differential equations of the second order of the type

$$y'' = \alpha_0 p(t) \varphi(y).$$

Any regularly varying function is a product of some power function and some slowly varying function. Therefore researches of equations with regularly varying nonlinearities have been connected with the wish to extend to such equations the results, that have been received during the 20th century for the equations with power nonlinearities, in particular, for the generalized equation of Emden–Fowler's type, particular cases of which appear in a lot of sciences of nature.

We call the solution y of the equation (1), that satisfies (2), the $P_\omega(Y_0, Y_1, Y_2, Y_3 \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the next condition takes place

$$\lim_{t \uparrow \omega} \frac{(y'''(t))^2}{y^{(4)}(t)\, y''(t)} = \lambda_0.$$

[1] If $\omega > 0$, we will take $a > 0$.

[2] If $Y_i = +\infty (Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$, correspondingly.

The improvement of mathematical models of physical phenomena contributed to the growth of the number of results for equations of greater than the general form. In the works by V. M. Evtukhov and A. V. Drozhzhyna (see, for example, [3]) the differential equation of general form

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

was investigated. Here $f : [a, \omega[\times \Delta_{Y_0} \times \cdots \times \Delta_{Y_{n-1}} \to \mathbb{R}$ is a continuous function, $-\infty < a < \omega \leq +\infty$, $\Delta_{Y_{i-1}}$ is some one-sided neighbourhood of Y_{i-1}, Y_{i-1} equals to zero or to $\pm\infty$, $i = 1, \dots, n$. The subject of the research is $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of this equation, conditions of their existence and also asymptotic as $t \uparrow \omega$ representations of such solutions and their derivatives up to the order $n-1$. The class of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions was introduced in the works by V. M. Evtukhov and it appeared to be an enough wide class of monotone solutions. It includes regularly, slowly and rapidly varying as $t \uparrow \omega$ solutions and also some types of singular solutions. Every of the mentioned above $n+2$ types of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the differential equation of the n-th order of general form is studied separately by the fulfillment of the condition $(RN)_{\lambda_0}$. The kernel of the condition is the fact that onto any of such solutions the equation is in some sense asymptotically near to the equation

$$y^{(n)} = \alpha_0 p(t) \prod_{j=1}^{n} \varphi_{j-1}(y^{(j-1)}), \tag{3}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $\varphi_{j-1} : \Delta_{Y_{j-1}} \to]0, +\infty[$ is a continuous regularly varying function of the order σ_{j-1} as $y^{(j)} \to Y_{j-1}$, $j = 1, \dots, n$.

In the equation (1) the nonlinearity is not near to the form (3) because of the type of the function

$$\exp\Big(\gamma \Big| \sum_{i=0}^{3} \ln |y^{(i)}| \Big|^{\mu}\Big).$$

It follows from the definition of $P_\omega(\lambda_0)$-solution that in cases $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}\}$ every $P_\omega(\lambda_0)$-solution of the equation (1) is regularly varying as $t \uparrow \omega$. In case of second order differential equation for all $P_\omega(\lambda_0)$-solutions of the equation of the type (1) the necessary and sufficient conditions of existence and asymptotic representations as $t \uparrow \omega$ were found (see, for example, [1, 2, 4–6, 8, 8]).

Let us introduce the subsidiary notations.

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-1} (n-j-1)\sigma_j, \quad \pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \quad a_{0i} = (n-i)\lambda_{n-1}^0 - (n-i-1) \ \ (i = 1, \dots, n),$$

$$C = \alpha_0 |\lambda_{n-1}^0 - 1|^{\mu_n} \prod_{k=0}^{n-2} \Big| \prod_{j=k+1}^{n-1} a_{0j} \Big|^{-\sigma_k} \operatorname{sign} y_{n-1}^0,$$

$$I_0(t) = \int_{A_\omega^0}^{t} C p(\tau) |\pi_\omega(\tau)|^{\mu_n} \, d\tau, \quad I_1(t) = \int_{A_\omega^1}^{t} \alpha_0 p(\tau) \, d\tau,$$

$$A_\omega^0 = \begin{cases} a & \text{if } \displaystyle\int_a^\omega p(\tau) |\pi_\omega(\tau)|^{\gamma_0} \, d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega p(\tau) |\pi_\omega(\tau)|^{\gamma_0} \, dtau < +\infty, \end{cases} \qquad A_\omega^1 = \begin{cases} a & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau < +\infty, \end{cases}$$

$$J(t) = \int\limits_{B_\omega}^{t} |\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}}\, d\tau, \quad B_\omega = \begin{cases} a & \text{if } \int\limits_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\, d\tau = +\infty, \\ \omega & \text{if } \int\limits_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\, d\tau < +\infty. \end{cases}$$

The following conclusions take place for the equation (1).

Theorem 1. *The next conditions are necessary for the existence of $P_\omega(Y_0, Y_1, Y_2, Y_3\lambda_0)$-solutions $(\lambda_0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}\})$ of the equation (1):*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = \frac{\gamma_0}{\lambda_{n-1}^0 - 1}, \quad \lim_{t\uparrow\omega} y_i^0 |\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_{n-1}^0 - 1}} = Y_i, \tag{4}$$

$$y_i^0 y_{i+1}^0 a_{0i+1}(\lambda_{n-1}^0 - 1)\pi_\omega(t) > 0 \ \text{ as } \ t \in [a, \omega[\,, \tag{5}$$

where $y_3^0 = \alpha_0$, $i = 0, \ldots, 3$.

If the equation

$$\sum_{k=0}^{3} \sigma_k \prod_{i=k+1}^{3} a_{0i} \prod_{i=1}^{k} (a_{0i} + \lambda) = (1 + \lambda) \prod_{i=1}^{3} (a_{0i} + \lambda)$$

has no roots with zero real part, then the conditions (4), (5) are sufficient for the existence of $P_\omega(Y_0, Y_1, Y_2, Y_3\lambda_0)$-solutions of the equation (1). For any such solution the next asymptotic representations as $t \uparrow \omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0} \exp\big(-\gamma |\sum_{i=0}^{3} \ln |y^{(i)}||^{\mu}\big)}{\prod\limits_{j=0}^{n-1} \theta_j(y^{(j)}(t))} = \gamma_0 I_0(t)[1 + o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \frac{[\,(\lambda_{n-1}^0 - 1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1} a_{0j}} [1 + o(1)],$$

where $i = 0, \ldots, 2$, take place.

By additional conditions on the functions $\varphi_0, \varphi_1, \ldots, \varphi_3$ the asymptotic representations as $t \uparrow \omega$ of $P_\omega(Y_0, Y_1, Y_2, Y_3\lambda_0)$-solutions and their derivatives from the first to third order are found in another form.

In order to formulate our following results, we present the next definition.

We call the slowly varying as $z \to Y$ $(z \in \Delta)$ function θ satisfies the condition S if for every continuously differentiable function $L : \Delta \to]0; +\infty[$ such that

$$\lim_{\substack{z\to Y \\ z\in\Delta}} \frac{zL'(z)}{L(z)} = 0,$$

the next representation takes place

$$\theta(zL(z)) = \theta(z)[1 + o(1)] \ \text{ as } \ z \to Y \ \ (z \in \Delta).$$

The next result follows from Theorem 1.

Theorem 2. *Let the functions $\theta_0,\dots,\theta_3$ satisfy the condition S. Then for any $P_\omega(Y_0,Y_1,Y_2,Y_3\lambda_0)$-solution ($\lambda_0\in\mathbb{R}\setminus\{0,1,\frac12,\frac23\}$) of the equation* (1) *the next asymptotic representations as $t\uparrow\omega$*

$$y^{(n-1)}(t)\exp\Big(-\frac{\gamma}{\gamma_0}\Big|\sum_{i=0}^{3}\ln|y^{(i)}|\Big|^{\mu}\Big)=\Big|\gamma_0I_0(t)\prod_{j=0}^{n-1}\theta_j\big(y_j^0|\pi_\omega(t)|^{\frac{a_{0j+1}}{\lambda_{n-1}^0-1}}\big)\Big|^{\frac{1}{\gamma_0}}\operatorname{sign}y_{n-1}^0[1+o(1)],$$

$$y^{(i)}(t)=y^{(n-1)}(t)\,\frac{[(\lambda_{n-1}^0-1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1}a_{0j}}\,[1+o(1)],\quad i=0,\dots,n-2$$

take place.

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Radial Properties of Stability and Instability of a Differential System

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For a natural number $n \in \mathbb{N}$ and a domain G of the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure mes, consider a differential system of the form

$$\dot{x} = f(t,x), \quad x \in G, \quad f(t,0) = 0, \quad t \in \mathbb{R}_+ \equiv [0,+\infty), \quad f, f'_x \in C(\mathbb{R}_+ \times G). \tag{1}$$

Let $B_\delta \equiv \{x_0 \in \mathbb{R}^n : \ 0 < |x_0| \le \delta\}$, $\mathcal{S}_\delta(f)$ be the set of nonextendable solutions of system (1) with initial values $x(0) \in B_\delta$, and $\mathcal{S}_{\delta,x_0}(f) \subset \mathcal{S}_\delta(f)$ – its subset consisting of solutions satisfying the additional condition $x(0) = cx_0$, $c > 0$.

The initial concepts of this report are such properties of the zero solution as stability, asymptotic stability and complete instability. They are massive [6] in the sense that in their description certain conditions are imposed on all solutions starting in a neighborhood of zero. In addition, each of them can be of one of the following three types: Lyapunov, Perron and upper-limit (the last two ones, introduced relatively recently [4, 5], admit contrasting combinations with the first one [3]).

Definition 1 ([7])**.** We say that system (1) (more precisely, its zero solution) has the *Lyapunov, Perron* or, respectively, *upper-limit*:

1) *stability* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the following corresponding requirement

$$\sup_{t\in\mathbb{R}_+} |x(t)| \le \varepsilon, \quad \underline{\lim}_{t\to+\infty} |x(t)| \le \varepsilon, \quad \overline{\lim}_{t\to+\infty} |x(t)| \le \varepsilon \tag{2}$$

(assuming by default that this solution is defined on the entire ray $\mathbb{R}_+$; otherwise, this requirement is assumed to fail to hold);

2) *asymptotic stability* if: in the *Perron* or *upper-limit* cases – there exists a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies corresponding requirement (2) for $\varepsilon = 0$, and in the *Lyapunov* case – it has both upper-limit asymptotic stability and Lyapunov stability;

3) *complete instability* if for some $\varepsilon, \delta > 0$ no solution $x \in \mathcal{S}_\delta(f)$ satisfies corresponding requirement (2).

Each of the properties introduced in Definition 1 can be associated with its radial analogue, in which the initial values of the perturbed solutions are taken not from the complete neighborhood of zero, but only along a given ray starting from zero.

Definition 2 ([10])**.** We say that system (1) has the *Lyapunov, Perron* or *upper-limit*:

4) *radial* property: *stability, asymptotic stability* or *complete instability* in the *direction* of nonzero vector $x_0 \in \mathbb{R}^n$ – if it has the corresponding property from points 1–3 of Definition 1 with the replacement of the set $\mathcal{S}_\delta(f)$ in it by the set $\mathcal{S}_{\delta,x_0}(f)$;

5) *total radial* property: *stability, asymptotic stability* or *complete instability* – if it has this property in each direction.

There is a simple, albeit one-sided, logical connection between the properties introduced in points 1–3 of Definition 1 and their total radial analogues from point 5 of Definition 2.

Theorem 1. *If system* (1) *has stability, asymptotic stability or complete instability of the Lyapunov, Perron or upper-limit type, then it has total radial stability, asymptotic stability or, respectively, complete instability of the same type.*

In the one-dimensional case, the statement of Theorem 1 is reversible.

Theorem 2. *If for $n = 1$ system* (1) *has a total radial property of the Lyapunov, Perron or upper-limit type: stability, asymptotic stability or complete instability, – then it has stability, asymptotic stability or, respectively, complete instability of the same type.*

In the special case of a linear homogeneous system

$$\dot{x} = A(t)x \equiv f(t,x), \quad t \in \mathbb{R}_+, \quad x \in G \equiv \mathbb{R}^n, \quad A \in C(\mathbb{R}_+, \operatorname{End}\mathbb{R}^n), \tag{3}$$

total radial properties have some peculiarities.

Theorem 3. *If linear system* (3) *has a total radial property of the Lyapunov, Perron or upper-limit type: asymptotic stability or complete instability, – then it has asymptotic stability or, respectively, complete instability of the same type.*

Theorem 4. *If linear system* (3) *has a total radial stability of the Lyapunov or upper-limit type, then it has stability of both of these types at once.*

Remark 1. The problem of the possibility to extending the statement of Theorem 4 also to similar Perron-type stability properties (separately from Lyapunov and upper-limit) from points 1, 5 of Definitions 1, 2 remains unresolved for now.

In the case in which system (1) does not have some of the initial properties of Definition 1, the question of whether this property holds at least to some extent, becomes meaningful. To answer this question, the following definition introduces characteristics (partly new), which are naturally called measures of these properties and have a probabilistic connotation.

Definition 3 ([8])**.** The *measures of Lyapunov, Perron* or *upper-limit stability* and *instability* of system (1) for $\varkappa = \lambda, \pi, \sigma$ are respectively defined by the formulas

$$\mu_\varkappa(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_\varkappa(f,\varepsilon,\delta)}{\operatorname{mes} B_\delta}\,, \quad \mu_{\overline{\varkappa}}(f) = \lim_{\varepsilon\to+0} \varlimsup_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_{\overline{\varkappa}}(f,\varepsilon,\delta)}{\operatorname{mes} B_\delta}\,, \tag{4}$$

and the *measures of asymptotic stability* of the same types are defined respectively by formulas

$$\mu_{\lambda_0}(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \frac{\operatorname{mes}(\mathrm{M}_\sigma(f,0,\delta) \cap \mathrm{M}_\lambda(f,\varepsilon,\delta))}{\operatorname{mes} B_\delta}\,, \quad \mu_{\varkappa_0}(f) = \varliminf_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_\varkappa(f,0,\delta)}{\operatorname{mes} B_\delta}\,, \tag{5}$$

where $\varkappa = \pi, \sigma$, $\mathrm{M}_{\overline{\varkappa}}(f,\varepsilon,\delta) \equiv 1 - \mathrm{M}_\varkappa(f,\varepsilon,\delta)$ and $\mathrm{M}_\varkappa(f,\varepsilon,\delta)$ – the set of initial values $x(0)$ of all solutions $x \in \mathcal{S}_\delta(f)$ satisfying the corresponding requirement (2).

The concepts introduced in Definition 3 are correct.

Theorem 5. *For any system* (1)*, all the sets in formulas* (4) *and* (5) *are measurable, and the limits as $\varepsilon \to +0$ exist and can be replaced in the formulas for stability measures (including the Lyapunov asymptotic) and instability, respectively, by the exact lower and upper bounds at $\varepsilon > 0$.*

There is a logical connection between total radial properties and their measures.

Theorem 6. *If system* (1) *has a total radial property of some type: stability, asymptotic stability or complete instability, – then its measure of stability, asymptotic stability or, respectively, instability of this type is equal to 1.*

For some properties, the statement of Theorem 6 can even be slightly strengthened.

Theorem 7. *If system* (1) *has a total radial property of some type: stability or asymptotic stability, – then in the corresponding formula for stability measures* (4) *or, respectively, asymptotic stability measures* (5) *of this type, the lower limit as* $\delta \to +0$ *coincides with the upper limit and is equal to 1.*

Definition 4. We say that system (1) has *Perron* or *upper-limit partial extreme instability* if for any $\delta > 0$ there exists a solution $x \in \mathcal{S}_\delta(f)$ satisfying the corresponding requirement

$$\varliminf_{t\to+\infty} |x(t)| = +\infty, \qquad \varlimsup_{t\to+\infty} |x(t)| = +\infty \tag{6}$$

(which is assumed to hold, in particular, if the solution is not defined on the entire ray $\mathbb{R}_+$).

Remark 2. The property given in Definition 4 is a type of extreme instability [1, 9]:

– reinforced by the fact that the limit (6) in it is infinite;

– weakened by the fact that requirement (6) is not necessarily satisfied here for all solutions $x \in S_\delta(f)$ but at least for one of them.

Definition 5 ([6]). If we assume in Definition 1 that not all solutions $x \in \mathcal{S}_\delta(f)$ satisfy requirement (2) in points 1–3, but *almost* all (i.e. with initial values $x(0)$ from the ball $B_\delta(f)$ minus a subset of measure zero), then the result will be the definition of the following properties of a system (1): *almost stability*, *almost asymptotic stability* and *almost complete instability* of the corresponding type.

The satisfiability of Theorem 6 assumptions for a system (1), does not ensure for it not only stability, asymptotic stability or, respectively, complete instability, but even almost stability, almost asymptotic stability or almost complete instability (in particular [10], for two-dimensional systems).

Theorem 8. *For any natural* $n > 1$ *there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously:*

– *has total radial asymptotic stability of all three types;*

– *has both Perron and upper-limit partial extreme instability;*

– *does not have almost stability of any type.*

Theorem 9. *For any natural* $n > 1$ *there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously:*

– *has total radial complete instability of all three types;*

– *has both Perron and upper-limit partial extreme instability;*

– *does not have almost complete instability of any type.*

Theorem 10. *For any natural $n > 1$ there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously has:*

– *total radial asymptotic stability of all three types;*

– *both Perron and upper-limit partial extreme instability;*

– *both Perron and upper-limit almost asymptotic stability.*

Remark 3. It does not seem possible to strengthen Theorem 10 by adding Lyapunov asymptotic almost stability to its formulation, since the presence of this stability implies the presence of Lyapunov stability in a system (see [3, Theorem 3]), which can not be implemented in a simultaneous combination with Perron partial extreme instability.

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On the Solvability of Linear Functional Differential Equations of the First and Second Orders

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We consider the functional differential equations, which can be written in the operator form:

$$(\mathcal{L}_1 x)(t) \equiv \dot{x}(t) - (T^+x)(t) + (T^-x)(t) = f(t), \quad t \in [0,1], \tag{1}$$

$$(\mathcal{L}_2 x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t) = f(t), \quad t \in [0,1], \tag{2}$$

where T^+ and T^- are linear positive operators acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ (positive operators map non-negative functions into non-negative ones), $f \in \mathbf{L}[0,1]$ is integrable. The solution of equation (1) (equation (2)) is an absolutely continuous function on $[0,1]$ (a function with an absolutely continuous derivative on $[0,1]$) that satisfies the equation for almost all $t \in [0,1]$.

Numerous studies have explored the solvability conditions of a wide range of boundary value problems associated with functional differential equations (1), (2), in particular, the periodic, Cauchy, antiperiodic, and other types of boundary value problems [1–3, 5].

The literature concerning solvability conditions for functional differential equations, as distinct from boundary value problems for these equations, is notably sparse, if not entirely absent. However, the question of the solvability of the functional differential equation itself is nontrivial, since we do not require the operators to be Volterra, hence, in particular, the Cauchy problem may not have a solution. To the best of the author's knowledge, simple coefficient conditions for the solvability of equations (1), (2) have not yet been formulated. Our goal is to fill this gap and obtain unimprovable sufficient conditions for solvability in terms of the norms of the positive operators T^+ and T^-.

Such operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ have the representation [4, p. 317] in the form of the Stieltjes integral

$$(Tx)(t) = \int_0^1 x(s)\, d_s r(t,s), \quad t \in [0,1],$$

where $r(t,\cdot) \in \mathbf{L}[0,1]$ is nondeccreasing for almost all $t \in [0,1]$, the function $t \to r(t,1) - r(t,0)$ is integrable on $[0,1]$. We assume that $r(t,0) \equiv 0$ for all $t \in [0,1]$. The norm of such an operator is defined by the equality

$$\|T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(t)\, dt = \int_0^1 r(t,1)\, dt,$$

where $\mathbf{1}$ is the unit function.

Definition. We will call equation (1) or (2) everywhere solvable if for each function $f \in \mathbf{L}[0,1]$ there is at least one solution.

For first order equations, it was relatively easy to show that if the following conditions (4) or (5) are satisfied, then for the equation $\mathcal{L}_1 x = f$ one of the boundary value problems

$$x(0) = x(1), \quad x(0) = 0, \quad x(1) = 0$$

is uniquely solvable. Thus, the equation $\mathcal{L}_1 x = f$ is solvable everywhere. If conditions (4) and (5) are not satisfied, then there exist linear positive operators T^+, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$, for which the equalities (3) are satisfied and $\mathcal{L}_1(\mathbf{AC}[0,1]) \neq \mathbf{L}[0,1]$.

Theorem 1. *The equation* (1) *is everywhere solvable for all linear positive operators* T^+, T^- : $\mathbf{C}[0,1] \to \mathbf{L}$ *satisfying equalities*

$$\| T^+ \|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^+, \quad \| T^- \|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^-, \tag{3}$$

if and only if non-negative numbers $\mathcal{T}^+$, $\mathcal{T}^-$ *satisfy the inequalities*

$$\mathcal{T}^+ < 1, \quad \mathcal{T}^- < 2\big(1 + \sqrt{1 - \mathcal{T}^+}\big) \tag{4}$$

or the inequalities

$$\mathcal{T}^- < 1, \quad \mathcal{T}^+ < 2\big(1 + \sqrt{1 - \mathcal{T}^-}\big). \tag{5}$$

In the study of the equation (2), the adjoint operator

$$\mathcal{L}_2^* : \mathbf{L}_\infty[0,1] \to (\mathbf{AC}^1[0,1])^* \simeq \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$$

is used.

Since $\mathcal{L}_2$ is a Noetherian operator of index 2, the equation $\mathcal{L}_2$ is everywhere solvable if and only if the homogeneous equation with the adjoint operator

$$\mathcal{L}_2^* g = 0_{(\mathbf{AC}^1[0,1])^*}, \quad g \in \mathbf{L}_\infty[0,1], \tag{6}$$

has only the trivial solution.

If the function $g \in \mathbf{L}_\infty[0,1]$ is a solution to the equation (6), then the function g is absolutely continuous and satisfies the following boundary value problem:

$$\dot g(t) = \int_0^1 r(s,t) g(s)\, ds, \quad t \in [0,1], \tag{7}$$

$$g(0) = 0, \quad g(1) = 0, \quad \int_0^1 r(s,1) g(s)\, ds = 0, \tag{8}$$

where $r(s,t) = r^+(s,t) - r^-(s,t)$,

$$(T^+ x)(t) = \int_0^1 x(s)\, d_s r^+(t,s), \quad (T^- x)(t) = \int_0^1 x(s)\, d_s r^-(t,s), \quad t \in [0,1].$$

When studying the system (7), (8), we find that if for given $\mathcal{T}^+$, $\mathcal{T}^-$ there exists a nontrivial solution of this system for some operators T^+, T^- satisfying the equalities (3), then the system (7), (8) has a piecewise linear solution (possibly for other operators T^+, T^- satisfying the equalities (3)). Such a solution corresponds to some operators T^+, T^- of the following form:

$$(T^+ x)(t) = \sum_{j=1}^n p_j^+(t) x(t_j), \quad (T^- x)(t) = \sum_{j=1}^n p_j^-(t) x(t_j), \quad t \in [0,1], \tag{9}$$

the integrable functions p_j^+, p_j^- are non-negative,

$$0 \le t_1 < t_2 < \cdots < t_n \le 1.$$

For operators of the form (9), for which the equalities

$$\sum_{j=1}^{n} \| p_j^+ \|_{\mathbf{L}} = \mathcal{T}^+, \quad \sum_{j=1}^{n} \| p_j^- \|_{\mathbf{L}} = \mathcal{T}^-,$$

are satisfied, the solvability conditions for problem (7), (8) are formulated explicitly.

Let's introduce the following notation:

$$G(k) = \left(1 + \sqrt{k} + \sqrt{k+1}\right)^2 (k+1),$$
$$H_1(k) = \frac{G(k) - \mathcal{T}^-}{k}, \quad H_2(k) = G(k) - k\,Q,$$
$$\widetilde{P}_1(\mathcal{T}^-) \equiv \min_{k \in (0,1]} H_1(k, \mathcal{T}^-), \quad \widetilde{P}_2(\mathcal{T}^-) \equiv \min_{k \in [0,1]} H_2(k, \mathcal{T}^-).$$

Remark. Note that $\widetilde{P}_1(\mathcal{T}^-)$ decreases on $[0, 4]$ and can be defined parametrically:

$$\mathcal{T}^- = G(k) - \frac{dG(k)}{dk}\,k, \quad \widetilde{P}_1(\mathcal{T}^-) = \frac{dG(k)}{dk}, \quad k \in [k_0, 1],$$

where $k_0 = k_1^2 \approx 0.43$, $k_1 \in [0, 1]$ is the only root of the equation $k^4 + 6k^3 + 5k^2 - k = 0$ on the interval $[0, 1]$.

The function $\widetilde{P}_2$ is equal to 4 on $[4, \widetilde{P}_1(4)]$, where $\widetilde{P}_1(4) \approx 17.7$; on the interval $[\widetilde{P}_1(4), 12 + 8\sqrt{2}]$ the function $\widetilde{P}_2(\mathcal{T}^-)$ decreases and can also be specified parametrically:

$$\widetilde{P}_2(\mathcal{T}^-) = G(k) - \frac{dG(k)}{dk}\,k, \quad \mathcal{T}^- = \frac{dG(k)}{dk}.$$

Theorem 2. *Let non-negative $\mathcal{T}^+$ and $\mathcal{T}^-$ be given. The equation* (2) *is everywhere solvable for all positive linear operators T^+, $T^- : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that equalities* (3) *hold, if and only if*

$$\mathcal{T}^- \in [0, 4], \quad \mathcal{T}^+ \le \widetilde{P}_1(\mathcal{T}^-),$$

or

$$\mathcal{T}^- \in (4, 12 + 8\sqrt{2}], \quad \mathcal{T}^+ \le \widetilde{P}_2(\mathcal{T}^-).$$

Corollary. *Let non-negative $\mathcal{T}$ be given. Each of the equations*

$$\ddot{x}(t) - (Tx)(t) = f(t), \quad \ddot{x}(t) + (Tx)(t) = f(t), \quad t \in [0, 1],$$

is everywhere solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that $\| T \|_{\mathbf{C} \to \mathbf{L}} = \mathcal{T}$, if and only if

$$\mathcal{T} \le 12 + 8\sqrt{2}.$$

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Adomian Decomposition Method in Theory of Nonlinear Periodic Boundary Value Problems

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For the nonlinear boundary-value problem for an ordinary differential equation in the critical and noncritical cases, we obtain constructive conditions of its solvability and the scheme for finding solutions by using Adomian decomposition method.

We investigate the problem of construction of solution [3, 6, 7]

$$z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1[0,T], \;\; z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0]$$

of the nonlinear periodic boundary-value problem

$$\frac{dz}{dt} = A\,z + f(t) + Z(z,t), \;\; \ell z(\,\cdot\,) := z(0) - z(T) = 0 \tag{1}$$

in a small neighborhood of the solution of the generating problem

$$\frac{dz_0}{dt} = A\,z_0, \;\; \ell z_0(\,\cdot\,) := z_0(0) - z_0(T) = 0, \tag{2}$$

where A is a constant $(n \times n)$-dimensional matrix, $Z(z,t)$ is a nonlinear vector function analytic in the unknown z in a small neighborhood of the solution of the generating problem (2). In addition, the vector function $Z(z,t)$ and the function $f(t)$ are continuous in the independent variable t on the segment $[a,b]$.

The urgency of investigation of the boundary-value problem (1) is explained by extensive applications of similar problems in the study of nonisothermal chemical reactions. An example of simulation of these reactions can be found in [2].

At the end of the present paper, we give an example of determination of approximations to a periodic solution of problem (1) obtained by using our iterative scheme. In [4, 5], approximations to the solutions of nonlinear boundary-value problems and, in particular, periodic boundary-value problems, were found by using the effective Newton–Kantorovich method [9].

In constructing solutions of nonlinear boundary-value problems, we encounter the problem of impossibility of representation of these solutions in terms of elementary functions, which, in turn, leads to the appearance of significant errors in the solutions of the analyzed problems. A similar situation was demonstrated for a periodic problem posed for the equation used to describe the motion of a satellite on the elliptic orbit [11].

In addition, the procedure of construction of the solutions of nonlinear boundary-value problems by the method of simple iterations is significantly complicated by the necessity of evaluation of the derivatives of nonlinearities [6]. In [4, 5], the rate of convergence of iterations was improved as a result of evaluation of the derivatives of nonlinearities in each step. In view of these difficulties, we can expect that the procedure of evaluation of the derivatives of nonlinearities can be simplified and the solutions of nonlinear boundary-value problems (and, in particular, of periodic boundary-value problems) can be found in terms of elementary functions by using the Adomian decomposition method [1]. An example of this simplification is presented in [8].

By $X(t)$ we denote a normal ($X(a) = I_n$) fundamental matrix of the generating problem (2). In the critical case, we have

$$\det Q = 0$$

and the generating problem (2) under the condition [6]

$$P_{Q_r^*} \ell K[f(s)](\,\cdot\,) = 0 \tag{3}$$

has an r-parameter family of solutions

$$z_0(t, c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r.$$

Here, the matrix $X_r(t)$ consists of r-linearly independent columns of the normal fundamental matrix $X(t)$. The matrix $P_{Q_r^*}$ is formed by r linearly independent rows of the matrix orthoprojector. Furthermore,

$$G[g(s)](t) := K[g(s)](t) - X(t)Q^+ \ell K[g(s)](\,\cdot\,)$$

is the generalized Green operator of the periodic boundary-value problem [6]

$$\frac{dy}{dt} = A\,y + g(t), \quad y(0) - y(T) = 0$$

in the critical case and Q^+ is the pseudoinverse Moore-–Penrose matrix. It is known that the critical case occurs if and only if the matrix A has eigenvalues on the imaginary axis, namely, imaginary numbers of the form

$$\lambda = \frac{2\pi i k}{T}\,, \ \ k = 0, 1, 2, \dots, \ \ i = \sqrt{-1}\,.$$

The necessary and sufficient condition for the solvability of problem (1)

$$P_{Q_r^*}\, \ell K\big[Z(z(s), s)\big](\,\cdot\,) = 0$$

leads to a necessary condition for the solvability of problem (2) in a small neighborhood of the solution of the generating T-periodic problem

$$F_0(c_r) := P_{Q_r^*}\, \ell K\big[A_0(z_0(s, c_r), s)\big](\,\cdot\,) = 0. \tag{4}$$

In what follows, equation (4) is called the equation for generating amplitudes of the $T-$periodic problem (1). Assume that the equation for generating amplitudes (4) has real roots. Fixing one of real solutions $c_r^* \in \mathbb{R}^r$ of equation (4), we arrive at the problem of construction of a solution to the nonlinear $T-$periodic problem (1) in a small neighborhood of the solution

$$z_0(t, c_r^*) = X_r(t)c_r^* + G[f(s)](t), \ \ c_r^* \in \mathbb{R}^r$$

of the generating $T-$periodic problem (2). The conventional condition of solvability of problem (1) in a small neighborhood of the generating T-periodic problem (2) is the requirement of simplicity of the roots [6]

$$\det B_0 \neq 0, \quad B_0 := F_0'(c_0) \in \mathbb{R}^{r \times r}$$

of the equation for generating amplitudes (4) of the T-periodic problem (1). The form of the matrix B_0 which plays the key role in the investigation of the T-periodic problem (1) with the use of the Adomian decomposition [1, c. 502], coincides with the conventional form [6]

$$B_0 = P_{Q_r^*}\,\ell K[\mathcal{A}_1(s)X_r(s)](\,\cdot\,), \quad \mathcal{A}_1(t) = \left.\frac{\partial Z(z,t)}{\partial z}\right|_{z=z_0(t,c_r^*)}$$

is an $(n\times n)$-dimensional matrix. We seek the solution of the periodic boundary-value problem (1) in the form

$$z(t) := z_0(t,c_r^*) + u_1(t) + \cdots + u_k(t) + \cdots\,.$$

Since the nonlinear vector function $Z(z,t)$ is analytic in the unknown z in a neighborhood of the solution $z_0(t,c_r^*)$ of the generating problem (2), the following decomposition is true in this neighborhood [1, p. 502]

$$Z(z(t),t) = A_0(z_0(t,c_r^*),t) + A_1\big(z_0(t,c_r^*),u_1(t),t\big) + \cdots + A_n\big(z_0(t,c_r^*),u_1(t),\ldots,u_n(t),t\big) + \cdots\,. \quad (5)$$

The first approximation to the solution of the nonlinear periodic boundary-value problem (1) in the critical case

$$z_1(t,c_r^*) := z_0(t,c_r^*) + u_1(t), \quad u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t), \;\; c_1\in\mathbb{R}^r$$

is given by the solution of the nonlinear periodic boundary-value problem of the first approximation

$$u_1'(t) = A\,u_1(t) + A_0(z_0(t,c_r^*)), \quad u_1(0) - u_1(T) = 0.$$

The periodicity of solution to the boundary-value problem of the first approximation is guaranteed by the choice of the solution $c_r^*\in\mathbb{R}^r$ of equation (4). The second approximation to the solution of the nonlinear periodic boundary-value problem (1) in the critical case

$$z_2(t,c_r^*) := z_0(t,c_r^*) + u_1(t,c_1) + u_2(t,c_2)$$

is given by the solution of the nonlinear periodic boundary-value problem of the second approximation

$$u_2'(t) = A\,u_2(t) + A_1\big(z_0(t,c_r^*),u_1(t,c_1)\big), \quad u_2(0) - u_2(T) = 0,$$

where

$$u_2(t) = X_r(t)c_2 + G\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](t), \;\; c_2\in\mathbb{R}^r.$$

The condition of solvability of the boundary-value problem of the second approximation

$$F_1(c_1) := P_{Q_r^*}\,\ell K\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](\,\cdot\,) = 0$$

is a linear equation

$$F_1(c_1) = B_0\,c_1 + d_1 = 0, \quad (6)$$

which is uniquely solvable in the case where the matrix B_0 is nondegenerate; here,

$$B_0 = F_1'(c_1)\in\mathbb{R}^{r\times r}, \quad d_1 := F_1(c_1) - B_0\,c_1.$$

Indeed, consider a vector function [10]

$$v(t,\varepsilon) := z_0(t,c_r^*) + \varepsilon\,u_1(t,c_1) + \cdots + \varepsilon^k\,u_k(t,c_k) + \cdots\,.$$

In this case,

$$F_1(c_1) := P_{Q_r^*}\,\ell K\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](\,\cdot\,) \\ = P_{Q_r^*}\,\ell K\big[Z'_\varepsilon(v(s,\varepsilon),s))\big](\,\cdot\,)\Big|_{\varepsilon=0} = P_{Q_r^*}\,\ell K\big[\mathcal{A}_1(s)u_1(s,c_1)\big](\,\cdot\,).$$

Thus,

$$B_0 = F_1'(c_1).$$

Therefore, under the condition of simplicity of roots of the equation for generating amplitudes (4) of the periodic problem (1), we obtain the following solution of the boundary-value problem of the first approximation:

$$u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t), \quad c_1 = -B_0^{-1}\,d_1.$$

The conditions of solvability of the boundary-value problems in the next approximations have the form of linear equations. A sequence of approximations to the solution on the nonlinear periodic boundary-value problem (1) in the critical case is given by the following iterative scheme:

$$\begin{gathered} z_1(t,c_r^*) := z_0(t,c_r^*) + u_1(t),\ u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t),\ \ c_1 = -B_0^{-1}\,d_1,\dots, \\ z_{k+1}(t,c_r^*) := z_0(t,c_r^*) + u_1(t,c_1) + \cdots + u_{k+1}(t,c_k),\ \ k = 0,1,2,\dots\,, \\ u_{k+1}(t) = X_r(t)c_{k+1} + G\big[A_k(z_0(s,c_r^*),u_1(s,c_1),\dots,u_k(s,c_k))\big](t),\ \ c_k = -B_0^{-1}\,d_k. \end{gathered} \tag{7}$$

Theorem. *In the critical case the generating periodic boundary-value problem* (2) *with condition* (3) *has an r-parameter family of solutions*

$$z_0(t,c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r.$$

Moreover, if the problem of construction of a solution to the nonlinear periodic boundary-value problem (1) *in a small neighborhood of the solution of the generating problem* (2) *is solvable in the critical case, then the equation for generating amplitudes* (4) *of the T-periodic problem* (1) *has real roots. In the case, where the matrix B_0 is nondegenerate, the iterative scheme* (7) *gives a sequence of approximations to the solution of the T-periodic boundary-value problem* (1) *in the critical case. If there exists a constant $0<\gamma<1$, for which the inequality*

$$\|u_1(t,c_1)\|_\infty \le \gamma\|z_0(t)\|_\infty, \quad \|u_{k+1}(t,c_{k+1})\|_\infty \le \gamma\|u_k(t,c_k)\|_\infty,\ \ k = 1,2,\dots, \tag{8}$$

is true, then the iterative scheme (7) *converges to the solution of the nonlinear periodic boundary-value problem* (1).

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The Structure of Strongly Irregular Solutions of the Quasiperiodic Linear Algebraic System

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We consider a linear homogeneous algebraic system

$$G(t)z = 0, \tag{1}$$

where $G(t)$ is in the general case a continuous almost periodic $n \times n$-matrix, z is an unknown n-dimensional vector function. The problem of the existence of quasiperiodic solutions of the quasiperiodic system (1) was first investigated by A. M. Samoilenko in connection with the problem of a periodic basis [20]. He gave conditions for the existence of the solutions with a frequency basis of the matrix $G(t)$.

Note that in most works on oscillation theory (see, for example, [6, 11–13, 16, 21, 22] and others) the case was studied when the frequency moduli of the differential system and its solution coincide. Questions about other possible relationships of frequency modules were not addressed. Although, in particular, for many applied problems, it is necessary to have information about the extent to which the specificity of the disturbance frequencies affects the nature of the oscillation frequencies of the system [15, 18, 19].

Apparently, the first to study this issue in more detail was H. L. Massera. In 1950, he showed that periodic differential systems can have periodic solutions with an irrational ratio of the periods of the solution and the system [17]. This result served as the beginning of a new direction in the theory of oscillations, which was subsequently developed for various classes of systems and their solutions in the works of J. Kurzweil and O. Veivoda [14], N. P. Erugin [4, 5], I. V. Gaishun [7], E. I. Grudo [8–10], V. T. Borukhov [1], and others. Solutions of this kind were called strongly irregular. As it turned out later, in many cases the solution to the problem of the existence of strongly irregular solutions of the original differential systems is reduced to a similar problem for system (1) (see, for example, [2, 5, 8]). However, the questions about the structure of the solution itself remained open.

In this report, the structure of a strongly irregular quasiperiodic solution of system (1) is investigated.

Let us present some well-known concepts of the theory of quasiperiodic functions. Let a finite set of real numbers $(\omega_1)^{-1}, \dots, (\omega_k)^{-1}$ be rationally linearly independent. A continuous function $f(t)$ is called quasiperiodic with periods $\omega_1, \dots, \omega_k$ if there exists a function of k variables $F^*(t_1, \dots, t_k)$, periodic in t_j with period ω_j $(j = \overline{1, m})$, which is diagonal for the original function, i.e. $f(t) \equiv F^*(t, \dots, t)$. The numbers $2\pi/\omega_1, \dots, 2\pi/\omega_k$ form the frequency basis of the quasiperiodic function $f(t)$. Further we assume that in system (1) the matrix $G(t)$ is quasiperiodic with periods $\omega_1, \dots, \omega_k$.

Some quasiperiodic solution $z = z(t)$ with periods $\Omega_1, \dots, \Omega_m$ of system (1) is called strongly irregular if the frequency bases of the solution and the matrix $G(t)$ are rationally linearly independent. In other words, the linear combination

$$q_1\Big(\frac{2\pi}{\omega_1}\Big) + \cdots + q_k\Big(\frac{2\pi}{\omega_k}\Big) + q_{k+1}\Big(\frac{2\pi}{\Omega_1}\Big) + \cdots + q_{k+m}\Big(\frac{2\pi}{\Omega_m}\Big)$$

with rational coefficients $q_1, \dots, q_{k+m}$ identically vanishes to zero if and only if all coefficients are zero.

The case $G(t) \equiv 0$ is degenerate and is of no interest for research. Therefore, in what follows, we assume that $G(t) \not\equiv 0$.

Our main result is the following

Theorem. *If the quasiperiodic system* (1) *has a non-trivial strongly irregular solution, then there is a linear dependence between the components of such solution.*

Proof. Let system (1) have a strongly irregular non-trivial periodic solution $z = z(t)$ with periods $\Omega_1, \dots, \Omega_m$, i.e. the identity holds

$$G(t)z(t) \equiv 0.$$

(i) First, we will indicate the condition that must be met for the existence of a solution $z(t)$ of system (1)

$$d_1 G_1(t) + \cdots + d_n G_n(t) \equiv 0, \quad (d_1^2 + \cdots + d_n^2) \neq 0,$$

where $G_1(t), \dots, G_n(t)$ are the columns of the matrix $G(t)$. The last identity means the linear dependence of the columns of the quasiperiodic matrix $G(t)$. This property of the matrix $G(t)$ will be further referred to as the Lin_G condition.

(ii) Let the condition Lin_G be satisfied and r be the largest number of linearly independent columns of the matrix $G(t)$, $1 \leq r < n$. The monograph [3, Ch. 2] describes an algorithm for constructing a constant non-singular $n \times n$-matrix Q, with the help of which the matrix of coefficients $G(t)$ is reduced to a special form with zero first $n - r$ columns

$$G(t)Q = \begin{bmatrix} 0 & \cdots & 0 & G_{n,r}(t) \end{bmatrix},$$

where the columns of the right block $G_{n,r}$ of dimension $n \times r$ are linearly independent.

(iii) In addition to (ii), we show that the condition Lin_G is not only necessary but also sufficient for both the existence and finding of a strongly irregular quasiperiodic solution $z(t)$. To do this, we perform a change of phase variable

$$z = Qy,$$

which reduces system (1) to the system

$$G(t)Qy = \begin{bmatrix} 0 & \cdots & 0 & G_{n,r}(t) \end{bmatrix} y = 0.$$

Let's introduce the notation

$$y = \mathrm{col}\left(y^{[n-r]}, y_{[r]}\right) \quad y^{[n-r]} = \mathrm{col}(y_1, \dots, y_{n-r}), \quad y_{[r]} = \mathrm{col}(y_{n-r+1}, \dots, y_n).$$

Then the last system can be written in a simpler form

$$G_{n,r}(t) y_{[r]} = 0,$$

in this case, the components of the vector $y^{[n-r]}$, as we see, remain arbitrary. Therefore, in order to solve the problem posed at this stage, we set

$$y^{[n-r]} = y^{[n-r]}(t) = \xi(t), \quad \xi(t) = \mathrm{col}(\xi_1(t), \dots, \xi_{n-r}(t)),$$

where $\xi_1(t), \dots, \xi_{n-r}(t)$ are arbitrary continuous quasiperiodic functions with periods $\Omega_1, \dots, \Omega_m$.

Since the columns of the coefficient matrix $G_{n,r}(t)$ are linearly independent by construction, then according to the result of stage (i) this system does not have non-trivial strongly irregular quasiperiodic solutions $y_{[r]} = y_{[r]}(t)$. Therefore $y_{[r]} \equiv 0$.

It is easy to verify that if the Lin_G condition is satisfied, system (1) will have a non-trivial strongly irregular quasiperiodic solution

$$z(t) = Q\,\mathrm{col}(y^{[n-r]}(t), 0, \dots, 0) = Q\,\mathrm{col}(\xi(t), 0, \dots, 0),$$
$$\xi(t) = \mathrm{col}(\xi_1(t), \dots, \xi_{n-r}(t)).$$

(iv) Let's establish the dependence between the components of the solution $z(t)$. Let Q_1 be a block of dimension $n \times (n-r)$, composed of the first $n-r$ columns of the matrix Q $(r < n)$. Then the last equality will take the form $z(t) = Q_1\xi(t)$. Since the constructed matrix Q is non-singular, the columns of its block Q_1, the number of which is equal to $n-r$, will be linearly independent. Therefore, the block Q_1 has a minor of order $n-r$, different from zero. Let this minor be in the rows with numbers $i_1 < i_2 < \dots < i_{n-r}$, and $i_{n-r+1} < i_{n-r+2} < \dots < i_n$ are the ordered numbers of the remaining rows. Let Q_1' be a block of dimension $(n-r) \times (n-r)$, formed by $n-r$ rows of the matrix Q_1 with numbers $i_1, \dots, i_{n-r}$. Note that by construction, this block is non-degenerate $\det Q_1' \neq 0$. By Q_1'' we denote the block of dimension $r \times (n-r)$ formed by the remaining r rows of the matrix Q_1, i.e. the rows with numbers $i_{n-r+1}, \dots, i_n$. Let

$$z'(t) = \mathrm{col}(z_{i_1}(t), \dots, z_{i_{n-r}}(t)), \quad z''(t) = \mathrm{col}(z_{i_{n-r+1}}(t), \dots, z_{i_n}(t)),$$

dimensionally consistent partitioning of vector $z(t) = \mathrm{col}(z_1(t), \dots, z_n(t))$. Then

$$z'(t) = Q_1'\xi(t), \quad z''(t) = Q_1''\xi(t).$$

Since the matrix Q_1' is non-singular, from the first equality we find $\xi(t) = (Q_1')^{-1}z'(t)$. Substituting this expression into the second equality, we obtain

$$z''(t) = Q_1''(Q_1')^{-1}z'(t).$$

Denoting the $r \times (n-r)$-matrix $Q_1''(Q_1')^{-1}$ by F, we find an explicit linear dependence between the components of the strongly irregular quasiperiodic solution

$$z''(t) = Fz'(t). \quad \square$$

Corollary 1. *For $n = 1$ the desired quasiperiodic solutions for system* (1) *are absent.*

Corollary 2. *The product of any two identically non-zero quasiperiodic functions, whose frequency bases form a rationally linearly independent set of numbers, does not vanish identically.*

Let us consider a fairly transparent example illustrating the obtained result. For a quasi-periodic linear system with periods $\omega_1 = 2\pi$, $\omega_2 = \sqrt{2}\,\pi$

$$G(t)z = \begin{pmatrix} \sin t & 2\sin t & \cos\sqrt{2}t \\ \sin\sqrt{2}\,t & 2\sin\sqrt{2}\,t & \sin t \\ \cos t & 2\cos t & 2\sin\sqrt{2}\,t + \cos t \end{pmatrix} \mathrm{col}(z_1, z_2, z_3) = 0$$

columns $G_1(t)$, $G_2(t)$, $G_3(t)$ of its coefficient matrix satisfy the identity

$$2G_1(t) + G_2(t) \equiv 0,$$

i.e. linearly dependent. Therefore, this system has a family of strongly irregular quasi-periodic solutions. One of such solutions will be the vector

$$z(t) = \operatorname{col}(z_1(t), z_2(t), z_3(t)), \quad z_1(t) = \xi(t), \quad z_2(t) = -0,5\xi(t), \quad z_3(t) \equiv 0,$$
$$\xi(t) = \sin\sqrt{3}\,t + \cos\sqrt{5}\,t,$$

the components of which are related by the following linear relationship

$$\operatorname{col}(z_2, z_3) = Fz_1, \quad F = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}.$$

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Boundary Value Problems on the Half-Line Involving Generalized Curvature Operators

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1 Introduction

Consider the second order differential equation

$$\big(a(t)\Phi_C(x')\big)' + b(t)F(x) = 0, \quad t \in I = [t_0, \infty), \tag{1.1}$$

where the functions a, b are continuous and positive on $I = [t_0, \infty)$, $t_0 \geq 0$, the function F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \neq 0$, $\Phi_R : (-1, 1) \to \mathbb{R}$ and $\Phi_C : \mathbb{R} \to (-1, 1)$ is the monotone homeomorphismus

$$\Phi_C(u) = \frac{|u|^{p-2}u}{(1 + |u|^p)^{(p-1)/p}}\,, \quad p > 1.$$

The operator Φ_C is called *generalized Euclidean mean curvature operator*. In [4], qualitative similarities between the linear equation

$$(a(t)y')' + b(t)y = 0 \tag{1.2}$$

and equations

$$\big(a(t)\Phi_E(x')\big)' + b(t)F(x) = 0 \ \text{ and } \ \big(a(t)\Phi_M(x')\big)' + b(t)F(x) = 0,$$

are pointed out, where

$$\Phi_E(u) = \frac{u}{\sqrt{1 + |u|^2}} \ \text{ and } \ \Phi_M(u) = \frac{u}{\sqrt{1 - |u|^2}}\,.$$

Operator Φ_C is called *Euclidean mean curvature operator* and Φ_M *Minkowski mean curvature operator*. Operator Φ_E is a special case of Φ_C. Similarly, Φ_M is a particular case of the so called *generalized relativistic operator*

$$\Phi_R(u) = \frac{|u|^{p-2}u}{(1 - |u|^p)^{(p-1)/p}}\,, \quad p > 1.$$

Curvature operators arise in studying some nonlinear fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids, as well as in the relativity theory when some extrinsic properties of the mean curvature of hypersurfaces are considered, see [1,5,8–10] and the references therein. In particular, in [10], it is observed that, as for small values of the variable the classical acceleration operator is an approximation of Φ_M and Φ_E; similarly, the p-Laplacian operator Φ_p

$$\Phi_p(u)=|u|^{p-2}u, \ \ p>1, \tag{1.3}$$

can be viewed, again for small values of the variable, as an approximation of Φ_R and Φ_C. Under suitable assumptions on the forcing term F, this similarity between the equation

$$\big(a(t)\Phi_R(x')\big)'+b(t)F(x)=0 \tag{1.4}$$

and

$$\big(a(t)\Phi_p(x')\big)'+b(t)F(x)=0$$

is highlighted in the search of periodic solutions, see [10], as well as in other different contexts, concerning the oscillation or the nonoscillation, see [2, Theorem 2.1] and [5, Section 5]. Moreover, in [5] also the existence of solutions x of (1.4) such that $x(t)x'(t)<0$ on the whole interval I, is considered, jointly with their convergence to zero as $t\to\infty$. These solutions are usually called *global Kneser solutions*. Moreover, their existence and asymptotic behavior have been investigated by many authors for a large variety of equations, see, e.g. [11] and the references therein.

Our aim here is to complete the results in [5, Theorem 4.1], by studying the existence of *global Kneser solution*s for (1.1). Further, also the decay of these solutions near infinity is examined. These results illustrate also that an asymptotic proximity between equations with generalized mean curvature operators and with the p-Laplacian continues to hold for Kneser solutions.

2 A fixed point result

The existence of global Kneser solutions to (1.1) is based on a fixed point result which originates from [3]. It concerns operators $\mathcal{T}$, which are defined in a Fréchet space by a Schauder's quasilinearization device. Roughly speaking, this method reduces the solvability of the given problem to the one of a possibly nonlinear problem, whose solutions have known properties. In particular, this approach does not require the knowledge of the explicit form of the fixed point operator. Moreover, it seems particularly useful when the problem is considered in a noncompact interval. In this case, it permits us also to overcome difficulties, which may originate from the check of topological properties of the fixed point operator, like the compactness, because they become a direct consequence of suitable *a-priori* bounds.

More precisely, we start by reducing the problem to an abstract fixed point equation $x=\mathcal{T}(x)$, where $\mathcal{T}$ is a possible nonlinear operator, defined in a subset of a suitable Fréchet space X. In this approach, an important tool is played by a nice property that connects the operators Φ_C and Φ_R. Indeed, when $p=2$, the inverse of Φ_E is Φ_M and vice-versa. When $p\neq 2$, denote by q the conjugate number of p, that is

$$q=\frac{p}{p-1}\,. \tag{2.1}$$

Thus a standard calculation shows that if

$$v=\Phi_C(u)=\frac{\Phi_p(u)}{(1+|u|^p)^{(p-1)/p}}\,, \tag{2.2}$$

then the inverse Φ_C^* of Φ_C is given by

$$u = \Phi_C^*(v) = \frac{\Phi_q(v)}{(1-|v|^q)^{(q-1)/q}}\,, \tag{2.3}$$

that is Φ_C^* reads as Φ_R where p is replaced by q. Indeed, from (2.2) we get

$$|v|^{p/p-1} = |u|^p\big(1+|u|^p\big)^{-1} \;\text{ or }\; 1-|v|^{p/p-1} = \big(1+|u|^p\big)^{-1}.$$

Thus

$$|u| = \frac{|v|^{1/(p-1)}}{(1-|v|^{p/(p-1)})^{1/p}}$$

and so from (2.1) the equality (2.3) follows. In a similar way, the inverse Φ_R^* of Φ_R is given by

$$\Phi_R^*(v) = \frac{\Phi_q(v)}{(1+|v|^q)^{(q-1)/q}}.$$

Using these properties, we can define the fixed point operator in the following way. Let $G : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the continuous function such that

$$G(t,\mu,\mu) = b(t)F(\mu) \;\text{ for any }\; (t,\mu)\in I\times\mathbb{R}, \tag{2.4}$$

that is F is the restriction to the diagonal of G. Setting $y = a(t)\Phi_C(x')$, equation (1.1) can be rewritten as the system

$$x' = \Phi_C^*\Big(\frac{y}{a}\Big), \quad y' = -bF(x), \tag{2.5}$$

where, for sake of simplicity, the dependence on the variable t is omitted. Using (2.3), the system (2.5) becomes

$$x' = \big(a^q - |y|^q\big)^{-(q-1)/q}\Phi_q(y), \quad y' = -bF(x). \tag{2.6}$$

Jointly with (2.6), consider the system

$$\xi' = \big(a^q - |v|^q\big)^{-(q-1)/q}\Phi_q(\eta), \quad \eta' = -G(t,u,\xi), \tag{2.7}$$

where the couple (u,v) belongs to a suitable set $\Omega\subset C(I,\mathbb{R}^2)$. If for any $(u,v)\in\Omega$, the system (2.7) has a unique solution (ξ_{uv},η_{uv}) which belongs to a subset $S\subset C(I,\mathbb{R}^2)$, defining $\mathcal{T}(u,v) = (\xi_{uv},\eta_{uv})$ and the operator $\mathcal{T}$ has a fixed point in Ω, then it is easy to verify that the fixed point $(\widehat{x},\widehat{y})$ of $\mathcal{T}$, if any, is a solution of (2.5). In other words, the algebraic aspect of the approach, consists in reducing our problem to one, whose solvability may be more easy. A special case, in which this fact occurs, is when the function F satisfies

$$\lim_{u\to 0}\frac{F(u)}{\Phi_p(u)} = F_0, \;\; 0\le F_0<\infty. \tag{2.8}$$

Indeed, by choosing as G the function $G(t,u,x) = b(t)\widetilde{F}(u(t))\Phi_p(x)$, where

$$\widetilde{F}(u) = \frac{F(u)}{\Phi_p(u)} \;\text{ if }\; u\ne 0, \;\text{ and }\; \widetilde{F}(0) = F_0, \tag{2.9}$$

a standard calculation shows that the system (2.7) is equivalent to the half-linear equation

$$\big(A_v(t)\Phi_p(\xi')\big)' + b(t)\widetilde{F}(u(t))\Phi_p(\xi) = 0, \tag{2.10}$$

where

$$A_v(t) = \Big(a^{p/(p-1)}(t) - |v|^{p/(p-1)}(t)\Big)^{(p-1)/p}. \tag{2.11}$$

For obtaining a fixed point of $\mathcal{T}$, we use the quoted result in [3, Theorem 1.1] and the Tychonoff fixed point theorem. The following holds.

Theorem 2.1. *Let S be a nonempty subset of the Fréchet space $C(I,\mathbb{R}^2)$. Assume that there exists a nonempty, closed, convex and bounded subset $\Omega \subset C(I,\mathbb{R}^2)$ such that, for any $(u,v)\in\Omega$, the system* (2.7) *has a unique solution $(\xi_{uv},\eta_{uv})\in S$. Let $\mathcal{T}$ be the operator $\Omega\to S$, given by $\mathcal{T}(u,v)=(\xi_{uv},\eta_{uv})$. Assume that*

(i_1) *$\mathcal{T}(\Omega)\subset\Omega$;*

(i_2) *if $\{(u_n,v_n)\}\subset\Omega$ is a sequence converging in Ω and $\mathcal{T}((u_n,v_n))\to(\xi_1,\eta_1)$, then $(\xi_1,\eta_1)\in S$.*

Then $\mathcal{T}$ has a fixed point $(\widehat{x},\widehat{y})\in\Omega\cap S$ and $\widehat{x}$ is a solution of (1.1).

An abstract fixed point theorem for equations involving a more general operator is given in [5, Theorem 2.1]. The assumption (i_2) is needed for proving the continuity of $\mathcal{T}$. Indeed, as spite of the fact that in many cases $\mathcal{T}$ turns out to be discontinuous, condition (i_1) becomes necessary and sufficient for the continuity $\mathcal{T}$ when $\mathcal{T}(\Omega)$ is bounded, see [3]. Moreover, condition (i_1) is verified if there exists a closed subset $S_1\subset S\cap\Omega$ such that for any $(u,v)\in\Omega$ the system (2.7) has a unique solution $(\xi_{uv},\eta_{uv})\in S_1$. As claimed, this fact illustrates how the compacteness of $\mathcal{T}$ can be a direct consequence of *a-priori* bounds.

3 Kneser solutions

Here we prove the existence of global Kneser solutions to (1.1), which converge to zero as $t\to\infty$.

Let q be defined by (2.1) and $\Phi_q(u)=|u|^{q-2}u$ be q-Laplacian operator. We assume (2.8),

$$J_a=\int_{t_0}^{\infty}\Phi_q(a^{-1}(s))\,ds<\infty,\quad \inf_{t\ge t_0}\Phi_q(a(t))\int_{t}^{\infty}\Phi_q(a^{-1}(s))\,ds=\lambda>0, \tag{3.1}$$

and

$$\int_{t_0}^{\infty}\Phi_q\Big(a^{-1}(t)\int_{t_0}^{t}b(s)\,ds\Big)\,dt<\infty\,,\quad \int_{t_0}^{\infty}b(t)\Phi_p\Big(\int_{t}^{\infty}\Phi_q(a^{-1}(s))\,ds\Big)\,dt<\infty. \tag{3.2}$$

Choose $0<c<\lambda$ and set

$$K=(1-c\lambda^{-1})^{(p-1)/p},\quad M_F=\max_{u\in[0,c]}\widetilde{F}(u), \tag{3.3}$$

where $\widetilde{F}$ is given in (2.9). Consider the half-linear equation

$$\big(Ka(t)\,\Phi_p(z')\big)'+M_Fb(t)\Phi_p(z)=0. \tag{3.4}$$

The following holds.

Theorem 3.1. *Let* (2.8), (3.1) *and* (3.2) *be satisfied. If* (3.4) *is nonoscillatory and its principal solution z_0 is positive decreasing on I, then* (1.1) *has infinitely many global Kneser solutions, which converge to zero as $t\to\infty$.*

The proof of Theorem 3.1 is similar to the one given in [5, Theorem 4.1] for proving the existence of global Kneser solutions of (1.4), with some modifications. It is based on Theorem 2.1 and on some comparison properties between principal solutions of half-linear equations. We start by recalling these properties. The notion of principal solution, introduced in 1936 by Leighton & Morse for the linear equation (1.2), has been extended to the half-linear equation

$$\big(a(t)\,\Phi_p(x')\big)'+b(t)\Phi_p(x)=0 \tag{3.5}$$

by Elbert & Kusano and independently by Mirzov, using the associated generalized Riccati equation, see [7] for more datails. More precisely, if (3.5) is nonoscillatory, then a nontrivial solution x_0 of (3.5) is said to be *the principal solution* if for every nontrivial solution x of (3.5) such that $x \neq \mu x_0$, $\mu \in \mathbb{R}$, the inequality

$$\frac{x_0'(t)}{x_0(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t$$

holds. The set of principal solutions of (3.5) is nonempty and principal solutions are determined up to a constant factor. If x is a solution of (3.5), we denote its *quasiderivative* $x^{[1]}$ by $x^{[1]}(t) = a(t)\,\Phi_\alpha(x'(t))$. The following comparison property plays a crucial role in the proof of Theorem 3.1.

Lemma 3.1. *Assume $J_a < \infty$. If* (3.5) *is nonoscillatory and its principal solution x_0, starting at $x_0(t_0) = k > 0$, is positive decreasing on I, then the principal solution y_0 of any minorant of* (3.5), *starting at $y(t_0) > 0$, is positive decreasing on the whole interval I and satisfies the inequality*

$$\frac{x_0^{[1]}(t)}{\Phi_p(x_0(t))} < \frac{y_0^{[1]}(t)}{\Phi_p(y_0(t))} \quad \textit{for any } \ t \in I.$$

Now, we give a sketch of the proof of Theorem 3.1. Set $H = \Phi_q(K)$, where K is given in (3.3) and, without loss of generality, suppose $z_0(t_0) = c^H$. In view of (3.2) we have $\lim_{t\to\infty} z_0(t) = 0$, see, e.g., [5, Proposition 3.2]. Let Ω be the set

$$\Omega = \Big\{(u,v) \in C(I,\mathbb{R}^2): \ \ 0 \le u(t) \le (z_0(t))^H, \ \ u(t_0) = 0, \ \ -\Phi_p(c\lambda^{-1})a(t) \le v(t) \le 0\Big\}.$$

For any $(u,v) \in \Omega$, the half-linear equation (2.10) is a minorant of (3.4). Thus, (2.10) is nonoscillatory and, from Lemma 3.1, its principal solution η_{uv} such that $\eta_{uv}(t_0) = c$, is positive decreasing on I. Moreover, we have for any $t \in I$

$$\frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))} \le \frac{K a(t)\,\Phi_p(z_0'(t))}{\Phi_p(z_0(t))}\,.$$

From this, using $Ka(t) \le A_v(t) \le a(t)$ and taking into account that $z_0'(t) < 0$, with a standard calculation we get

$$\eta_{uv}(t) \le (z_0(t))^H\,. \tag{3.6}$$

Let w_0 be the principal solution of equation $(a(t)\Phi_p(w'))' = 0$ such that $w_0(t_0) = c$, i.e.,

$$w_0(t) = c\Big(\int_{t_0}^{\infty} \Phi_q(a^{-1}(s))\,ds\Big)^{-1} \int_t^{\infty} \Phi_q(a^{-1}(s))\,ds.$$

Again from Lemma 3.1, we obtain

$$\frac{-1}{\Phi_p(w_0(t))} = \frac{w_0^{[1]}(t)}{\Phi_p(w_0(t))} \le \frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))}\,.$$

From this, since $\eta_{uv}(t) \le \eta_{uv}(t_0) = c$, we have

$$|\eta_{uv}^{[1]}(t)| \le \Phi_p\Big(c\Big(\int_t^{\infty} \Phi_q(a^{-1}(s))\,ds\Big)^{-1}\Big)$$

$$= a(t)\Phi_p\Big(c\Big(\Phi_q(a(t)\int_t^{\infty} \Phi_q(a^{-1}(s))\,ds\Big)^{-1}\Big) \le \Phi_p(c\lambda^{-1})a(t).$$

From this and (3.6) we have $(\eta_{uv}, \eta_{uv}^{[1]}) \in \Omega$. Using (3.2) and the same argument to the one given in [5, Theorem 4.1], for any $(u, v) \in \Omega$, the couple $(\eta_{uv}, \eta_{uv}^{[1]})$ is the only pair $(\xi, \xi^{[1]})$ with ξ solution of (2.10), that belongs to Ω. Let $\mathcal{T}$ be the operator $\mathcal{T}(u, v) = (\eta_{uv}, \eta_{uv}^{[1]})$. By choosing $S = \Omega$, conditions (i_1) and (i_2) of Theorem 2.1 are verified. Thus, $\mathcal{T}$ has a fixed point, and the assertion follows.

We close the paper with some comments.

(1) Theorem 3.1 requires that (3.5) is nonoscillatory and its principal solution starting at a positive value at t_0, is positive decreasing for any $t \in I$. To check this property, we may use the generalized Euler equations

$$\big(t^p \Phi_p(x')\big)' + p^{-p}\Phi_p(x) = 0, \ \ t \ge t_0 > 0 \tag{3.7}$$

or

$$(t^n \Phi_p(x'))' + \Big(\frac{n-p+1}{p}\Big)^p t^{n-p}\Phi_p(x) = 0, \ \ p > 2, \ \ n > p-1, \ \ t \ge t_0 > 0, \tag{3.8}$$

see [5, Corollary 4.3 and (4.18)] and [6, Corollary 1], respectively. The principal solution of (3.7) is $\varphi(t) = t^{-p}$ and that of (3.8) is $\varphi(t) = t^{-p}$. Thus, using (3.7) and applying Lemma 3.1, equation (3.5) is nonoscillatory and its principal solution starting at a positive value at t_0, is positive decreasing for any $t \in I$, if for $t \ge t_0$

$$Ka(t) \ge t^p \ \text{ and } \ M_F b(t) \le p^{-p}.$$

Clearly, a similar result can be formulated by using (3.8).

(2) The proof of Theorem 3.1 yields also the rate of the decay to zero for global Kneser solutions of (1.1). Indeed, using (3.2) and [5, Proposition 3.2], for any $(u, v) \in \Omega$, the principal solution $\eta_{uv}^{[1]}$ satisfies $\lim\limits_{t\to\infty} |\eta_{uv}^{[1]}(t)| = \ell_\eta$, $0 < \ell_\eta < \infty$. From this, it is easy to obtain

$$\eta_{uv}(t) = O\bigg(\int\limits_t^\infty \Phi_q(a^{-1}(s))\,ds\bigg) \ \text{ for large } t.$$

(3) Another interesting case in which Theorem 2.1 can be applied is when the function G in (2.4) is

$$G(t, u, x) = b(t)\widetilde{F}(u(t))\Phi_r(x), \ \ r \ne p.$$

In this case the system (2.7) becomes equivalent to the generalized Emden–Fowler equation

$$\big(a(t)\Phi_p(x')\big)' + b(t)F(x) = 0.$$

Thus, the asymptotic behavior of solutions of (1.1) can be examined via properties of solutions of a suitable Emden–Fowler type equation. This will be done in a forthcoming paper.

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Asymptotic Representation for Solutions of Systems of Differential Equations with Rapidly Varying Nonlinearities

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We consider the system of differential equations

$$\begin{cases} y_1' = \alpha_1 p_1(t)\varphi_2(y_2), \\ y_2' = \alpha_2 p_2(t)\varphi_1(y_1), \end{cases} \tag{1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = 1, 2)$, $p_i : [a, \omega[\, \to]0, +\infty[$ $(i = 1, 2)$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = 1, 2)$ ($\Delta(Y_i^0)$ is a one-sided neighborhood of Y_i^0, Y_i^0 equals either 0, or $\pm\infty$) are twice continuously differentiable functions that satisfy the conditions

$$\varphi_i'(z) \ne 0 \ \text{ when } \ z \in \Delta(Y_i^0), \qquad \lim_{\substack{z \to Y_i \\ z \in \Delta(Y_i^0)}} \varphi_i(z) = \Phi_i^0 \in \{0, +\infty\},$$

$$\lim_{\substack{z \to Y_i \\ z \in \Delta(Y_i^0)}} \frac{\varphi_i''(z)\varphi_i(z)}{[\varphi_i'(z)]^2} = \gamma_i \ \ (i = 1, 2).$$

Such system of differential equations when $\varphi_i(y_i) = |y_i|^{\sigma_i}$ $(i = \overline{1, n})$ is called the system of differential equations of Emden–Fowler type. While $t \uparrow \omega$, the asymptotic representations for its non-oscillating solutions were established in [2,6]. When $\gamma_i \ne 1$ $(i = 1, 2)$, system (1) is the system with regularly warying nonlinearities. Such system of differential equations had been investigated in [4].

This work considers situation, when $\gamma_1 = 1$, that means function φ_1 is rapidly warying when $y_1 \to Y_1^0$ [1,5]. In this situation, special case of system (1) is a two-term non-autonomous differential equation with rapidly warying nonlinearity (see [3]).

A solution $(y_i)_{i=1}^2$ of system (1), defined on the interval $[t_0, \omega[\, \subset [a, \omega[$, is called $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solution, if functions $u_i(t) = \varphi_i(y_i(t))$ $(i = 1, 2)$ satisfy the following conditions:

$$\lim_{t \uparrow \omega} u_i(t) = \Phi_i^0, \quad \lim_{t \uparrow \omega} \frac{u_i(t)u_{i+1}'(t)}{u_i'(t)u_{i+1}(t)} = \Lambda_i \ \ (i = 1, 2).$$

Note that the second condition in the definition of $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solution implies

$$\prod_{i=1}^{2} L_i = 1.$$

For system (1) in case, when $\Lambda_i \ne 0$ $(i = 1, 2)$, the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solutions are established, as well as the asymptotic representation for these solutions when $t \uparrow \omega$.

In order to formulate the theorem, we introduce several auxiliary notations:

$$I_i(t) = \begin{cases} \int\limits_{A_1}^{t} p_1(\tau)\,d\tau & \text{for } i = 1, \\ \int\limits_{A_2}^{t} I_1(\tau)p_2(\tau)\,d\tau & \text{for } i = 2, \end{cases} \qquad \beta_i = \begin{cases} -\Lambda_1, & \text{if } i = 1, \\ -1, & \text{if } i = 2, \end{cases}$$

where limits of integration $A_i \in \{\omega, a\}$ are chosen in such a way that corresponding integral I_i aims either to zero, or to ∞ when $t \uparrow \omega$.

$$A_i^* = \begin{cases} 1, & \text{if } A_i = a, \\ -1, & \text{if } A_i = \omega \end{cases} \qquad (i = 1, 2).$$

Theorem. *Let* $\Lambda_i \in \mathbb{R} \setminus \{0\}$ $(i = 1, 2)$ *and* $\gamma_1 = 1$. *Then for the existence of* $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$ *– solutions of* (1) *it is necessary and, if algebraic equation*

$$\nu\big[\nu + (1 - \gamma_2)\Lambda_1\big] = 1$$

does not have roots with zero real part, it is also sufficient that for each $i = 1, 2$

$$\lim_{t\uparrow\omega} \frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)} = \Lambda_i\,\frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied

$$A_i^*\beta_i > 0 \ \text{ when } \ \Phi_i^0 = +\infty, \quad A_i^*\beta_i < 0 \ \text{ when } \ \Phi_i^0 = 0,$$
$$\operatorname{sign}\big[\alpha_i A_i^*\beta_i\big] = \operatorname{sign}\varphi'_i(z).$$

Moreover, components of each solution of that type admit the following asymptotic representation when $t \uparrow \omega$

$$\frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t))\varphi_{i+1}(y_{i+1}(t))} = \alpha_i\beta_i I_i(t)[1 + o(1)], \quad \text{if } i = 1,$$
$$\frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t))\varphi_{i+1}(y_{i+1}(t))} = \alpha_i\beta_i\,\frac{I_i(t)}{I_1(t)}\,[1 + o(1)], \quad \text{if } i = 2.$$

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On Attainability on the Potential from the Space $W_2^{-1}[0,1]$ of the Lower Bound of the First Eigenvalue of a Sturm–Liouville Problem

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Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold:

$$\int_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \gamma \neq 0, \tag{3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{4}$$

A function y is a *solution* of problem (1), (2) if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

It was proved that if condition (4) does not hold, then for any $0 \le p \le \infty$, there is no non-trivial solution y of equation (1) with properties $y(0) = 0$, $y'(0) = p$ ([4, Theorem 1].

If $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, then the set $T_{\alpha,\beta,\gamma}$ is empty; for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty [7, Chapter 1, § 2, Theorem 3]. Since for $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$ there exists no function Q satisfying (3) and (4) taken together, we do not consider the problem for these parameters.

Consider the functional

$$R[Q,y] = \frac{\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}.$$

If condition (4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$ [5]. It was proved [4, 5] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

In this paper we describe estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q)$$

for some values of parameters α, β, γ. The result of the paper is a generalization of a result obtained by one of the authors in [2, 3]. In order to implement the ideas, used in this paper, the authors follow the technique applied in [6] where a similar problem was considered.

Let $\gamma = 1$, $0 \leqslant \alpha, \beta < 1$. For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$\int_0^1 Q(x)y^2\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta} \int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx$$

$$\leqslant \sup_{[0,1]} x^{1-\alpha}(1-x)^{1-\beta} \sup_{[0,1]} \frac{y^2}{x(1-x)} \leqslant \frac{(1-\alpha)^{1-\alpha}(1-\beta)^{1-\beta}}{(2-\alpha-\beta)^{2-\alpha-\beta}} \int_0^1 y'^2\,dx$$

and

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - \frac{(1-\alpha)^{1-\alpha}(1-\beta)^{1-\beta}}{(2-\alpha-\beta)^{2-\alpha-\beta}}\Big) \cdot \pi^2 > 0.$$

If $0 \leqslant \alpha, \beta < 1$, $Q \in T_{\alpha,\beta,\gamma}$, then

$$R[Q,y] \geqslant \frac{\int_0^1 y'^2\,dx - \sup_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta}}{\int_0^1 y^2\,dx} = L[y].$$

Functional L is bounded below, thus, there exists

$$\inf_{y \in H_0^1(0,1)\setminus\{0\}} L[y] = m.$$

Theorem. *If $0 \leqslant \alpha, \beta < 1$, then for a point $x_0 \in (0,1)$ and a number $K = {x_0}^{-\alpha}(1-x_0)^{-\beta}$ we have*

$$m_{\alpha,\beta,1} = m,$$

where m is a solution of the equation

$$\tan\sqrt{m}\,(1-x_0) = \frac{\sqrt{m}}{K\sin\sqrt{m}\,x_0 - \sqrt{m}\cos\sqrt{m}\,x_0},$$

and $m_{\alpha,\beta,1}$ is attained on the potential $K\delta(x-x_0)$.

Proof. Following [6], we consider $W_2^{-1}[0,1]$, the Hilbert space that is a completion of $L_2[0,1]$ in the norm

$$\|y\|_{W_2^{-1}[0,1]} \rightleftharpoons \sup_{\|z\|_{W_2^1[0,1]}=1} \int_0^1 yz\,dx.$$

For $y \in W_2^{-1}[0,1]$, we denote by $\int_0^1 yz\,dx$ the result

$$\langle y, z\rangle \rightleftharpoons \lim_{n\to\infty} \int_0^1 y_n z\,dx \quad \Big(\text{where } y = \lim_{n\to\infty} y_n,\ \ y_n \in L_2[0,1]\Big)$$

of applying the linear functional y to the function $z \in W_2^1[0,1]$. According to [6], for any function $Q \in T_{\alpha,\beta,\gamma}$ and for any $\lambda \in \mathbb{R}$ we consider the map

$$\begin{gathered} M : W_2^1[0,1] \to L_{loc}[0,1], \\ y \mapsto y'' + (Q+\lambda)y \end{gathered}$$

that for $Q \in W_2^{-1}[0,1]$ can be extended to the operator

$$\begin{gathered} T_Q(\lambda) : W_2^1[0,1] \to W_2^{-1}[0,1], \\ y \mapsto y'' + (Q+\lambda)y. \end{gathered}$$

The result of applying this operator $T_Q(\lambda)$ to a function $z \in W_2^1[0,1]$ is

$$\langle T_q(\lambda)y, z\rangle = \int\limits_0^1 [-y'z' + \lambda yz]\,dx + \langle Qy, z\rangle = \int\limits_0^1 [-y'z' + \lambda yz]\,dx + \lim_{n\to\infty} \int\limits_0^1 (Qy)_n z\,dx,$$

where $\{Qy\}_n$ is a sequence of functions from $L_2[0,1]$.

Let

$$m = \inf_{H_0^1(0,1)\setminus\{0\}} \frac{\int\limits_0^1 y'^2\,dx - \sup\limits_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta}}{\int\limits_0^1 y^2\,dx} = L[u],$$

where $u \in H_0^1(0,1)$ is the minimizer of L, x_0 is one of the points such that

$$\sup_{[0,1]} \frac{u^2}{x^\alpha(1-x)^\beta} = \frac{u^2(x_0)}{x_0^\alpha(1-x_0)^\beta}\,.$$

Denote

$$K = \frac{1}{{x_0}^\alpha(1-x_0)^\beta}\,.$$

Consider the equation

$$y'' + K\delta(x-x_0)y + my = 0 \tag{5}$$

and the boundary conditions

$$y(0) = y(1) = 0. \tag{6}$$

Let us consider the equivalent to (5), (6) boundary value problem

$$y'' + my = 0, \quad (0,x_0)\cup(x_0,1), \tag{7}$$

$$y'(x_0+0) - y'(x_0-0) = -Ky(x_0), \tag{8}$$

$$y(0) = y(1) = 0. \tag{9}$$

Since $y'' = \{y''\} + [y']_{x_0}\delta(x-x_0)$ and $-K\delta(x-x_0)y = -Ky(x_0)$, we have

$$-K\delta(x-x_0)y - my = -my + (y'(x_0+\varepsilon) - y'(x_0-\varepsilon))\delta(x-x_0)$$

and

$$y'(x_0+\varepsilon) - y'(x_0-\varepsilon) = -Ky(x_0).$$

On $[0, x_0)$ we have

$$y = C \sin \sqrt{m}\, x, \quad y' = C\sqrt{m}\, \cos \sqrt{m}\, x,$$
$$y'(x_0 - 0) = C\sqrt{m}\, \cos \sqrt{m}\, x_0.$$

On $(x_0, 1]$ we have

$$y = C \sin \sqrt{m}\, x, \quad y' = C\sqrt{m}\, \cos \sqrt{m}\, x,$$
$$y'(x_0 - 0) = C\sqrt{m}\, \cos \sqrt{m}\, x_0,$$
$$y = D_1 \cos \sqrt{m}\, x + D_2 \sin \sqrt{m}\, x,$$
$$D_1 = -D_2 \tan \sqrt{m}\,,$$
$$y = -D_2 \tan \sqrt{m}\, \cos \sqrt{m}\, x + D_2 \sin \sqrt{m}\, x = -D_2\, \frac{\sin \sqrt{m}\,(1 - x)}{\cos \sqrt{m}}\,,$$
$$y' = \frac{D_2}{\cos \sqrt{m}}\, \sqrt{m}\, \cos \sqrt{m}\,(1 - x),$$
$$y'(x_0 + 0) = \frac{D_2\sqrt{m}}{\cos \sqrt{m}} \cos \sqrt{m}\,(1 - x_0).$$

By virtue of (9), we have

$$\frac{D_2\sqrt{m}}{\cos \sqrt{m}}\, \cos \sqrt{m}\,(1 - x_0) - C\sqrt{m}\, \cos \sqrt{m}\, x_0 = -KC \sin \sqrt{m}\, x_0,$$
$$C = \frac{D_2\sqrt{m}\, \cos \sqrt{m}\,(1 - x_0)}{\cos \sqrt{m}\,\left(\sqrt{m}\, \cos \sqrt{m}\, x_0 - K \sin \sqrt{m}\, x_0\right)}\,,$$

and

$$y = \begin{cases} \dfrac{D_2\sqrt{m}\, \cos \sqrt{m}\,(1 - x_0)}{\cos \sqrt{m}\,\left(\sqrt{m} \cos \sqrt{m}\, x_0 - K \sin \sqrt{m}\, x_0\right)} \sin \sqrt{m}\, x, & x \in [0, x_0], \\ \dfrac{-D_2 \sin \sqrt{m}\,(1 - x)}{\cos \sqrt{m}}\,, & x \in (x_0, 1]. \end{cases}$$

Since y is continuous at x_0, we have

$$\frac{\sqrt{m}\, \cos \sqrt{m}\,(1 - x_0)}{\sqrt{m}\, \cos \sqrt{m}\, x_0 - K \sin \sqrt{m}\, x_0}\, \sin \sqrt{m}\, x_0 = -\sin \sqrt{m}\,(1 - x_0)$$

or

$$\tan \sqrt{m}\,(1 - x_0) = \frac{\sqrt{m}}{K \sin \sqrt{m}\, x_0 - \sqrt{m}\, \cos \sqrt{m}\, x_0}\,.$$

In particular [2, 3], for $\alpha = \beta = 0$, $K = 1$, $x_0 = \frac{1}{2}$, m is the solution of the equation

$$\tan \frac{\sqrt{m}}{2} = 2\sqrt{m}\,,$$

attained on the potential $\delta(x - \frac{1}{2})$,

$$y = \begin{cases} C \sin \sqrt{m}\, x, & x \in \left[0, \dfrac{1}{2}\right], \\ C \sin \sqrt{m}\,(1 - x), & x \in \left(\dfrac{1}{2}\,, 1\right], \end{cases}$$

where C is a constant.

By virtue of

$$\langle T_q(\lambda)y,z\rangle=\int\limits_0^1\big[-y'z'+\lambda yz\big]\,dx+\langle Qy,z\rangle=\int\limits_0^1\big[-y'z'+\lambda yz\big]\,dx+\lim_{n\to\infty}\int\limits_0^1(Qy)_n z\,dx,$$

we have

$$\langle T_q(\lambda)y,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+\langle Qy,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+K{y_0}^2,$$

because if we consider the sequence

$$Q_n(x)=\begin{cases}K\cdot n, & x\in\Big[x_0-\dfrac{1}{2n},x_0+\dfrac{1}{2n}\Big],\\ 0, & x\in\Big[0,x_0-\dfrac{1}{2n}\Big)\cup\Big(x_0+\dfrac{1}{2n},1\Big]\end{cases}$$

and the sequence $\{Qy\}_n$ of functions belonging to $L_2[0,1]$ such that $(Qy)_n=Q_ny$, then

$$\langle Qy,y\rangle=\lim_{n\to\infty}\int\limits_0^1(Qy)_n y\,dx=\lim_{n\to\infty}\int\limits_0^1 Q_ny^2\,dx=K\cdot y^2(x_0).$$

Note that by the mean-value theorem, for any fixed n there exists $x_*\in(x_0-\frac{1}{2n},x_0+\frac{1}{2n})$ such that

$$\frac{1}{2n}\int\limits_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}K\cdot y^2\,dx=K\cdot\frac{1}{2n}\cdot 2n\cdot y^2(x_*)=K\cdot y^2(x_*).$$

If

$$\langle T_Q(\lambda)y,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+\langle Qy,y\rangle=0,$$

then

$$\int\limits_0^1\big[-y'^2+my^2\big]\,dx+K{y_0}^2=0$$

or

$$\frac{\int\limits_0^1 y'^2\,dx-K{y_0}^2}{\int\limits_0^1 y^2\,dx}=m.$$

Therefore, for the found weak solution y of equation (5), we have

$$m=\frac{\int\limits_0^1 y'^2\,dx-K{y_0}^2}{\int\limits_0^1 y^2\,dx}\geqslant\frac{\int\limits_0^1 u'^2\,dx-K{u_0}^2}{\int\limits_0^1 u^2\,dx}=m,$$

and the weak solution of equation (5) is the minimizer of the functional L. □

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On the Stability Conditions of a Nonlinear Biological Epidemic Model

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We consider the dynamical system

$$\begin{cases} \dfrac{dS}{dt} = (1-p)a - dS - \dfrac{\beta IS}{1+\sigma I^k} + \delta V, \\ \dfrac{dE}{dt} = \dfrac{\beta IS}{1+\sigma I^k} - (d+\varepsilon+\eta)E, \\ \dfrac{dI}{dt} = \varepsilon E - (d+\tau)I, \\ \dfrac{dV}{dt} = pa + \tau I + \eta E - (d+\delta)V, \end{cases} \tag{1}$$

which arises in an epidemiological model incorporating an incubation period and temporary immunity. The population N is divided into four categories: susceptible (S), exposed (E), infected (I), and vaccinated/recovered (V). All parameters of system (1) are non-negative, and their biological meanings are interpreted as follows: individuals are born at a rate a and enter the susceptible class S, while a fraction of newborns is effectively vaccinated at a rate p. Susceptible individuals become infected at a rate β. Temporary immunity (caused by an ideal vaccine, disease, or asymptomatic infections) wanes at a rate δ. All individuals in every class experience the same natural mortality rate d. Individuals in the exposed class E can transition to the infected class I at a rate ε, as well as to the vaccinated/recovered class V at a rate η (due to the acquisition of natural immunity). Infected individuals effectively recover at a rate τ, and the parameters σ and k will be described below.

The $SEIVS$ and $SIRS$ models with various incidence rates have been studied in papers [1–3, 5–8, 10]: in [2, 5, 6, 8], the $SEIVS$ models were analyzed using a geometric approach to establish asymptotic stability and global asymptoticity of equilibrium states depending on the control reproduction number R_c. In [7], the geometric criterion for global asymptoticity was generalized. In [1, 3], diffusion effects of epidemic spread in a population were considered for the $SIRS$ models. In [10], an SIR model with a specific type of infectious force was examined.

Below, a new infectious force is considered

$$\varphi(I) := \frac{\beta I}{1+\sigma I^k}\,,$$

where the parameters σ and k account for inhibitory or psychological effects caused by public. System (1) has an equilibrium point for any parameter values given by

$$Q_0 = (S_0, 0, 0, V_0), \quad S_0 \equiv \frac{a((1-p)d+\delta)}{d(d+\delta)}\,, \quad V_0 \equiv \frac{pa}{d+\delta}\,,$$

which corresponds to the absence of infected individuals in the population.

System (1) admits a biologically feasible region

$$\mathcal{D} = \Big\{(S,E,I,V) \in \mathbb{R}^4_+ : \;\; S \leqslant S_0, \;\; V \leqslant V_0, \;\; E \leqslant S_0 + V_0, \;\; I \leqslant S_0 + V_0, \;\; S + E + I + V \leqslant \frac{a}{d}\Big\},$$

which is positively invariant. The control reproduction number, depending on the parameters of the model, is defined as

$$R_c := \frac{\varepsilon S_0 \varphi'(0)}{(d+\tau)(d+\epsilon+\eta)}\,.$$

Definition. An equilibrium point x^* is called:

- *asymptotically stable* if all solutions starting sufficiently close to it not only remain near x^* for all time but also converge to x^* as time tends to infinity;

- *globally asymptotically stable* if every solution, regardless of the initial condition, converges to x^* as time tends to infinity.

Theorem 1. *The equilibrium point Q_0 of system* (1) *is globally asymptotically stable if $R_c \leqslant 1$ and unstable if $R_c > 1$.*

Proof. The local stability of Q_0 is established using the next-generation operator method developed in [9]. Using the notation from [9], the matrices F and V for the model take the form

$$F = \begin{bmatrix} 0 & S_0\varphi'(0) \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} d+\varepsilon+\eta & 0 \\ -\varepsilon & d+\tau \end{bmatrix},$$

so that the control reproduction number for the model is given by

$$R_c = \rho(FV^{-1}) = \frac{\varepsilon S_0 \varphi'(0)}{(d+\tau)(d+\varepsilon+\eta)}\,,$$

where ρ is the spectral radius of the matrix.

Following Theorem 2 in [9], we obtain the first part of the theorem statement.

The Lyapunov function is $V(t) = E + \frac{(d+\varepsilon+\eta)I}{\varepsilon}$. Since $\varphi'(I) \le \frac{\varphi(I)}{I}$, this indicates the monotonic non-increase of $\frac{\varphi(I)}{I}$ for $I > 0$, so that

$$\frac{\varphi(I)}{I} \le \lim_{I \to 0^+} \frac{\varphi(I)}{I} = \varphi'(0).$$

Along the trajectories of system (1), the time derivative of $V(t)$ can be computed as

$$\begin{aligned} \frac{dV(t)}{dt} &= I\Big(S\frac{\varphi(I)}{I} - \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\Big) \\ &\le I\Big(S_0\varphi'(0) - \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\Big) = (R_c - 1)\,\frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\, I \le 0. \end{aligned}$$

Therefore, by LaSalle's invariance principle [4] and the local stability of Q_0, it follows that Q_0 is globally asymptotically stable in $\mathcal{D}$ when $R_c \le 1$. □

Theorem 2. *If $R_c > 1$, then system* (1) *has another equilibrium point Q^* in the region $\mathcal{D}$, distinct from Q_0, which is asymptotically stable.*

Proof. For system (1), the coordinates of the positive equilibrium are determined as follows:

$$\begin{cases} 0 = (1-p)A - dS - S\varphi(I) + \delta V, \\ 0 = S\varphi(I) - (d+\varepsilon+\eta)E, \\ 0 = \varepsilon E - (d+\tau)I, \\ 0 = pA + \tau I + \eta E - (d+\delta)V. \end{cases} \tag{2}$$

For convenience, let

$$\theta_1 := \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon} \ \text{ and } \ \theta_2 := \frac{(d+\tau)(d+\varepsilon+\eta+\delta)+\varepsilon\delta}{\varepsilon(d+\delta)}\,.$$

Using the third equation in (2), we obtain $E = \frac{(d+\tau)I}{\varepsilon}$. Adding the first two equations and using the last equation, we get

$$\begin{aligned} (1-p)A - dS - \theta_1 I + \delta V = 0, \\ pA + \tau\varepsilon + \frac{(d+\tau)\eta}{\varepsilon}\, I - (d+\delta)V = 0. \end{aligned} \tag{3}$$

Eliminating V from these equations leads to $S = S_0 - \theta_2 I$. Given $S \geq 0$, we obtain $I \leq S_0/\theta_2$. Using the second equation in (2), we obtain

$$\Phi(I) := (S_0 - \theta_2 I)\varphi(I) - \theta_1 I = 0, \ \ 0 < I \leq \frac{S_0}{\theta_2}\,. \tag{4}$$

The existence and uniqueness of the positive solution to equation (4) proceed in the following three steps.

Step 1. Existence of a positive solution for $R_c > 1$. In fact, from

$$\Phi'(I) = -\theta_2\varphi(I) + (S_0 - \theta_2 I)\varphi'(I) - \theta_1$$

and since $\varphi(0) = 0$, we have

$$\Phi'(0) = \lim_{I\to 0^+} S_0\varphi'(I) - \theta_1 = \theta_1(R_c - 1),$$

which can be achieved. Given $R_c > 1$, it is easy to show that $\Phi(I) > 0$ for sufficiently small values of I, since $\Phi'(0) > 0$, $\Phi(0) = 0$, and $\Phi(S_0/\theta_2) < 0$. This means that at least one positive solution to equation (4) exists. Let us denote this solution by I^*.

Step 2. It can be verified that the positive solution I^* is unique for $R_c > 1$. Without loss of generality, assume that another positive root, closest to I^*, exists and is denoted by $I^\dagger$. Then, the inequality $\Phi'(I^\dagger) \geq 0$ follows from the continuity of $\Phi(I)$. Using the properties of the function φ, we obtain:

$$\Phi'(I^\dagger) = (S^\dagger)\varphi'(I^\dagger) - \theta_2\varphi(I^\dagger) - \frac{(S^\dagger)\varphi(I^\dagger)}{I^\dagger} < 0. \tag{5}$$

This leads to a contradiction and confirms the uniqueness of I^*.

Step 3. We prove the absence of a positive root for (4) in the case $R_c \leq 1$ by contradiction. Assume that there exists a smallest positive root I^+. Then, it is evident that $\Phi'(I^+) < 0$ according to (5). Since $\Phi(0) = 0$ and $\Phi'(0) \leq 0$, we have $\Phi(I) \leq 0$ for sufficiently small values of I. Thus, the continuous function $\Phi(I)$ increases from a non-positive value to 0, which implies that $\Phi'(I^+) \geq 0$, leading to a contradiction.

Therefore, from Steps 1–3, we conclude that model (1) has a unique endemic equilibrium point $Q^* = (S^*, E^*, I^*, V^*)$ if and only if $R_c > 1$, where S^*, E^*, V^* can be uniquely determined according to the results derived above.

The Jacobian matrix of model (1) is given by

$$J = \begin{bmatrix} -(d+\varphi(I)) & 0 & -S\varphi'(I) & \delta \\ \varphi(I) & -(d+\varepsilon+\eta) & S\varphi'(I) & 0 \\ 0 & \varepsilon & -(d+\tau) & 0 \\ 0 & \eta & \tau & -(d+\delta) \end{bmatrix},$$

so that the characteristic equation at the point Q^* is given by

$$(\chi+d)\Big[(\chi+d+\tau)(\chi+d+\varepsilon+\eta)(\chi+d+\delta+\varphi(I^*))+ \\ +\delta\varphi(I^*)(\chi+d+\tau+\varepsilon)-\varepsilon S^*\varphi'(I^*)(\chi+d+\delta)\Big]=0. \quad (6)$$

Clearly, $\chi_1 = -d < 0$. As for the remaining eigenvalues of the equation:

$$(\chi+d+\varepsilon+\eta)(\chi+d+\delta+\varphi(I^*))+\delta\varphi(I^*)(\chi+d+\tau+\varepsilon)=\varepsilon S^*\varphi'(I^*)(\chi+d+\delta). \quad (7)$$

Case I. $\varphi'(I^*) > 0$. It is claimed that all eigenvalues of equation (7) have negative real parts. Otherwise, there exists at least one eigenvalue $\widetilde{\chi}$ such that $\operatorname{Re}\widetilde{\chi} \geq 0$. From this, it follows that

$$(d+\tau)(d+\varepsilon+\eta) \\ < \left|(\widetilde{\chi}+d+\tau)(\widetilde{\chi}+d+\varepsilon+\eta)\Big(1+\frac{\varphi(I^*)}{\widetilde{\chi}+d+\delta}\Big)+\delta\varphi(I^*)\,\frac{\widetilde{\chi}+d+\tau+\varepsilon}{\widetilde{\chi}+d+\delta}\right| \\ = \varepsilon S^*\varphi'(I^*) \leq \frac{\varepsilon S^*\varphi(I^*)}{I^*} = (d+\tau)(d+\varepsilon+\eta). \quad (8)$$

Therefore, each eigenvalue χ of equation (6) satisfies $\operatorname{Re}\chi < 0$.

Case II. $\varphi(I^*) \leq 0$. Equation (7) can be reformulated as $\chi^3 + H_1\chi^2 + H_2\chi + H_3 = 0$, where H_1, H_2, and H_3 are defined by the relations

$$H_1 = h_1+h_2+h_3, \quad H_2 = h_1h_2+h_1h_3+h_2h_3+\delta\varphi(I^*)-\varepsilon S^*\varphi'(I^*), \\ H_3 = h_1h_2h_3+\delta\varphi(I^*)h_4-\varepsilon S^*\varphi'(I^*)h_5,$$

where

$$h_1 = d+\tau, \quad h_2 = d+\varepsilon+\eta, \quad h_3 = d+\delta+\varphi(I^*), \quad h_4 = d+\tau+\varepsilon, \quad h_5 = d+\delta.$$

According to the Routh–Hurwitz stability criterion, the necessary and sufficient conditions for the stability of Q^* are:

(i) $H_i > 0$, $i = 1, 2, 3$;

(ii) $H_1H_2 - H_3 > 0$.

It is evident that (i) holds, as $h_i > 0$. Moreover, (ii) can be guaranteed by

$$H_1H_2 - H_3 = \Big[(h_1+h_2+h_3)(h_1h_2+h_1h_3+h_2h_3)-h_1h_2h_3\Big] \\ +\delta\varphi(I^*)(h_1+h_2+h_3-h_4)-\varepsilon S^*\varphi(I^*)(h_1+h_2+h_3-h_5) > 0.$$

By combining cases I and II, it can be concluded that Q^* is locally asymptotically stable if and only if $R_c > 1$. □

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Asymptotic Behaviour of Rapidly Changing Solutions of Fourth-Order Differential Equation with Rapidly Changing Nonlinearity

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The following differential equation is considered

$$y^{(4)} = \alpha_0 p_0(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p_0 : [a, \omega[\to]0, +\infty[$ – continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ – is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ at } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 is equal to either 0, or $\pm\infty$, Δ_{Y_0} – unilateral dislocation Y_0. It directly follows from condition (2) that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

According to these conditions, the function φ and its first-order derivative (see the monograph by M. Maric [5, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92]) are rapidly changing functions as $y \to Y_0$.

Definition 1. A solution y of the differential equation (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0} \ \text{ at } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y(t) = Y_0,$$

$$\lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either} \ \ 0, \\ \text{or} \ \ \pm\infty, \end{cases} \quad (k = 1, 2, 3), \quad \lim_{t \uparrow \omega} \frac{[y'''(t)]^2}{y''(t) y^{(4)}(t)} = \lambda_0.$$

Earlier, the asymptotic behaviour of $P_\omega(Y_0, \lambda_0)$-solutions of equation (1) was investigated in the case when $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}, 1\}$ in [3]. The purpose of this paper is to study the existence and asymptotic behaviour of $P_\omega(Y_0, \lambda_0)$-solutions in the special case when $\lambda_0 = 1$. In this case, due to the a priori asymptotic properties of the $P_\omega(Y_0, \lambda_0)$-solutions (see [1, Section 3, Subsection 10]), the following asymptotic relations hold for each $P_\omega(Y_0, 1)$-solution

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \frac{y'''(t)}{y''(t)} \sim \frac{y''''(t)}{y'''(t)} \ \text{ at } \ t \uparrow \omega, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = \pm\infty,$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{or } \ \omega = +\infty, \\ t - \omega, & \text{or } \ \omega < +\infty. \end{cases}$$

It follows, in particular, that the $P_\omega(Y_0,1)$-solution of equation (1) and its derivatives up to and including the third order are rapidly changing functions at $t\uparrow\omega$.

Let us introduce the necessary auxiliary notation and assume that the domain of the function φ in equation (1) is defined as follows

$$\Delta_{Y_0}=\Delta_{Y_0}(y_0), \ \text{ where } \ \Delta_{Y_0}(y_0)=\begin{cases}[y_0,Y_0[& \text{if } \ \Delta_{Y_0} \ \text{ left neighbourhood } Y_0,\\ ]Y_0,y_0] & \text{if } \ \Delta_{Y_0} \ \text{ right neighbourhood } Y_0,\end{cases}$$

and $y_0\in\Delta_{Y_0}$ such that $|y_0|<1$ at $Y_0=0$ and $y_0>1$ ($y_0<-1$) at $Y_0=+\infty$ (at $Y_0=-\infty$).

Next, let's assume that

$$\mu_0=\operatorname{sign}\varphi'(y), \quad \nu_0=\operatorname{sign}y_0, \quad \nu_1=\begin{cases}1 & \text{if } \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \ \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$J_0(t)=\int\limits_{A_0}^{t}p^{\frac14}(\tau)\,d\tau, \quad \Phi(y)=\int\limits_{B}^{y}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}\,,$$

where $p:[a,\omega[\,\to\,]0,+\infty[$ is a continuous or continuously differentiable function as $t\uparrow\omega$,

$$A_0=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^{\omega}p^{\frac14}(\tau)\,d\tau<+\infty,\\ a & \text{if } \displaystyle\int\limits_a^{\omega}p^{\frac14}(\tau)\,d\tau=+\infty,\end{cases} \qquad B=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=const,\\ y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=\pm\infty.\end{cases}$$

Let's pay attention to some properties of the function Φ. It keeps its sign at Δ_{Y_0}, goes either to zero or to $\pm\infty$ at $y\to Y_0$, and increases at Δ_{Y_0}, since in this interval $\Phi'(t)=|y|^{-\frac34}\varphi^{-\frac14}(y)>0$. Therefore, there exists an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where, due to the second condition (2) and the monotonic growth of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases}$$

$$\Delta_{Z_0}=\begin{cases}[z_0,Z_0[& \text{if } \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0] & \text{if } \ \Delta_{Y_0}=]Y_0,y_0],\end{cases} \qquad z_0=\Phi(y_0),$$

Given Definition 1, we note that the numbers ν_0 and ν_1 determine the signs of any $P_\omega(Y_0,1)$-solution and the first derivative in some left neighbourhood of ω. The conditions

$$\nu_0\nu_1<0, \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1>0, \ \text{ if } \ Y_0=\pm\infty,$$

are necessary for the existence of such solutions.

In addition to the above designations, we will also introduce auxiliary functions:

$$H(t)=\frac{\Phi^{-1}(\nu_1J_0(t))\varphi'(\Phi^{-1}(\nu_1J_0(t)))}{\varphi(\Phi^{-1}(\nu_1J_0(t)))}\,,$$

$$J_1(t)=\int\limits_{A_1}^{t}p(\tau)\varphi(\Phi^{-1}(\nu_1J_0(\tau)))\,d\tau, \quad J_2(t)=\int\limits_{A_2}^{t}J_1(\tau)\,d\tau, \quad J_3(t)=\int\limits_{A_3}^{t}J_2(\tau)\,d\tau,$$

where

$$A_1 = \begin{cases} t_1 & \text{if } \int\limits_{t_1}^{\omega} p(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{t_1}^{\omega} p(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau < +\infty, \end{cases} \qquad t_1 \in [a,\omega],$$

$$A_2 = \begin{cases} t_1 & \text{if } \int\limits_{t_1}^{\omega} J_1(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{t_1}^{\omega} J_1(\tau)\,d\tau < +\infty, \end{cases} \qquad A_3 = \begin{cases} t_1 & \text{if } \int\limits_{t_1}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{t_1}^{\omega} J_2(\tau)\,d\tau < +\infty, \end{cases}$$

The following statement is true for equation (1).

Theorem. *For the existence of $P_\omega(Y_0, 1)$-solutions of the differential equation* (1)*, the following inequalities must be satisfied*

$$\nu_1\mu_0 J_0(t) < 0 \;\; at \;\; t \in]a,\omega[\,,$$
$$\alpha_0\nu_1 < 0 \;\; if \;\; Y_0 = 0, \quad \alpha_0\nu_1 > 0 \;\; if \;\; Y_0 = \pm\infty,$$

and conditions

$$\frac{\alpha_0 J_3(t)}{\Phi^{-1}(\nu_1 J_0(t))} \sim \frac{J_1'(t)}{J_1(t)} \sim \frac{J_2'(t)}{J_2(t)} \sim \frac{J_3'(t)}{J_3(t)} \sim \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \;\; as \;\; t \uparrow \omega,$$

$$\lim_{t\uparrow\omega} H(t) = \pm\infty, \quad \nu_1 \lim_{t\uparrow\omega} J_0(t) = Z_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t)))} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_0'(t)}{J_0(t)} = \pm\infty.$$

In addition, for each such solution, the following asymptotic representations are obtained

$$y(t) = \Phi^{-1}(\nu_1 J_0(t))\Big[1 + \frac{o(1)}{H(t)}\Big],$$
$$y'(t) = \alpha_0 J_3(t)[1+o(1)], \quad y''(t) = \alpha_0 J_2(t)[1+o(1)], \quad y'''(t) = \alpha_0 J_1(t)[1+o(1)] \;\; as \;\; t \uparrow \omega,$$

The sufficiency of the obtained conditions is proved by imposing an additional condition. Namely,

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} \left|\frac{y\varphi'(y)}{\varphi(y)}\right|^{\frac{3}{4}} = 0.$$

The question of the actual existence of solutions with these asymptotic images is established under some additional conditions, by transforming the found asymptotic images to a system of quasilinear equations using Theorem 2.2 from the work by Evtukhov V. M., Samoilenko A. M. [4] on the existence of solutions tending to zero.

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The Inverse Problem for Periodic Travelling Waves of the Linear 1D Shallow-Water Equations

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1 Introduction

The motion of small amplitude waves of a water layer with variable depth along the x-axis is described by the equations of the shallow water theory

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[h(x)u] = 0, \quad \frac{\partial u}{\partial t} + g\,\frac{\partial \eta}{\partial x} = 0,$$

where $\eta(x,t)$ is the vertical water surface elevation, $u(x,t)$ is the depth-averaged water flow velocity (also called wave velocity), $h(x)$ is the unperturbed water depth and g is the gravity acceleration (see Fig. 1). From now on, we assume without loss of generality that $g = 1$.

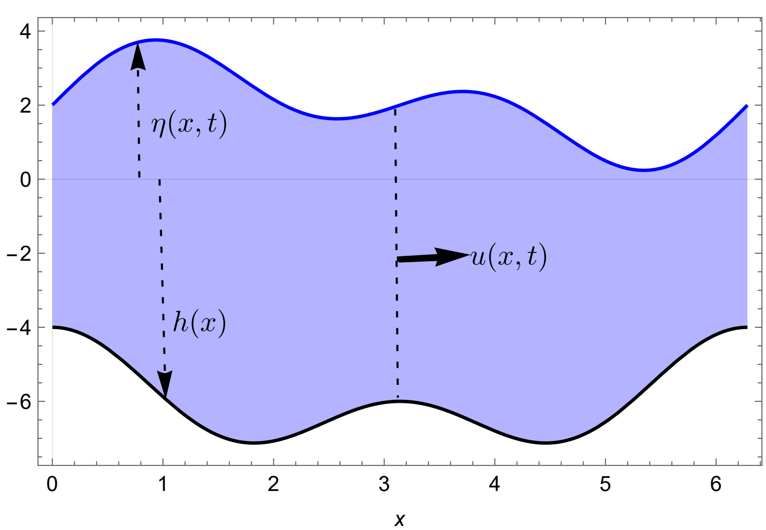


Figure 1. Graphical description of the model.

The shallow water equations constitute a system of coupled PDEs of first order that can be easily decoupled into a single wave equation for the surface displacement

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[h(x)\,\frac{\partial \eta}{\partial x}\right] = 0, \tag{1.1}$$

or for the water velocity

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}\,[h(x)u] = 0. \tag{1.2}$$

There is a considerable number of papers devoted to finding sufficient conditions on the bottom profile $h(x)$ to ensure the existence of travelling waves or other explicit solutions [1–7, 9, 10]. A travelling wave is a special solution of the form $q(x)\exp i[\omega t - \Psi(x)]$, where both q and Ψ are scalar real-valued functions. In the related literature, $q(x)$ is known as the amplitude of the travelling wave, ω is the frequency and $\Psi(x)$ is the phase, which is called non-trivial if it is non-constant. In this paper, we are going to study the following inverse problem: *given a prescribed amplitude $q(x)$, can we determine a suitable bottom profile $h(x)$ allowing the equation to admit a travelling wave with amplitude $q(x)$?*

2 The inverse problem for water velocity

From now on, C_T^+ will denote the space of continuous scalar T-periodic functions with positive values. In this section, we study the inverse problem for the water velocity. Given a fixed $q \in C_T^+$, we wonder if there exists $h \in C_T^+$ such that Eq. (1.2) has a travelling wave

$$u(x,t) = q(x)\exp i\big[\omega t - \Psi(x)\big]. \tag{2.1}$$

Inserting (2.1) into (1.2) and separating real and imaginary parts, we get the equations

$$(hq)'' + \omega^2 q - hq\Psi'^2 = 0, \tag{2.2}$$

$$2(hq)'\Psi' + hq\Psi'' = 0. \tag{2.3}$$

From (2.3), we deduce that $[(hq)^2\Psi']' = 0$, and $(hq)^2\Psi'$ is a conserved quantity, which is actually an energy flux, and it is in total analogy with the angular momentum in systems with radial symmetry. This means that there exists $\alpha \in \mathbb{R}$ such that

$$\big[h(x)q(x)\big]^2\Psi'(x) = \alpha, \ \ \forall\, x \in \mathbb{R}. \tag{2.4}$$

Now, we insert (2.4) into (2.2) and arrive to a single second order ODE

$$(hq)'' + \omega^2 q = \frac{\alpha^2}{(hq)^3}\,. \tag{2.5}$$

Recall that for this equation, the unknown is $h(x)$, where $q(x)$ is given. The main result of this section is the following.

Theorem 2.1. *There exists a solution $h \in C_T^+$ of (2.5) for any $\alpha \neq 0$, $\omega \neq 0$.*

Proof. By introducing the change of variables $y = hq$ into (2.5), we get the equation

$$y'' + \omega^2 q = \frac{\alpha^2}{y^3}\,.$$

Now, the result is a direct consequence of [8, Theorem 3.12]. □

3 The inverse problem for surface elevation

This section is devoted to studying the inverse problem for the surface elevation. Given a fixed $q \in C_T^+$, the problem is to find $h \in C_T^+$ such that Eq. (1.1) has a travelling wave of the form

$$\eta(x,t) = q(x) \exp i\big[\omega t - \Psi(x)\big]. \tag{3.1}$$

Following the steps of the previous section, we insert (3.1) into (1.1) and separate real and imaginary parts to obtain

$$(hq')' + \omega^2 q - hq\Psi'^2 = 0, \tag{3.2}$$

$$(hq\Psi')' + hq'\Psi' = 0. \tag{3.3}$$

Now, the conserved quantity (energy flux) coming from (3.3) is

$$h(x)q(x)^2\Psi'(x) = \alpha, \ \ \forall\, x \in \mathbb{R}.$$

Using this information in (3.2), we arrive to the equation

$$(hq')' + \omega^2 q = \frac{\alpha^2}{hq^3}\,. \tag{3.4}$$

Again, the unknown is h and q is a prescribed function. Although this equation may look similar to (2.5), they are indeed totally different. The fundamental difference is that now we have a first-order differential equation, with the difficulty that q' will change its sign, hence we are dealing with a differential equation that is singular not only in the dependent variable h but also in the independent variable x.

Theorem 3.1. *Let us assume that q is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0,T]$, all of them non-degenerate, that is, if $q'(x)=0$, then $q''(x) \neq 0$. Then, there exists a threshold $\lambda_0 > 0$ such that*

(i) *there exists a positive T-periodic solution h of (3.4) provided $0 < |\frac{\alpha}{\omega^2}| < \lambda_0$,*

(ii) *no positive T-periodic solution of (3.4) exists provided $|\frac{\alpha}{\omega^2}| > \lambda_0$.*

Moreover,

$$\frac{q_*^5}{4|q_0|} < \lambda_0^2 \leq \min\Big\{\frac{q^5(b)}{4|q''(b)|} : \ \ q'(b)=0, q''(b)<0\Big\},$$

where

$$q_* \overset{def}{=} \min\big\{q(x) : \ x \in [0,T]\big\}, \quad q_0 \overset{def}{=} \min\big\{q''(x) : x \in [0,T]\big\}.$$

3.1 Sketch of Proof

We assume that q is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0,T]$, all of them non-degenerate, that is, if $q'(x)=0$, then $q''(x) \neq 0$. Under this assumption, we can divide the interval $[0,T]$ into subintervals $[a,b]$ such that q' is of a constant sign on (a,b) and $q'(a) = q'(b) = 0$. Then, the substitution

$$u(x) = \frac{(h(x)q'(x))^2}{2\omega^4} \ \ \text{for} \ \ x \in (a,b) \tag{3.5}$$

transforms (3.4) into the equation

$$u'(x) = \frac{\lambda^2 q'(x)}{q^3(x)} - q(x)\operatorname{sgn}(q')\sqrt{2u(x)} \ \text{ for } \ x \in (a,b), \tag{3.6}$$

where $\lambda = \alpha/\omega^2$.

At first we consider an interval $[a,b]$ where

$$q'(a) = 0, \quad q'(b) = 0, \quad q'(x) > 0 \ \text{ for } \ x \in (a,b), \quad q''(a) > 0, \quad q''(b) < 0. \tag{3.7}$$

In such an interval, eq. (3.6) reads

$$u'(x) = \frac{\lambda^2 q'(x)}{q^3(x)} - q(x)\sqrt{2u(x)} \ \text{ for } \ x \in (a,b). \tag{3.8}$$

For technical reasons, we embed this equation into

$$u'(x) = \frac{\lambda^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|u(x)|}\,\operatorname{sgn} u(x) \ \text{ for } \ x \in [a,b]. \tag{3.9}$$

Obviously, non-negative solutions of (3.8) and (3.9) are the same.

A solution of (3.9) is understood in the classical sense, that is, a function $u \in C^1([a,b];\mathbb{R})$ satisfying (3.9) for every $x \in [a,b]$. We will investigate the properties of a solution to (3.9) subject to the initial condition

$$u(a) = 0. \tag{3.10}$$

Lemma 3.1. *There exists a unique solution u of the initial value problem* (3.9), (3.10). *Moreover, if $\lambda \neq 0$, then*

$$u(x) > 0 \ \text{ for } \ x \in (a,b).$$

Lemma 3.2. *Let $\lambda \neq 0$ and let u be the solution to* (3.9), (3.10). *Then, there exists one-sided limit*

$$\ell_a \overset{def}{=} \lim_{x \to a+} \frac{\sqrt{2u(x)}}{q'(x)}, \tag{3.11}$$

it is finite, and ℓ_a is the unique positive root of the quadratic equation

$$x^2 + \frac{q(a)}{q''(a)}\,x - \frac{\lambda^2}{q^3(a)q''(a)} = 0. \tag{3.12}$$

Lemma 3.3. *Let u be a solution of* (3.9) *satisfying*

$$u(x) > 0 \ \text{ for } \ x \in (x_0, b)$$

for some $x_0 \in (a,b)$ and $u(b) = 0$. Then, there exists one-sided limit

$$\ell_b \overset{def}{=} \lim_{x \to b-} \frac{\sqrt{2u(x)}}{q'(x)}, \tag{3.13}$$

it is finite, and ℓ_b is a root of the quadratic equation

$$x^2 - \frac{q(b)}{|q''(b)|}\,x + \frac{\lambda^2}{q^3(b)|q''(b)|} = 0. \tag{3.14}$$

Lemma 3.4. *There exists a threshold $\lambda_{ab} > 0$ such that*

(i) *if* $0 < |\lambda| < \lambda_{ab}$, *the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) = 0$. *Moreover,* ℓ_a, ℓ_b *defined by* (3.11) *and* (3.13) *are respectively the unique positive root of* (3.12) *and the smaller root of* (3.14)*;*

(ii) *if* $|\lambda| = \lambda_{ab}$, *the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) = 0$. *Moreover,* ℓ_a, ℓ_b *defined by* (3.11) *and* (3.13) *are respectively the unique positive root of* (3.12) *and a root of* (3.14)*;*

(iii) *if* $|\lambda| > \lambda_{ab}$, *the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) > 0$.

The case when

$$q'(a) = 0, \quad q'(b) = 0, \quad q'(x) < 0 \ \text{ for } \ x \in (a, b), \quad q''(a) < 0, \quad q''(b) > 0$$

can be transformed by $\widetilde{q}(x) = q(-x)$ to the previous case.

The threshold λ_0 is then a minimum of the thresholds λ_{ab} that correspond to each subinterval (a, b). Further, the change (3.5) is inverted by defining

$$h(x) = \frac{\omega^2\sqrt{2u(x)}}{|q'(x)|} \ \text{ for } \ x \in (a, b), \quad h(a) = \omega^2\ell_a, \quad h(b) = \omega^2\ell_b,$$

on every subinterval (a, b). By construction, h is a positive absolutely continuous T-periodic function.

3.2 Estimation of the threshold λ_0

Theorem 3.1 includes a general quantitative estimate of the threshold value λ_0. In this subsection, we develop a technique that permits a significant improvement of the estimates in concrete examples. Since λ_0 is the minimum of the thresholds λ_{ab} corresponding to each subinterval (a, b), we only focus on estimating the latter. As in the previous subsection we formulate the results for the case when (3.7) is valid.

Theorem 3.2. *Let there exist positive constants* λ_1 *and* λ_2 *such that* $\lambda_1 \leq \lambda_2$, *and let* $v, w \in AC([a, b]; \mathbb{R})$ *satisfy*

$$\begin{gathered}
v'(x) \geq \frac{\lambda_1^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|v(x)|}\,\operatorname{sgn} v(x) \ \text{ for a.e. } \ x \in [a, b], \\
w'(x) \leq \frac{\lambda_2^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|w(x)|}\,\operatorname{sgn} w(x) \ \text{ for a.e. } \ x \in [a, b], \\
v(a) \geq 0 \geq w(a), \quad v(b) = 0 = w(b), \\
\liminf_{x \to b-} \frac{\sqrt{2|w(x)|}\,\operatorname{sgn} w(x)}{q'(x)} > x_1(\lambda_2),
\end{gathered}$$

where $x_1(\lambda_2)$ *is the smaller root of* (3.14) *with* $\lambda = \lambda_2$. *Then, the threshold* λ_{ab} *admits the estimate*

$$\lambda_1 \leq \lambda_{ab} \leq \lambda_2. \tag{3.15}$$

If we put

$$v(x) \overset{def}{=} \frac{(\ell_1(x)q'(x))^2}{2}, \quad w(x) \overset{def}{=} \frac{(\ell_2(x)q'(x))^2}{2} \ \text{ for } \ x \in [a, b],$$

then Theorem 3.2 yields the following assertion.

Corollary 3.1. *Let there exist positive constants λ_1 and λ_2 such that $\lambda_1 \le \lambda_2$, let $q \in C^2([a,b];\mathbb{R})$, and let $\ell_1, \ell_2 \in C^1([a,b];\mathbb{R})$ satisfy*

$$\ell_i(x) > 0 \ \text{ for } \ x \in [a,b] \ \ (i=1,2), \tag{3.16}$$

$$\ell_1(x)\big(\ell_1'(x)q'(x) + \ell_1(x)q''(x)\big) \ge \frac{\lambda_1^2}{q^3(x)} - q(x)\ell_1(x) \ \text{ for } \ x \in [a,b], \tag{3.17}$$

$$\ell_2(x)\big(\ell_2'(x)q'(x) + \ell_2(x)q''(x)\big) \le \frac{\lambda_2^2}{q^3(x)} - q(x)\ell_2(x) \ \text{ for } \ x \in [a,b], \tag{3.18}$$

$$\ell_2(b) > x_1(\lambda_2), \tag{3.19}$$

where $x_1(\lambda_2)$ is the smaller root of (3.14) *with $\lambda = \lambda_2$. Then, the threshold λ_{ab} admits the estimate* (3.15)*.*

3.3 A concrete example

Theorem 3.1 provides a general quantitative estimate for the threshold value λ_0. However, such an estimate can be improved for particular cases by a suitable construction of upper and lower functions. To illustrate this idea, consider $q(x) = 2 - \cos x$ for $x \in [0, 2\pi]$. Then local extremes of q divide the interval $[0, 2\pi]$ into two subintervals, in particular, we set $T = 2\pi$, $x_1 = 0$, $x_2 = \pi$, $x_1 + T = 2\pi$ in order to apply Theorem 3.1. Then we have

$$q'(x) > 0 \ \text{ for } \ x \in (0,\pi), \quad q'(x) < 0 \ \text{ for } \ x \in (\pi, 2\pi),$$
$$q'(0) = q'(\pi) = q'(2\pi) = 0, \quad q''(0) = q''(2\pi) = 1, \quad q''(\pi) = -1.$$

Moreover, since q is symmetric with respect to π, we can easily conclude that the thresholds corresponding to each subinterval has the same value, i.e., $\lambda_0 = \lambda_{0,\pi} = \lambda_{\pi,2\pi}$. Thus, according to Theorem 3.1, the threshold λ_0 satisfies the inequalities

$$0.25 = \frac{1}{4} < \lambda_0^2 \le \frac{243}{4} = 60.75.$$

Let us see how to improve this estimate by constructing a specific couple of upper and lower functions.

According to Corollary 3.1, it is sufficient to find suitable functions $\ell_1(x)$ and $\ell_2(x)$ that satisfy (3.16)–(3.19). Obviously, we can start with positive constant functions. Then, if we put

$$\lambda_1^2 \overset{def}{=} \min\big\{(\ell_1^2 q''(x) + \ell_1 q(x))q^3(x) : \ \ x \in [0,\pi]\big\},$$
$$\lambda_2^2 \overset{def}{=} \max\big\{(\ell_2^2 q''(x) + \ell_2 q(x))q^3(x) : \ \ x \in [0,\pi]\big\},$$

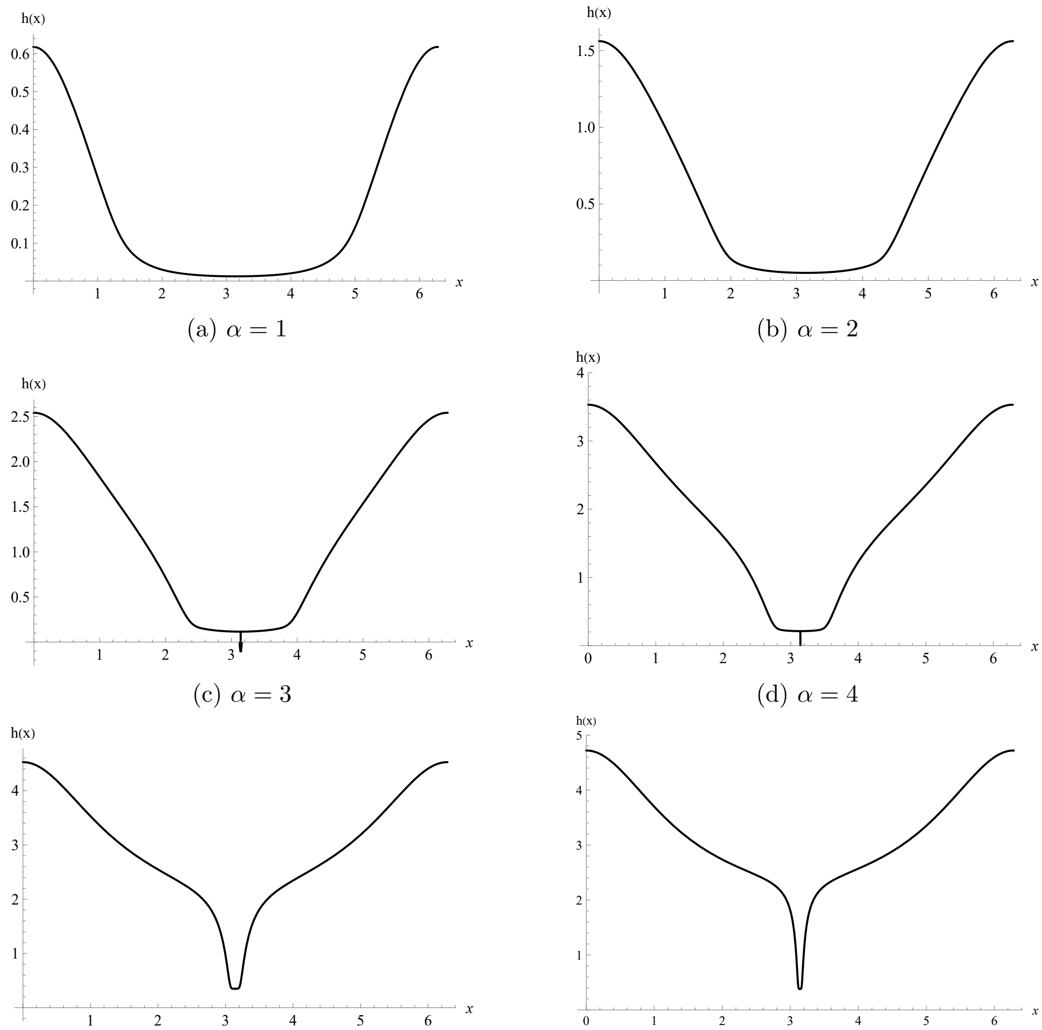
we can easily verify that inequalities (3.17) and (3.18) with $a = 0$, $b = \pi$ are fulfilled. Consequently, if also (3.19) is fulfilled, then we can conclude that (3.15) holds.

Analyzing the function $x \mapsto (\ell^2 q''(x) + \ell q(x))q^3(x)$ in detail, one can show that the optimal values for constant functions ℓ_1 and ℓ_2 are

$$\ell_1 = \frac{20}{7}\,, \quad \ell_2 = \frac{5}{2}\,.$$

Then, we get

$$\lambda_1^2 = \frac{540}{49} \approx 11.020408163, \quad \lambda_2^2 = \frac{3125}{64} = 48.828125, \qquad x_1(\lambda_2) \approx 0.835507015894,$$



(a) $\alpha = 1$ (b) $\alpha = 2$ (c) $\alpha = 3$ (d) $\alpha = 4$ (e) $\alpha = 5$ (f) $\alpha = \sqrt{27}$

Figure 2. Numerically computed water depth function $h(x)$ for an amplitude $q(x) = 2 - \cos x$ of the travelling wave. We fixed $\omega = 1$ and moved the energy flux parameter α.

and we have the estimate

$$\frac{540}{49} \leq \lambda_0^2 \leq \frac{3125}{64}\,.$$

Let us pass to nonconstant functions $\ell_1(x)$ and $\ell_2(x)$. Then the choice

$$\ell_i(x) \overset{def}{=} a_i + b_i \cos x + c_i \sin x + d_i \sin x \cos x \;\text{ for }\; x \in [0, \pi] \;\; (i = 1, 2),$$

where

$$a_1 = 4.265, \quad b_1 = 1.639, \quad c_1 = -1.075, \quad d_1 = -0.778,$$
$$a_2 = 3.605, \quad b_2 = 1.025, \quad c_2 = -0.408, \quad d_2 = -0.222,$$

guarantees that $\ell_1(x)$ and $\ell_2(x)$ satisfy (3.16)–(3.19) with $a = 0$, $b = \pi$, $\lambda_1^2 = 26.4$, and $\lambda_2^2 = 31.68$. Furthermore, note also that

$$\ell_1(\pi) < x_2(\lambda_1), \quad x_2(\lambda_2) < \ell_2(\pi), \tag{3.20}$$

where $x_2(\lambda_i)$ is the greater root of (3.14) with $\lambda = \lambda_i$ $(i = 1, 2)$. Indeed,

$$\ell_1(\pi) = 2.626, \quad x_2(\lambda_1) \approx 2.62792828771, \quad x_2(\lambda_2) \approx 2.53762549442, \quad \ell_2(\pi) = 2.58.$$

The condition (3.20) is stronger than (3.19) and allows strict inequalities in the threshold estimate. Therefore, according to Corollary 3.1 we have

$$26.4 < \lambda_0^2 < 31.68.$$

We conducted several numerical calculations to approximately solve the relevant equations and determine the water depth function $h(x)$ associated with the amplitude $q(x) = 2 - \cos x$. The results are illustrated in Fig. 2. Notably, as λ approaches the critical value λ_0, a singularity arises in $h(x)$.

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Bounded Solutions of a Linear Differential Equation with Piecewise Constant Operator Coefficients

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Abstract

By passing to the corresponding difference equation, a criterion is obtained for the existence of a unique solution bounded on the entire real axis of a linear differential equation with piecewise constant operator coefficients.

1 Introduction

Let $(X, \|\cdot\|)$ be a complex separable Banach space, $L(X)$ be the Banach space of linear continuous operators acting from X into X, I and O be the identity and null operators in X, and $C_b(\mathbb{R}, X)$ be the Banach space of functions $x : \mathbb{R} \to X$ continuous and bounded on $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|.$$

Let us fix a natural number p, operators A, B; A_n, $1 \le n \le p$, from $L(X)$, real numbers $t_0 < t_1 < \dots < t_p$ and consider the differential equation

$$\begin{cases} x'(t) = Ax(t) + y(t), & t \ge t_p, \\ x'(t) = A_n x(t) + y(t), & t_{n-1} \le t < t_n, \ \ 1 \le n \le p, \\ x'(t) = Bx(t) + y(t), & t < t_0, \end{cases} \tag{1.1}$$

in which y is a fixed function from $C_b(\mathbb{R}, X)$. A bounded solution of equation (1.1) is a function $x \in C_b(\mathbb{R}, X)$ such that for each $t \in \mathbb{R} \setminus \{t_0, t_1, \dots, t_p\}$ there exists $x'(t)$ and equality (1.1) holds.

Our goal is to obtain necessary and sufficient conditions on the operator coefficients A, B; A_n, $1 \le n \le p$, for the following condition to be satisfied.

Boundedness condition. For each function $y \in C_b(\mathbb{R}, X)$ the differential equation (1.1) has a unique bounded solution.

2 Auxiliary statements

Consider the corresponding to (1.1) difference equation

$$\begin{cases} u_{n+1} = e^{A} u_n + v_n, & n \ge p, \\ u_{n+1} = e^{A_{n+1}(t_{n+1}-t_n)} u_n + v_n, & 0 \le n \le p-1, \\ u_{n+1} = e^{B} u_n + v_n, & n \le -1, \end{cases} \tag{2.1}$$

in which $\{v_n, n \in \mathbb{Z}\}$ is a given and $\{u_n, n \in \mathbb{Z}\}$ is a sought sequence of elements of the space X. We will say that the difference equation (2.1) satisfies the boundedness condition if it has a unique bounded solution $\{u_n, n \in \mathbb{Z}\}$ for each bounded sequence $\{v_n, n \in \mathbb{Z}\}$.

The following theorem holds.

Theorem 2.1. *For the differential equation* (1.1) *to satisfy the boundedness condition, it is necessary and sufficient that difference equation* (2.1) *also satisfy the boundedness condition.*

Let $S = \{z \in \mathbb{C} \mid \ |z| = 1\}$. Let T be an operator from $L(X)$ such that $\sigma(T) \cap S = \varnothing$; $\sigma_-(T)$, $\sigma_+(T)$ are the parts of the spectrum of T that lie inside and outside the circle S, respectively; $P_-(T)$ and $P_+(T)$ are the Riesz projectors corresponding to $\sigma_-(T)$ and $\sigma_+(T)$. Then the space X can be decomposed into a direct sum $X = X_-(T) \dot{+} X_+(T)$ of subspaces $X_\pm(T) = P_\pm(T)(X)$ that are invariant with respect to T (see, for example, [3, pp. 32–34]).

For brevity, we denote $E_k = e^{A_k(t_k - t_{k-1})}$, $E_{jk} = E_k E_{k-1} \dots E_j$, $1 \le j \le k \le p$. Since the operator E_{jk} is continuously invertible, the image $E_{jk}(G)$ of an arbitrary subspace G of a Banach space X is also a subspace. Therefore, by Theorem 3 of [4], the following theorem holds.

Theorem 2.2. *For the difference equation* (2.1)*, the boundedness condition is satisfied if and only if the following conditions are satisfied:*

(i1) $\sigma(e^A) \cap S = \varnothing$, $\sigma(e^B) \cap S = \varnothing$;

(i2) $X = X_-(e^A) \dot{+} E_{1p}(X_+(e^B))$.

3 Main results

Now let $i\mathbb{R} = \{it \mid \ t \in \mathbb{R}\}$, $V \in L(X)$, $\sigma(V) \cap i\mathbb{R} = \varnothing$, $\widetilde{\sigma}_-(V)$, $\widetilde{\sigma}_+(V)$ be the parts of the spectrum of the operator V that lie in the left and right half-planes of $\mathbb{C}$, respectively. Then, as for the operator T, the space X decomposes into a direct sum $X = \widetilde{X}_-(V) \dot{+} \widetilde{X}_+(V)$ of subspaces $\widetilde{X}_\pm(V) = \widetilde{P}_\pm(T)(V)$ that are invariant with respect to the operator V, where $\widetilde{P}_\pm(V)$ are the Riesz projections corresponding to $\widetilde{\sigma}_\pm(V)$. The above statements allow us to prove the following theorems.

Theorem 3.1. *For the differential equation* (1.1) *to satisfy the boundedness condition, it is necessary and sufficient that the following conditions be satisfied:*

(j1) $\sigma(A) \cap i\mathbb{R} = \varnothing$, $\sigma(B) \cap i\mathbb{R} = \varnothing$;

(j2) $X = \widetilde{X}_-(A) \dot{+} E_{1p}(\widetilde{X}_+(B))$.

Theorem 3.2. *Assume that conditions j*1), *j*2) *of Theorem 3 are satisfied. Then the following statements hold:*

(b1) *for each* $0 \le k \le p$,
$$X = E^{-1}_{(k+1)p}(\widetilde{X}_-(A)) \dot{+} E_{1k}(\widetilde{X}_+(B)), \tag{3.1}$$
where $E_{(p+1)p} = E_{10} = I$;

(b2) *corresponding to the function* $y \in C_b(\mathbb{R}, X)$ *the unique bounded solution* x *of the differential equation* (1.1) *has the following form:*

if $t \ge t_p$, *then*

$$x(t) = \int\limits_{t_p}^{t} e^{A(t-s)} P_-(A) y(s)\, ds - \int\limits_{t}^{+\infty} e^{A(t-s)} P_+(A) y(s)\, ds$$
$$+ e^{A(t-t_p)} P_p^- \Bigg(\int\limits_{t_p}^{+\infty} e^{A(t_p - s)} P_+(A) y(s)\, ds + \sum_{k=1}^{p} E_{kp} \int\limits_{t_{k-1}}^{t_k} e^{A_k(t_{k-1}-s)} y(s)\, ds$$

$$+ E_{1p} P_0^- \int_{-\infty}^{t_0} e^{B(t_0-s)} P_-(B) y(s)\, ds \Big);$$

if $1 \le k \le p$, *then (sequentially in descending order from* $k = p$ *to* $k = 1$*)*

$$x(t) = -\int_t^{t_k} e^{A_k(t-s)} y(s)\, ds + e^{A_k(t-t_k)} x(t_k), \quad t \in [t_{k-1}; t_k);$$

if $t < t_0$, *then*

$$\begin{aligned} x(t) = \int_{-\infty}^{t} e^{B(t-s)} P_-(B) y(s)\, ds - \int_t^{t_0} e^{B(t-s)} P_+(B) y(s)\, ds \\ - e^{B(t-t_0)} \Big(E_{1p}^{-1} P_p^+ \int_{t_p}^{+\infty} e^{A(t_p-s)} P_+(A) y(s)\, ds \\ + \sum_{k=1}^{p} E_{1(k-1)}^{-1} P_{k-1}^+ \int_{t_{k-1}}^{t_k} e^{A_k(t_{k-1}-s)} y(s)\, ds + P_0^+ \int_{-\infty}^{t_0} e^{B(t_0-s)} P_-(B) y(s)\, ds \Big). \end{aligned}$$

Here for each $0 \le k \le p$ P_k^-, P_k^+ *are the projectors corresponding to representation* (3.1).

For a differential equation with a variable operator coefficient, the boundedness condition was studied, in particular, in [1–3,5] using the exponential dichotomy condition on $\mathbb{R}$ for the corresponding homogeneous differential equation. In the general case, checking the exponential dichotomy condition is not trivial. Theorem 3.1 contains necessary and sufficient conditions directly on the operator coefficients that ensure that the exponential dichotomy condition is satisfied for the homogeneous differential equation corresponding to equation (1.1).

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Anti-Perron Effect of Changing Characteristic Exponents in Differential Systems

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The anti-Perron effect [1–3] (opposite to the well-known Perron one [4, 5]) presupposed the change of all positive characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ of linear approximation

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq t_0, \tag{1}$$

with the bounded infinitely differentiable coefficients to negative in (some) nontrivial solutions of the differential system

$$\dot{y} = A(t)y + f(t,y), \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{2}$$

also with an infinitely differentiable vector-function from the known classes of small perturbations. This effect is of great interest in its applications as compared with the Peron effect (devoted in a cycle of author's works). In the present report, we give an account of the results obtained by the author for the realization of anti-Prron effect.

1^0. In a class of linear exponentially decreasing perturbations the following theorem is valid.

Theorem 1 ([1])**.** *For any parameters $\lambda_n \geq \cdots \geq \lambda_1 > 0$, $\theta > 1$, $0 < \sigma < \lambda_1 + \theta^{-1}\lambda_2$, there exist:*

1) *system* (1) *with exponents $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$;*

2) *a linear perturbation $f(t,y) \equiv Q(t)y$ with the exponent $\lambda[Q] \leq -\sigma < 0$ such that system* (2) *has exactly $n-1$ linearly independent solutions $Y_1(t), \dots, Y_{n-1}(t)$ with the Liapounov exponents*

$$\lambda[Y_i] = \big[\theta(\sigma - \lambda_1) - \lambda_{i+1}\big](\theta - 1)^{-1}, \ \ i = \overline{1, n-1}.$$

Remark 1. The variant $\lambda_1(A) > 0$, $\lambda_n(A+Q) < 0$, $\lambda[Q] < 0$ remains open.

2^0. In the case of linear perturbations $Q(t) \to 0$, as $t \to +\infty$, the following theorem is valid.

Theorem 2 ([2])**.** *For any parameters $0 < \lambda_1 \leq \cdots \leq \lambda_n$, $\mu_1 \leq \cdots \leq \mu_n < 0$, there exist:*

1) *system* (1) *with the exponents $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$;*

2) *the perturbation $Q(t) \to 0$, $t \to +\infty$ such that $\lambda_i(A+Q) = \mu_i$, $i = \overline{1,n}$.*

3^0. In the case of nonlinear m-perturbations

$$\|f(t,y)\| \le C_f\|y\|^m,\ \ m>1,\ \ y\in\mathbb{R}^n,\ \ t\ge t_0, \tag{3}$$

the following theorem holds.

Theorem 3 ([3])**.** *For any parameters $m>1$, $\theta>1$ and $\lambda>0$, there exist:*

1) *two-dimensional system* (1) *with exponents $\lambda_1(A)=\lambda_2(A)=\lambda>0$;*
2) *an infinitely differentiable perturbation* (3) *such that the nonlinear system* (2) *has the solution $Y(t)$ with the exponent*

$$\lambda[Y]=-\frac{\lambda(\theta+1)}{m\theta-1}\,.$$

The anti-Perron effect in the case under consideration is realized for a great number of solutions of the perturbed system. These systems belong to the spatially-time octants

$$R_1^2=\big\{y\in\mathbb{R}^2:\ \ y_1\ge 0,\ \ y_2\ge 0\big\}\times T_0,\quad R_2^2=\big\{y\in\mathbb{R}^2:\ \ y_1\le 0,\ \ y_2\ge 0\big\}\times T_0,$$
$$R_3^2=\big\{y\in\mathbb{R}^2:\ \ y_1\le 0,\ \ y_2\le 0\big\}\times T_0,\quad R_4^2=\big\{y\in\mathbb{R}^2:\ \ y_1\ge 0,\ \ y_2\le 0\big\}\times T_0,$$

in which $y=(y_1,y_2)\in\mathbb{R}^2$, $T_0=[t_0,+\infty)$, $t_0\ge 0$.

The following theorem is valid.

Theorem 4. *For any parameters $\lambda>0$, $m_4\ge m_3\ge m_2\ge m_1>1$, $\theta>1$, there exist:*

1) *two-dimensional linear system* (1) *with the characteristic exponents $\lambda_1(A)=\lambda_2(A)=\lambda>0$;*
2) *an infinitely differentiable m_1-perturbation $f(t,y):[t_0,+\infty)\times\mathbb{R}^2\to\mathbb{R}^2$ which is simultaneously an m_i-perturbation in the octant R_i^2 for any $i=\overline{1,4}$ such that the perturbed system* (2) *has the solutions $Y_i\subset R_i^2$, $i=\overline{1,4}$, with exponents*

$$\lambda[Y_i]=-\lambda\frac{\theta+1}{m_i\theta-1}<0.$$

Remark 2. An analogous to Theorem 3 statement on the existence of two-dimensional systems (1) with all positive exponents and (2) with perturbation (3) having 4 nontrivial solutions with negative different Liapounov exponents, is valid.

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On One Diffusion System of Nonlinear Partial Differential Equations

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The present note is devoted to one one-dimensional system of nonlinear partial differential equations (SNPDE). Many mathematical models in physics, biology, engineering and so on are described by such type of models (see, for example, [1–3, 6, 9, 11, 12, 15, 17, 18] and the references therein). In this article the initial-boundary value problems are considered and some features of solutions are stated. The finite difference scheme is constructed for the investigated problem and the question of its convergence is given. A lots of scientific works are dedicated to the investigation and numerical resolution of such models (see, for example, [1–13, 15, 17, 18] and the references therein).

As a model, let us consider the SNPDE of the following type:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(A(V)\,\frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\Big(C(V)\,\frac{\partial V}{\partial x}\Big) + F\Big(V, \frac{\partial U}{\partial x}\Big), \end{aligned} \tag{1}$$

where A, C and F are the given functions of their arguments.

The numerous diffusion problems are reduced to (1) SNPDE. In particular, if

$$C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

(1) SNPSE meets at the modeling of penetration of an electromagnetic field into a medium, whose coefficient of electroconductivity depends on temperature, without taking into account the heat conductivity [11]. If

$$C(V) \not\equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -D(V) + B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

where B and D are given functions of their arguments, then system (1) describes the process of penetration of an electromagnetic field into the medium, taking into account the heat conductivity [11].

If

$$A(V) \equiv V, \quad C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -V + G\Big(V\,\frac{\partial U}{\partial x}\Big),$$

where $0 < g_0 \le G(\xi) \le G_0, g_0$ and G_0 are constants, and G is a smooth enough function, then (1) represents an one-dimensional analogue of system which arises in studying the process of vein formation in young leaves of higher plants [15].

Let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha\,\frac{\partial U}{\partial x}\Big), \tag{2}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2, \tag{3}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \tag{4}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{5}$$

where $\psi = const > 0$, and $U_0 = U_0(x), V_0 = V_0(x)$ are the given functions.

If $U(x,0) = \psi x$ and $V(x,0) = \delta_0 = const > 0$, as it is mentioned in [7] the pair of functions:

$$U(x,t) = \psi x, \quad V(x,t) = \big[\delta_0^{1-\alpha} + (1-\alpha)\psi^2 t\big]^{\frac{1}{1-\alpha}}, \tag{6}$$

is the solution of the initial-boundary value problem (2)–(5) for any $\alpha \neq 1$. However, if $\alpha > 1$, then for a finite time $t_0 = \delta_0^{1-\alpha}/(\psi^2(\alpha-1))$, the function V becomes unbounded. This example shows that the solutions of a system such as (2), (3) with smooth initial and boundary conditions can blow-up at a finite time.

Note that the functions U and V, determined by formulas (6), also satisfy the system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \tag{7}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}, \tag{8}$$

with the boundary and initial conditions (4), (5) and adding to them the following boundary conditions:

$$\frac{\partial V}{\partial x}\Big|_{x=0} = \frac{\partial V}{\partial x}\Big|_{x=1} = 0. \tag{9}$$

From this we can conclude that if $\alpha > 1$, then for problem (4), (5), (7)–(9), the theorem on the existence of the global solution also does not hold.

The question of the stability of the stationary solution for appropriate diffusion problems is interesting for a mathematical explanation. In this connection we consider the following initial-boundary value problem:

$$\begin{gathered}
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \\
\frac{\partial V}{\partial t} = -V^\beta + V^\gamma \Big(\frac{\partial U}{\partial x}\Big)^2, \\
U(0,t) = 0, \quad V^\alpha \frac{\partial U}{\partial x}\Big|_{x=1} = \psi, \\
U(x,0) = U_0(x), \quad V(x,0) = V_0(x).
\end{gathered} \tag{10}$$

It is easy to be convinced that the stationary solution of problem (10) has the form

$$\big(\psi^{\frac{\beta-\gamma}{2\alpha+\beta-\gamma}} x, \psi^{\frac{2}{2\alpha+\beta-\gamma}}\big).$$

The following statement is true.

Theorem 1. *If $\alpha \neq 0, 2\alpha+\beta-\gamma > 0$, then the stationary solution $(\psi^{\frac{\beta-\gamma}{2\alpha+\beta-\gamma}} x, \psi^{\frac{2}{2\alpha+\beta-\gamma}})$ of problem (10) is linearly stable if and only if the following condition takes place*

$$(\gamma-\beta)\psi^{\frac{2(\beta-\alpha-1)}{2\alpha+\beta-\gamma}} < \frac{\pi}{4}\,. \tag{11}$$

Remark 1. If $\gamma - \beta \leq 0$, then the stationary solution of problem (10) is always linearly stable.

Remark 2. Let
$$\gamma-\beta>0,\quad \beta-\alpha-1\neq 0,\quad \psi_c=\Big[\frac{\pi^2}{4(\gamma-\beta)}\Big]^{\frac{2\alpha+\beta-\gamma}{2(\beta-\alpha-1)}}.$$
Applying (11), if $0<\psi<\psi_c$, the stationary solution of problem (10) is linearly stable, and if $\psi>\psi_c$, it becomes unstable. For $\psi=\psi_c$, there is the possibility of occurrence of the Hopf-type bifurcation [14]. Small perturbations of the stationary solution can be transformed into a periodic in time self-oscillation.

Based on [5], the analogous investigations for more general models are given in [6, 8, 10].

Let us now consider the global stability of a solution of problem (10) for one particular case. Consider the following problem:
$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}=-V+\Big(\frac{\partial U}{\partial x}\Big)^2,\\ U(0,t)=0,\quad V\,\frac{\partial U}{\partial x}\Big|_{x=1}=\psi,\\ U(x,0)=U_0(x),\quad V(x,0)=V_0(x).\end{gathered}\tag{12}$$

It is obvious that the stationary solution of problem (12) looks like to $(\psi^{1/3}x,\psi^{2/3})$. Introduce the notations:
$$y(x,t)=U(x,t)-\psi^{1/3}x,\quad z(x,t)=V(x,t)-\psi^{2/3},$$
where (U,V) is a solution of problem (12). We finally arrive at
$$\int_0^1\big[y^2(x,t)+z^2(x,t)\big]\,dx\le e^{-Kt}\int_0^1\Big\{\big[U_0(x)-\psi^{1/3}x\big]^2+\big[V_0(x)-\psi^{2/3}\big]^2\Big\}\,dx,$$
where K is a positive constant.

Thus, the following statement is true.

Theorem 2. *For the stationary solution of problem* (12) $(\psi^{1/3}x,\psi^{2/3})$ *there takes place the global and monotone stability in* $L_2(0,1)$.

Note that it is not difficult to get a certain generalization of the results considered here for the diffusion model, where the process of heat conductivity is taken into account.

Note also that some results regarding solvability, uniqueness, asymptotic behavior of solutions and properties of the difference schemes of corresponding integro-differential models with different kinds of boundary conditions for the above-mentioned equations and systems are studied in many works (see, for example, [6, 9] and the references therein).

Now, let us consider the convergence of difference schemes for the following problem:
$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V^\alpha\,\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}=V^{\alpha-1}\Big(\frac{\partial U}{\partial x}\Big)^2,\end{gathered}\tag{13}$$
$$\begin{gathered}U(0,t)=U(1,t)=0,\quad U(x,0)=U_0(x),\\ V(x,0)=V_0(x)\ge\sigma_0=const>0.\end{gathered}\tag{14}$$

The grid-function $u=\{u_i\}$ corresponding to U is considered in usual grid, whereas the function $v=\{v_i\}$ approximating V is considered at the centers of grid points.

Using usual notations [16], let us consider the following two-parameterized finite difference scheme:

$$
\begin{gathered}
u_t + \beta\tau u_{\bar{t}t} = \big[(v^{(\sigma)})^{\alpha} u_{\overline{x}}^{(\sigma)}\big]_x,\\
v_t + \beta\tau v_{\bar{t}t} = (v^{(\sigma)})^{\alpha-1}(u_{\overline{x}}^{(\sigma)})^2,\\
u(0,t) = u(1,t) = 0, \quad u(x,0) = U_0(x),\\
u(x,\tau) = U_0(x) + \tau\big[(V^{(\sigma)})^{\alpha} U_{\overline{x}}^{(\sigma)}\big]_{x,t=0},\\
v(x,0) = V_0(x), \quad v(x,\tau) = V_0(x) + \tau\big[(V^{(\sigma)})^{\alpha-1}(U_{\overline{x}}^{(\sigma)})^2\big]_{t=0}.
\end{gathered}
\tag{15}
$$

Here,

$$v^{(\sigma)} = \sigma v^{j+1} + (1-\sigma)v^j.$$

The scheme (15), for the sufficiently smooth solution of the problem (13), (14), has the following order of approximation:

$$O(\tau^2 + h^2 + (\sigma - 0,5 - \beta)\tau).$$

Using the method of energy inequalities [16] for investigation of the difference schemes, the following statement is proved.

Theorem 3. *If problem* (13), (14) *has the sufficiently smooth solution, then the solution of the difference scheme* (15) *tends to the solution of problem* (13), (14) *and the rate of the convergence is* $O(\tau^2 + h^2 + (\sigma - 0,5 - \beta)\tau)$.

Remark 3. If $\sigma = 1/2, \beta = 0$, the two-layer difference scheme with accuracy of order $O(\tau^2 + h^2)$ is constructed. The same accuracy takes place if $\sigma = 1$ and $\beta = 1/2$. In this case, (15) is the three-layer scheme.

The difference scheme (15) is the system of the nonlinear algebraic equations. To be convinced of the solvability, it is enough to use an a-priori estimation which follows after the multiplication of equations (15) by u and v, respectively, and apply the Brouwer fixed-point lemma (see, e.g., [13]). Note that applying the same technique as we use in proving the convergence Theorem 3, it is not difficult to prove the uniqueness of the solution and the stability of the scheme (15).

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Oscillation Theory of Nonlinear Differential Equations of Emden–Fowler Type with Variable Exponents

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1 Introduction

We consider nonlinear differential equations of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \tag{A}$$

under the following assumptions:

(a) the coefficients $p(t)$ and $q(t)$ are positive continuous functions on $I = [a,\infty)$, $a \geq 0$;

(b) the exponents $\alpha(t)$ and $\beta(t)$ are positive continuous functions on I having the limits $\alpha(\infty)$ and $\beta(\infty)$ as $t \to \infty$ in the extended real number system;

(c) the symbol $\varphi_{\gamma(t)}$ with a positive continuous function $\gamma(t)$ on I denotes the operator in $C(I)$ defined by

$$\varphi_{\gamma(t)}(u(t)) = |u(t)|^{\gamma(t)} \operatorname{sgn} u(t), \quad u \in C(I).$$

Since the prototype of (A) is the differential equation

$$\big(p(t)\varphi_{\alpha}(x')\big)' + q(t)\varphi_{\beta}(x) = 0, \tag{A_0}$$

α and β being positive constants, which is well-known as the Emden–Fowler equation, (A) is often referred to as a generalized Emden–Fowler equation or an Emden–Fowler type equation with variable exponents.

We are concerned exclusively with nontrivial solutions $x(t)$ of (A) which are defined on an infinite interval of the form $[T,\infty)$, $T \geq a$. A solution is called *oscillatory* if it has an infinite sequence of zeros tending to infinity and *nonoscillatory* otherwise. Given a solution $x(t)$ of (A), we define

$$D_\alpha x(t) = p(t)\varphi_{\alpha(t)}(x'(t)),$$

and call it the *quasi-derivative* of $x(t)$. In this notation, the dependence of the operator D_α on $p(t)$ is omitted for simplicity.

Historically, a vast literature has been published on oscillation theory of the standard Emden–Fowler differential equation (A_0). A remarkable result in the theory is the fact that the situation in which all solutions of (A_0) with $\alpha \neq \beta$ are oscillatory can be characterized completely by the impressive integral conditions formulated in terms of the exponents $\{\alpha, \beta\}$ and the coefficients $\{p(t), q(t)\}$.

Equation (A_0) is called strongly superlinear or strongly sublinear according as $\alpha < \beta$ or $\alpha > \beta$, respectively. Use is made of the following notations and functions:

$$I_p = \int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt, \quad I_q = \int_a^\infty q(t)\,dt,$$

$$P(t) = \int_a^t p(s)^{-\frac{1}{\alpha}}\,ds \ \text{ if } \ I_p = \infty, \quad \pi(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds \ \text{ if } \ I_p < \infty,$$

$$Q(t) = \int_a^t q(s)\,ds \ \text{ if } \ I_q = \infty, \quad \rho(t) = \int_t^\infty q(s) \ \text{ if } \ I_q < \infty.$$

The following facts are well-known.

(i) All solutions of (A_0) are oscillatory if $I_p = I_q = \infty$.

(ii) Assume that $I_p = \infty$ and $I_q < \infty$. Let (A_0) be strongly superlinear. All of its solutions are oscillatory if and only if

$$\int_a^\infty \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha}}\,dt = \infty.$$

(iii) Assume that $I_p = \infty$ and $I_q < \infty$. Let (A_0) be strongly sublinear. All of its solutions are oscillatory if and only if

$$\int_a^\infty q(t)P(t)^\beta\,dt = \infty.$$

(iv) Assume that $I_p < \infty$ and $I_q = \infty$. Let (A_0) be strongly superlinear. All of its solutions are oscillatory if and only if

$$\int_a^\infty q(t)\pi(t)^\beta\,dt = \infty.$$

(v) Assume that $I_p < \infty$ and $I_q = \infty$. Let (A_0) be strongly sublinear. All of its solutions are oscillatory if and only if

$$\int_a^\infty \big(p(t)^{-1}Q(t)\big)^{\frac{1}{\alpha}}\,dt = \infty.$$

For the proofs of these theorems see e.g. Elbert and Kusano [1] and Kusano et al. [3].

Now, a question naturally arises: Is it possible to characterize the oscillation of all solutions of the generalized Emden–Fowler equations with variable exponents? The aim of the present work is to give an affirmative answer to this question by showing that the results (ii)–(v) for (A_0) mentioned above can be properly generalized to equation (A) which is *strongly superlinear* or *strongly sublinear* in the sense defined below.

Any generalized Emden–Fowler equation (A) is made up by the two crucial components. One is the pair of the exponents $\{\alpha(t), \beta(t)\}$ which determines the nonlinearity of (A), and the other is the pair of the coefficients $\{p(t), q(t)\}$ which implies, so to speak, the size or magnitude of (A).

The concept of superlinearity and sublinearity of (A_0) is extended to equation (A) as follows.

Definition 1.1.

(i) Equation (A) is said to be *strongly superlinear* if the pair of exponents $\{\alpha(t), \beta(t)\}$ has the property that $\alpha(t)$ is nonincreasing, $\beta(t)$ is nondecreasing and there is a constant $\lambda > 1$ such that
$$\beta(t) \geq \lambda\alpha(t) \ \text{ for } \ t \geq a.$$

(ii) Equation (A) is said to be *strongly sublinear* if the pair of exponents $\{\alpha(t), \beta(t)\}$ has the property that $\alpha(t)$ is nonincreasing, $\beta(t)$ is nondecreasing and there is a positive constant $\mu > 1$ such that
$$\alpha(t) \geq \mu\beta(t) \ \text{ for } \ t \geq a.$$

We measure the size of the coefficients $p(t)$ and $q(t)$ by their integrals defined by

$$I(p) = \int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\, dt \ \text{ and } \ I(q) = \int_a^\infty q(t)\, dt.$$

There are four different combinations of $I(p)$ and $I(q)$, of which the following three cases will be the main object of our analysis.

Definition 1.2. Equation (A) is said to be of category I if $I(p) = \infty$ and $I(q) < \infty$, of category II if $I(p) < \infty$ and $I(q) = \infty$, and of category III if $I(p) = \infty$ and $I(q) = \infty$.

The category IV ($I(p) < \infty$, $I(q) < \infty$) is excluded from our consideration because equation (A) of this category always possesses nonoscillatory solutions.

Our main objective in this paper is to generalize the propositions (ii)–(v) listed above regarding the standard Emden–Fowler equation (A_0) to the corresponding Emden–Fowler equation with variable exponents (A).

In Section 2 we focus our attention on equation (A) of category I and show by way of direct asymptotic analysis that necessary and sufficient conditions for oscillation of all of its solutions can be established for both strongly superlinear and strongly sublinear cases. Equation (A) of category II is considered in Section 3. There, we avoid analyzing the equation directly as in Section 2, and make use of an uncommon means named *Duality Principle* which makes it possible to derive the desired oscillation theorems for the category II equation almost automatically from the results on the category I equation already known in Section 2. Thus it turns out that our results obtained in Sections 2 and 3 combined are an exact generalization of the propositions (ii)–(v) which are the typical oscillation theorems for the standard Emden–Fowler equation (A_0).

2 Oscillation of equation (A) of category I

We begin with an oscillation theorem which generalizes the proposition (i) for (A_0) to equation (A) of of category III.

Theorem 2.1. *Consider equation* (A) *with* $\alpha(\infty) > 0$ *and* $\beta(\infty) > 0$. *All of its solutions are oscillatory if* $p(t)$ *and* $q(t)$ *have the property that* $I(p) = \infty$ *and* $I(q) = \infty$.

Proof. Assume for contradiction that (A) has a nonoscillatory solution $x(t)$ on $J = [T, \infty)$. Without loss of generality we may suppose that $x(t) > 0$ on J. Since (A) is written as

$$(D_\alpha x)'(t) = -q(t)x(t)^{\beta(t)} < 0,$$

$D_\alpha x(t)$ is decreasing on J. We claim that $D_\alpha x(t) > 0$ on J. In fact, if it is negative at some point of $t_* \in J$ then there is a negative constant $-k = D_\alpha x(t_*)$ such that

$$D_\alpha x(t) = -p(t)(-x'(t))^{\alpha(t)} \le -k \ \text{ for } \ t \ge t_*.$$

Rewriting the above as

$$-x'(t) \ge k^{\frac{1}{\alpha(t)}} p(t)^{-\frac{1}{\alpha(t)}} \ \text{ for } \ t \ge t_*,$$

and integrating the above inequality from t_* to t, we have

$$x(t_*) - x(t) \ge \int_{t_*}^{t} k^{\frac{1}{\alpha(s)}} p(s)^{-\frac{1}{\alpha(s)}}\, ds, \ \ t \ge t_*,$$

from which, since $k^{\frac{1}{\alpha(t)}} \ge k_0$, $t \ge t_*$, for some constant $k_0 > 0$ because of $\alpha(\infty) > 0$, it follows that $x(t) \to -\infty$ as $t \to \infty$. This, however, contradicts the assumed positivity of $x(t)$, and hence we must have $D_\alpha x(t) > 0$ on J. This means that $x(t)$ is increasing on J.

Now, we integrate (A) from T to t to obtain

$$\int_{T}^{t} q(s)x(s)^{\beta(s)}\, ds = D_\alpha x(T) - D_\alpha x(t) \le D_\alpha x(T), \ \ t \ge T,$$

which implies that $\int_{T}^{\infty} q(s)x(s)^{\beta(s)}\, ds < \infty$. Combining this inequality with the fact that $x(t)^{\beta(t)}$ with $\beta(\infty) > 0$ is greater than some positive constant on J, we conclude that $\int_{T}^{\infty} q(s)\, ds < \infty$ contrary to the assumption $I(q) = \infty$. This completes the proof. □

Note that in Theorem 2.1 neither the superlinearity nor the sublinearity is required for (A).

Let there be given equation (A) of category I whose coefficients $p(t)$ and $q(t)$ satisfy $I(p) = \infty$ and $I(q) < \infty$, respectively. Use is made of the functions

$$P_\alpha(t) = \int_{a}^{t} p(s)^{-\frac{1}{\alpha(s)}}\, ds \ \text{ and } \ \rho(t) = \int_{t}^{\infty} q(s)\, ds.$$

It is clear that $P_\alpha(t) \to \infty$ and $\rho(t) \to 0$ as $t \to \infty$.

The main results of this section are stated in the following two theorems. They guarantee that the situation in which all solutions of equation (A) of category I are oscillatory is completely characterized provided that (A) is either strongly superlinear or strongly sublinear.

Theorem 2.2. *Let equation* (A) *with* $\alpha(\infty) > 0$ *be of category* I *and strongly superlinear. Then, all solutions of* (A) *are oscillatory if and only if*

$$\int_{a}^{\infty} \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha(t)}}\, dt = \infty. \tag{2.1}$$

Theorem 2.3. *Let equation* (A) *with* $\alpha(\infty) > 0$ *be of category* I *and strongly sublinear. Then, all solutions of* (A) *are oscillatory if and only if*

$$\int_{a}^{\infty} q(t)P_\alpha(t)^{\beta(t)}\, dt = \infty.$$

Each of these theorems is proved by reductio ad absurdum. In proving Theorem 2.2, for example, to verify the "if" part, first we assume (2.1) to hold but (A) has a nonoscillatory solution of (A) and after a sensitive computational process we are finally forced to admit the contrary conclusion that

$$\int_a^\infty \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha(t)}}\,dt < \infty. \tag{2.2}$$

Likewise, to verify the "only if" part of Theorem 2.2, we have to show that the condition (2.2) implies the existence of a nonoscillatory solution for equation (A). As a matter of fact, one such positive solution $x(t)$ such that $x(\infty) = 1$ can be obtained as a solution of the integral equation with variable exponents

$$x(t) = 1 - \int_T^t \left(p(s)^{-1}\int_s^\infty q(r)x(r)^{\beta(r)}\,dr\right)^{\frac{1}{\alpha(s)}} ds, \quad t \geq T, \tag{2.3}$$

for some sufficiently large $T > a$. It should be noted that the solvability of (2.3) is assured for a much wider class of equations of the form (A) including both strongly superlinear and sublinear equations as special cases.

The procedure of the proof of Theorem 2.3 by reductio ad absurdum is essentially the same as for Theorem 2.2.

What is said above suggests that in studying oscillation theory of generalized Emden–Fowler equations preliminary knowledge of nonoscillation theory for them is indispensable. See [2].

3 Oscillation of equation (A) of category II via Duality Principle

Now we turn our attention to equation (A) of category II whose coefficients $p(t)$ and $q(t)$ satisfy the integral conditions

$$\int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\,dt < \infty, \qquad \int_a^\infty q(t)\,dt = \infty.$$

Equation (A) is assumed to be either strongly superlinear or strongly sublinear.

For such an equation (A) the functions

$$\pi_\alpha(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\,ds \ \text{ and } \ Q(t) = \int_a^t q(s)\,ds,$$

are well-defined and play a major role throughtout this section. It is clear that $\pi_\alpha(t) \to 0$ and $Q(t) \to \infty$ as $t \to \infty$.

Our aim is to find explicit oscillation criteria for equation (A) of category II which are similar to those given in Theorems 2.1 and 2.2 for equation (A) of category I. We are so bold as to make use of an uncommon method (named *Duality Principle*) which enables us to precisely formulate the desired results for equations of category II almost automatically (without additional serious computations) from the corresponding known results for equations of category I.

Let (A) be a generalized Emden–Fowler equation with the exponents $\{\alpha(t), \beta(t)\}$ and the coefficients $\{p(t), q(t)\}$. Putting

$$y(t) = -p(t)\varphi_{\alpha(t)}(x'(t)),$$

equation (A) is split into the first-order differential system with variable exponents

$$x'(t) = -p(t)^{-\frac{1}{\alpha(t)}}\varphi_{\frac{1}{\alpha(t)}}(y(t)), \quad y'(t) = q(t)\varphi_{\beta(t)}(x(t)). \tag{3.1}$$

It is easy to see that elimination of $\{y(t), y'(t)\}$ from (3.1) gives the original second-order differential equation (A), and that elimination of $\{x(t), x'(t)\}$ from (3.1) gives a new second-order differential equation

$$\big(q(t)^{-\frac{1}{\beta(t)}}\varphi_{\frac{1}{\beta(t)}}(y')\big)' + p(t)^{-\frac{1}{\alpha(t)}}\varphi_{\frac{1}{\alpha(t)}}(y) = 0. \tag{B}$$

Equation (B) is called the *reciprocal equation* of (A). Equation (B) is structurally the same as equation (A) and has the exponents $\{\frac{1}{\beta(t)}, \frac{1}{\alpha(t)}\}$ and the coefficients $\{q(t)^{-\frac{1}{\beta(t)}}, p(t)^{-\frac{1}{\alpha(t)}}\}$. It is obvious that (A) is the reciprocal equation of (B).

If we denote the exponents of (B) by $\{\widetilde{\alpha}(t), \widetilde{\beta}(t)\}$, and the coefficients of (B) by $\{\widetilde{p}(t), \widetilde{q}(t)\}$, then it is easily verified that the nonlinearity of (B) is the same as that of (A), and that

$$\int_a^\infty \widetilde{p}(s)^{-\frac{1}{\widetilde{\alpha}(s)}}\,ds = \int_a^\infty q(s)\,ds, \quad \int_a^\infty \widetilde{q}(s)\,ds = \int_a^\infty p(s)^{-\frac{1}{\alpha(s)}}\,ds.$$

Thus it is confirmed that the transition from equation (A) to its reciprocal equation (B) keeps the strong superlinearity or strong sublinearity of (A) unchanged, but changes the category of (A) from I to II, or from II to I. Such a close interrelationship between (A) and its reciprocal equation (B) is worthy of being remembered as a principle:

Duality Principle. Let equation (B) be the reciprocal equation of (A).

(i) If (A) is strongly superlinear (or strongly sublinear), then so is (B).

(ii) If (A) is of category I (resp. category II), then (B) is of category II (resp. category I).

(iii) All solutions of (A) are oscillatory if and only if all solutions of (B) are oscillatory.

Let us return to equation (A) of category II which is either strongly superlinear or strongly sublinear, and demonstrate that the Duality Principle makes it possible to find the desired oscillation criteria for (A) almost automatically from the already known oscillation criteria for (B) which is category I.

It is known that since (A) has the coefficients $\{p(t), q(t)\}$ and the exponent $\{\alpha(t), \beta(t)\}$, the components of the coefficients $\{\widetilde{p}(t), \widetilde{q}(t)\}$ and the exponents $\{\widetilde{\alpha}(t), \widetilde{\beta}(t)\}$ of (B) are expressed as

$$\widetilde{p}(t) = q(t)^{-\frac{1}{\beta(t)}}, \quad \widetilde{q}(t) = p(t)^{-\frac{1}{\alpha(t)}}, \quad \widetilde{\alpha}(t) = \frac{1}{\beta(t)}\,, \quad \widetilde{\beta}(t) = \frac{1}{\alpha(t)}\,.$$

Suppose that (A) is strongly superlinear. In addition suppose that $\beta(\infty) < \infty$. Then, (B) is also strongly superlinear and $\widetilde{\alpha}(\infty) = 1/\beta(\infty) > 0$, Since (B) is of category I, Theorem 2.2 is applicable to to (B) and ensures that all solutions of (B) are oscillatory if and only if

$$\int_a^\infty (\widetilde{p}(t)^{-1}\widetilde{\rho}(t))^{\frac{1}{\widetilde{\alpha}(t)}}\,dt = \infty.$$

Noting that

$$\widetilde{p}(t)^{-1} = q(t)^{\frac{1}{\alpha(t)}} \ \text{ and } \ \widetilde{\rho}(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\,ds = \pi_\alpha(t),$$

we are led to the following oscillation theorem for strongly superlinear equation (A) of category II.

Theorem 3.1. *Let* (A) *be a strongly superlinear equation with* $\beta(\infty) < \infty$ *and of category* II. *All of its solutions are oscillatory if and only if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\,dt = \infty.$$

Next, suppose that (A) is strongly sublinear with $\beta(\infty) < \infty$. Then, (B) is also strongly sublinear with $\widetilde{\alpha}(\infty) > 0$ and so applying Theorem 2.3 to (B) we see that all solutions of (B) with $\widetilde{\alpha}(\infty) > 0$ are oscillatory if and only if

$$\int_a^\infty \widetilde{q}(t)\widetilde{P}_\alpha(t)^{\widetilde{\beta}(t)}\,dt = \infty. \tag{3.2}$$

Noting (3.2) that $\widetilde{q}(t) = p(t)^{-\frac{1}{\alpha(t)}}$ and

$$\widetilde{P}_{\widetilde{\alpha}(t)}(t)^{\widetilde{\beta(t)}} = \Bigg(\int_a^t q(s)\,ds\Bigg)^{\frac{1}{\alpha(t)}} = Q(t)^{\frac{1}{\alpha(t)}}.$$

we are led to the following oscillation theorem for strongly sublinear equation of category II.

Theorem 3.2. *Let* (A) *be a strongly sublinear equation with* $\beta(\infty) < \infty$ *and of category* II. *All of its solutions are oscillatory if and only if*

$$\int_a^\infty (p(t)^{-1}Q(t))^{\frac{1}{\alpha(t)}}\,dt = \infty.$$

Concluding Remarks. Recenly there has been an increasing interest in the study of differential equations with variable exponents. To the best of our knowledge the pioneer of oscillation theory of such equations is Koplatadze who published the papers [4, 5]. Koplatadze's results are closely related to ours specialized to equation (A) with $\alpha(t) \equiv 1$ and $p(t) \equiv 1$. For other related topics see e.g.the papers [2, 7, 8].

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Properties of Oscillatory Solutions of Second Order Half-Linear Differential Equations

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1 Introduction

This report is a continuation of the note [1] that was submitted to the QUALITDE-2021 in which the distribution of zeros and extrema of oscillatory solutions were studied for second order half-linear differential equations of the form

$$\big(p(t)\varphi_\alpha(x')\big)' + q(t)\varphi_\alpha(x) = 0, \tag{HL}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive continuously differentiable functions on $[a,\infty)$, and φ_α stands for the odd function on $\mathbb{R}$ given by

$$\varphi_\alpha(u) = |u|^\alpha \operatorname{sgn} u = |u|^{\alpha-1}u, \ \ u \in \mathbb{R}.$$

We assume that equation (HL) is oscillatory, that is, all of its nontrivial solutions are oscillatory. Let $x(t)$ be a solution of (HL) on $[a,\infty)$. Let $\{\sigma_k\}_{k=1}^\infty$ $(\sigma_k < \sigma_{k+1})$ be the sequence of all zeros of $x(t)$, and let $\{\tau_k\}_{k=1}^\infty$ $(\tau_k < \tau_{k+1})$ be the sequence of all points at which $x(t)$ takes on its local extrema. It is clear that $x'(\tau_k) = 0$ for all k. The value $|x'(\sigma_k)|$ is called the *slope* of $x(t)$ at $t = \sigma_k$, while the value $|x(\tau_k)|$ is called the *amplitude* of $x(t)$ at $t = \tau_k$. The sets of the amplitudes and slopes of $x(t)$ determine the following

$$\mathcal{A}^*[x] = \sup_k |x(\tau_k)|, \quad \mathcal{A}_*[x] = \inf_k |x(\tau_k)|, \tag{1.1}$$

$$\mathcal{S}^*[x] = \sup_k |x'(\sigma_k)|, \quad \mathcal{S}_*[x] = \inf_k |x'(\sigma_k)|, \tag{1.2}$$

which provide helpful information about the oscillatory behavior of $x(t)$. Since it is difficult to analyze the equation (HL) with general positive functions $p(t)$ and $q(t)$, we restrict our analysis to the equation in which both $p(t)$ and $q(t)$ are positive monotone functions on $[a,\infty)$. The four possible cases

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$;

(iii) $p'(t) \geqq 0$, $q'(t) \geqq 0$;

(iv) $p'(t) \leqq 0$, $q'(t) \leqq 0$

should be distinguished.

2 Known results

In this section we state known results for the convenience of the reader (see [1]).

Theorem A. *Let* (HL) *be oscillatory and let $x(t)$ be a solution of it satisfying the initial condition*

$$x(a) = l, \quad x'(a) = m, \tag{2.1}$$

where l and m are any given constants such that $(l, m) \neq (0, 0)$.

(i) *Suppose that $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } q(\infty) > 0, \tag{2.2}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty) < \infty. \tag{2.3}$$

(ii) *Suppose that $p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$.* Then,

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty) > 0, \tag{2.4}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } q(\infty) < \infty. \tag{2.5}$$

(iii) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.6}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty. \tag{2.7}$$

(iv) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0, \tag{2.8}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.9}$$

Since the constants l and m in (2.1) are arbitrary, the above inequalities (2.2)–(2.9) guarantee under the indicated conditions on $p(\infty)$ and/or $q(\infty)$ that $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ for all solutions $x(t)$ of (HL). Then, noting that $\mathcal{A}^*[x] < \infty$ gives the boundedness of $x(t)$ on $[a, \infty)$ and more $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ implies the non-decaying boundedness of $x(t)$ on $[a, \infty)$, we have the following propositions.

Corollary A. *Suppose that* (HL) *is oscillatory. All of its solutions are bounded on* $[a, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Corollary B. *Suppose that* (HL) *is oscillatory. All of its solutions are non-decaying bounded on* $[a, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $p(\infty) < \infty$, $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$, $q(\infty) < \infty$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Theorem B. *Let* (HL) *be oscillatory and let* $x(t)$ *be a solution of it satisfying* (2.1).

(i) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.10}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty) < \infty \ \textit{ and } \ q(\infty) > 0. \tag{2.11}$$

(ii) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty) > 0 \ \textit{ and } \ q(\infty) < \infty, \tag{2.12}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.13}$$

(iii) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } q(\infty) < \infty, \tag{2.14}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \textit{if } p(\infty) < \infty. \tag{2.15}$$

(iv) *Suppose that $p'(t) \leqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \ \text{ if } p(\infty) > 0, \tag{2.16}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a)q(a)}\Big]^{\frac{1}{\alpha+1}} \ \text{ if } q(\infty) > 0. \tag{2.17}$$

Corollary C. *Let* (HL) *be oscillatory. If $p(t)$ and $q(t)$ are monotone functions such that $0 < p(\infty) < \infty$ and $0 < q(\infty) < \infty$, then $\mathcal{S}^*[x] < \infty$ and $\mathcal{S}_*[x] > 0$ for all solutions $x(t)$ of* (HL).

3 Main results

Our first result concerns the estimation of the derivatives of oscillatory solutions of (HL).

Theorem 3.1. *Let* (HL) *be oscillatory and let $x(t)$ be the solution of it satisfying the initial condition* (2.1).

(i) *$p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}},$$

$$\lim_{t\to\infty} x'(t) = 0 \ \text{ if } p(\infty) = \infty.$$

(ii) *$p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \ \text{ if } p(\infty) > 0 \ \text{ and } \ q(\infty) < \infty.$$

(iii) *$p'(t) \geqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(a)q(a)}\Big]^{\frac{1}{\alpha+1}} \ \text{ if } q(\infty) < \infty,$$

$$\lim_{t\to\infty} x'(t) = 0 \ \text{ if } \lim_{t\to\infty} \frac{q(t)}{p(t)} = 0.$$

(iv) *$p'(t) \leqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \ \text{ if } p(\infty) > 0.$$

Our next result in this section concerns the sequences of zeros of solutions of (HL). We are interested in explicit laws or rules, if any, governing the arrangement of this sequences. Assume that (HL) is oscillatory. Let $x(t)$ be any of its solutions on $[a, \infty)$ and let $\{\sigma_k\}$ represent the sequences of zeros of $x(t)$.

Theorem 3.2. *The sequence $\{\sigma_{k+1} - \sigma_k\}$ is decreasing or increasing according to $p'(t) \leqq 0$ and $q'(t) \geqq 0$, or $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$.*

4 Example

Consider the half-linear differential equation

$$\big((\coth(t+\tau))^{\alpha}\varphi_\alpha(x')\big)' + k\tanh(t+\tau)\varphi_\alpha(x) = 0 \tag{4.1}$$

on $[0,\infty)$, where $\tau \geqq 0$ and $k > 0$ are constants. Equation (4.1) is oscillatory since the functions $p(t) = (\coth(t+\tau))^{\alpha}$ and $q(t) = k\tanh(t+\tau)$ are not integrable on $[0,\infty)$. It is clear that $p(t)$ and $q(t)$ satisfy $p'(t) \leqq 0$, $q'(t) \geqq 0$, $(p(t)^{\frac{1}{\alpha}}q(t))' = 0$, $p(0) = (\coth\tau)^{\alpha}$, $p(\infty) = 1$, $q(0) = k\tanh\tau$ and $q(\infty) = k$, all nontrivial solutions of equation (4.1) are bounded and non-decaying by (ii) and (iii) of Corollary A and Corollary B, respectively. As regards the estimates for upper and lower amplitudes and upper and lower slopes of solutions of (4.1), we obtain, for example,

$$\begin{aligned}
\mathcal{A}^*[x] &\leqq \Big[\coth\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}(\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{A}_*[x] &\geqq \Big[\tanh\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}(\coth\tau)^{\alpha}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem A, and

$$\begin{aligned}
\mathcal{S}^*[x] &\leqq \Big[\frac{k}{\alpha}\coth\tau\,|l|^{\alpha+1} + (\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{S}_*[x] &\geqq \Big[\frac{k}{\alpha}(\tanh\tau)^{\alpha+1}|l|^{\alpha+1} + |m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem B. If in particular $\tau \to \infty$ and $k = \alpha$, then the upper and lower amplitudes and slopes coincide, that is,

$$\mathcal{A}^*[x] = \mathcal{A}_*[x] = \mathcal{S}^*[x] = \mathcal{S}_*[x] = \big[\,|l|^{\alpha+1} + |m|^{\alpha+1}\big]^{\frac{1}{\alpha+1}}.$$

This value may well be called the amplitude $\mathcal{A}[x]$ and the slope $\mathcal{S}[x]$ of the solution $x(t)$ of the equation

$$(\varphi_\alpha(x'))' + \alpha\varphi_\alpha(x) = 0. \tag{4.2}$$

Equation (4.2) is known as a differential equation generating a generalized trigonometric function. Its solution $x(t)$ determined by the initial condition $x(0) = 0$, $x'(0) = 1$ is the generalized sine function $x(t) = S(t)$ which exists on $\mathbb{R}$, is periodic with period $2\pi_\alpha$, $\pi_\alpha = \frac{2\pi}{\alpha+1}/\sin(\frac{\pi}{\alpha+1})$, and vanishes at $t = n\pi_\alpha$, $n \in \mathbb{Z}$, whose amplitude and slope are given by $\mathcal{A}[x] = 1$ and $\mathcal{S}[x] = 1$, respectively. Moreover, from the first statement of Theorem 3.2 applied to equation (4.2), it follows that the sequences $\{\sigma_k\}$ of zeros of any solution of it are arranged in such a way that $\{\sigma_{k+1} - \sigma_k\}$ is decreasing.

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On the Dirichlet Type Problem for the Inhomogeneous Equation of String Oscillation

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Let Ω be a convex piecewise smooth domain on the plane of independent variables x and t. In the domain Ω for the inhomogeneous equation of the string vibration

$$\square\, u := u_{tt} - u_{xx} = F(x,t), \tag{1}$$

consider the Dirichlet type problem

$$u\big|_{\partial\Omega} = \varphi, \tag{2}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and $\partial\Omega$ is the boundary of the domain Ω.

Numerous works are devoted to the investigation of Dirichlet type problem for the homogeneous string vibration equation. A. Sommerfeld [11] was the first who draw attention to this problem, pointing out the difference between this problem and Dirichlet problem for Laplace equation. A systematic study of this problem for the string vibration homogeneous equation dates back to the works of J. Adamard [5–7], which was later developed in the work of A. Huber [8]. Of particular note is the work of F. John [9], in which Dirichlet type problem for a fairly wide class of domains is reduced to the same problem for a rectangle. In the case of a rectangular domain, this problem was the subject of research of D. G. Bourgin and R. Duffin [2], N. N. Vakhania [12]. In these works, questions of uniqueness and existence of solutions are closely related to the algebraic properties of the ratio λ of sides of a rectangle. In particular, in the case when λ is irrational, there is a unique solution to Dirichlet problem. In the case when λ is rational, the uniqueness of the solution to this problem is violated, and some special cases of solvability of this problem are studied in the works of D. W. Fox and C. Pucci [4], L. L. Campbell [3].

In our work, although a special case is considered when $\lambda = 1$ for the inhomogeneous equation of forced oscillations of a string, necessary and sufficient conditions for the solvability of Dirichlet type problem with inhomogeneous boundary conditions are established, under which the solutions to this problem are written in quadratures. In particular, it is shown that the corresponding homogeneous problem have an infinite number of linearly independent solutions, which are given out explicitly.

Below, for simplicity and clarity of the obtained results, we will limit ourselves to considering the case, when the domain $\Omega := \{(x,t) \in \mathbb{R}^2 :\ 0 < x < l,\ 0 < t < l\}$ is a square. Rewrite the corresponding (2) boundary conditions on $\partial\Omega$ as follows

$$u(x,0) = \varphi(x), \quad u(x,l) = \varphi_1(x), \ \ 0 \le x \le l, \tag{3}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \ \ 0 \le t \le l. \tag{4}$$

Considering regular solutions of the class $C^2(\overline{\Omega})$, we will require the following conditions of smoothness and consistency at the vertices of the square Ω to be satisfied for problem (1), (3), (4)

$$F \in C^1(\overline{\Omega}),\ \ \varphi,\ \varphi_1, \mu_i \in C^2([0,l]),\ \ i = 1,2,$$

$$\varphi(0)=\mu_1(0),\quad \varphi(l)=\mu_2(0),\quad \varphi_1(0)=\mu_1(l),\quad \varphi_1(l)=\mu_2(l),$$
$$\mu_1''(0)-\varphi''(0)=F(0,0),\quad \mu_1''(l)-\varphi_1''(0)=F(0,l),$$
$$\mu_2''(l)-\varphi_1''(l)=F(l,l),\quad \mu_2''(0)-\varphi''(l)=F(l,0).$$

Let $D:=PP_1P_3P_2$ be any characteristic rectangle, lying in Ω, where $P=P(x_0,t_0)$, $P_i=P_i(x_i,t_i)$, $t_0>t_i$, $i=1,2,3$, and the segments P_1P; P_3P_2 and P_1P_3; PP_2 belong to the families of characteristics $x-t=const$ and $x+t=const$, respectively.

Auxiliary statement. Let $\gamma=\gamma_1\cup\gamma_2$ be a simple piecewise smooth curve dividing the characteristic rectangle $PP_1P_3P_2$ into two simply connected domains D_1 and D_2, and γ_1 consists of the characteristic segments of equation (1), and γ_2 does not have a characteristic direction at any of its points. Next, let $u\in C^2(\overline{D}\setminus\gamma)\cap C(\overline{D})$ be a solution of equation (1) in $\overline{D}\setminus\gamma$, and the functions

$$u_1:=u\big|_{\overline{D}_1}\in C^2(\overline{D}_1)\ \text{ and }\ u_2:=u\big|_{\overline{D}_2}\in C^2(\overline{D}_2),$$

on the line transition γ are related by the following relations

$$u_1\big|_\gamma=u_2\big|_\gamma,\quad \frac{\partial u_1}{\partial\nu}\bigg|_{\gamma_2}=\frac{\partial u_2}{\partial\nu}\bigg|_{\gamma_2}, \tag{5}$$

where $\frac{\partial}{\partial\nu}$ is the derivative with respect to the direction of the outer unit normal $\nu:=(\nu_x,\nu_t)$ to the boundary one of the domains D_1 or D_2.

Then the equality holds

$$u(P)=u(P_1)+u(P_2)-u(P_3)+\frac{1}{2}\int\limits_{PP_1P_3P_2}F\,dx\,dt. \tag{6}$$

For investigation of the boundary value problem (1), (3), (4) below will be needed solution in quadratures of the following mixed problem: in the domain Ω find the solution $u\in C^2(\overline{\Omega})$ of equation (1) according to the initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{7}$$

and boundary

$$u(0,t)=\mu_1(t),\quad u(l,t)=\mu_2(t),\quad 0\le t\le l, \tag{8}$$

conditions, where the functions $F,\ \varphi,\ \psi,\ \mu_1$ and μ_2 satisfy the following smoothness and consistency conditions

$$\begin{gathered}F\in C^1(\overline{\Omega}),\ \ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \mu_1,\ \mu_2\in C^2([0,l]),\\ F(0,0)=\mu_1''(0)-\varphi''(0),\quad F(l,0)=\mu_2''(0)-\varphi''(l),\\ \mu_1(0)=\varphi(0),\quad \mu_1'(0)=\psi(0),\quad \mu_2(0)=\varphi(l),\quad \mu_2'(0)=\psi(l).\end{gathered} \tag{9}$$

In order to solve these problem in quadratures let us divide the domain Ω, which is a square with vertices at the points $A(0,0)$, $B(0,l)$, $C(l,l)$ and $D(l,0)$, into four rectangular triangles $\Omega_1:=\Delta AOD$, $\Omega_2:=\Delta AOB$, $\Omega_3:=\Delta DOC$ and $\Omega_4:=\Delta BOC$, where the point $O(\frac{l}{2},\frac{l}{2})$ is the center of the square Ω (see, for example, [10]).

By virtue of d'Alembert's formula (see, for example, [1]) the solution of problem (1), (7) is given by the following equality

$$u(x,t)=\frac{1}{2}\big[\varphi(x-t)+\varphi(x+t)\big]+\frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{\Omega^1_{x,t}}F\,d\xi\,d\tau,\quad (x,t)\in\Omega_1, \tag{10}$$

where $D^1_{x,t}$ is a triangle with vertices at the points (x,t), $(x-t,0)$ and $(x+t,0)$.

Let the point $P = P(x,t) \in \Omega_2$, and $PP_1P_3P_2$ be the characteristic rectangle, where $P_1 = P_1(0,t-x)$, $P_2 = P_2(x+t,0)$, $P_3 = P_3(t,-x)$. Let us denote by $\widetilde{P}_2 = \widetilde{P}_2(x-t,0)$ the point of intersection of the side P_1P_3 of the rectangle $PP_1P_3P_2$ with the side AD of the square Ω. For $t<0$, we introduce the function u_2 as a solution to the equation $\Box u_2 = 0$ with the initial conditions (7), i.e.,

$$u_2(x,t) = \frac{1}{2}\big[\varphi(x-t) + \varphi(x+t)\big] + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\tau)\,d\tau, \ \ (x,t) \in \Delta AED, \tag{11}$$

where $E = E(\frac{l}{2}, -\frac{l}{2})$. For the characteristic rectangle $PP_1P_3P_2$ we use equality (6), in which $\gamma = \widetilde{P}_2P_2$, $D_1 = PP_1\widetilde{P}_2P_2$, $D_2 = \widetilde{P}_2P_3P_2$, $u_1 := u\big|_{PP_1\widetilde{P}_2P_2}$, and the function u_2 is given by equality (11). Due to the above reasoning, bonding conditions (5) will be satisfied, and then, taking into account the Dirichlet boundary conditions (8), equality (6) for our case will take the form

$$u(x,t) = \mu_1(t-x) + \frac{1}{2}\big[\varphi(t+x) - \varphi(t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{t+x} \psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{PP_1\widetilde{P}_2P_2} F\,d\xi\,d\tau, \ \ (x,t) \in \Omega_2. \tag{12}$$

Carrying out similar reasoning in the case of $P = P(x,t) \in \Omega_3$ and $P = P(x,t) \in \Omega_4$ for solution $u = u(x,t)$ of problem (1), (7), (8) we will have

$$u(x,t) = \mu_2(x+t-l) + \frac{1}{2}\big[\varphi(x-t) - \varphi(2l-x-t)\big] + \frac{1}{2}\int\limits_{x-t}^{2l-x-t} \psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{D^3_{x,t}} F\,d\xi\,d\tau, \ \ (x,t) \in \Omega_3, \tag{13}$$

and

$$\begin{aligned} u(x,t) &= \mu_1(t-x) + \mu_2(x+t-l) \\ &\quad - \frac{1}{2}\big[\varphi(t-x) + \varphi(2l-t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{2l-t-x} \psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{D^4_{x,t}} F\,d\xi\,d\tau, \ \ (x,t) \in \Omega_4, \end{aligned} \tag{14}$$

respectively.

Here $D^3_{x,t}$ – quadrilateral with vertices: $P(x,t) \in \Omega_3$, $P^3_1(x-t,0)$, $P^3_2(2l-x-t,0)$ and $P^3_3(l, x+t-l)$, and $D^4_{x,t}$ – pentagon with vertices: $P(x,t) \in \Omega_4$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$ and $P^4_4(l, x+t-l)$.

Thus, due to the conditions of smoothness and consistency (9), the unique classical solution $u \in C^2(\overline{\Omega})$ of problem (1), (7), (8) is given by formulas (10), (12)–(14).

From the above reasoning the following theorem follows.

Theorem. *Let the smoothness and consistency conditions* (9) *be satisfied. Then for the solvability of Dirichlet problem* (1), (3), (4) *it is necessary and sufficient the following condition*

$$\varphi_1(x) = \mu_1(l-x) + \mu_2(x) - \varphi(l-x) + \frac{1}{2}\int\limits_{PP_1P_3P_2} F\,d\xi\,d\tau, \quad 0 \le x \le l \tag{15}$$

to be satisfied, where

$$P = P(x,l), \ \ P_1 = P_1(0, l-x), \ \ P_3 = P_3(l-x, 0), \ \ P_2 = P_2(l,x).$$

Moreover, if condition (15) *is satisfied, all solutions to this problem are given by formulas* (10), (12)–(14)*, where ψ is an arbitrary function from the class $C^1([0,l])$.*

From this theorem it follows that the kernel

$$K := \Big\{ v \in C^2(\overline{\Omega}),\ \ \Box\, v = 0,\ \ v\big|_{\partial\Omega} = 0 \Big\}$$

of problem (1), (3), (4) *is infinite-dimensional and can be described by the formula*

$$v(x,t) = \begin{cases} \dfrac{1}{2} \displaystyle\int\limits_{x-t}^{x+t} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{t-x}^{t+x} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{x-t}^{2l-x-t} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{t-x}^{2l-t-x} \psi(\tau)\,d\tau, & (x,t) \in \Omega_4, \end{cases} \tag{16}$$

where ψ is an arbitrary function of the class $C^1([0,l])$.

Remark. Taking into account that Dirichlet type problem (1), (3), (4) for the string vibration inhomogeneous equation, as well as Dirichlet problem for Poisson equation $\Delta u = F$ is self-adjoint, then a necessary condition for the solvability of problem (1), (3), (4) in the case of homogeneous boundary conditions (3), (4) is the following equality

$$\int\limits_{\Omega} Fv\,dx\,dt = 0 \ \ \forall\, v \in K,$$

where the function v is the given by equality (16).

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Nonlinear Functional Integral Itô Equations: Existence and Uniqueness

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Nonlinear deterministic and stochastic integral equations of the Hammerstein type have a long history. These equations are known to play a major role in classical problems of physics and engineering. Due to the expansion of the scope of applications of integral equations, in particular, to problems in biology and mathematical economics, various generalizations of the Hammerstein equations are becoming increasingly popular in the literature.

We consider the following Hammerstein-type stochastic equation with singular and non-singular kernels and nonlinear Volterra operators:

$$x(t) = \kappa(t) + \sum_{i=1}^{m} \int_0^t K_i(t,s)(F_i x)(s)\,ds + \sum_{i=1}^{m}\sum_{j=1}^{m_i} \int_0^t K_{ij}(t,s)(G_{ij}x)(s)\,dB_i(s), \tag{1}$$

where $x(t)$, $\kappa(t)$ are random n-dimensional processes, B_i are jointly independent scalar Wiener processes, K_i, K_{ij} are deterministic Borel functions with values in the space of $n\times n$-matrices, and F_i and G_{ij} are Volterra operators ensuring the dependence of solutions of the equations on the prehistory. Here the first integral is the Lebesgue integral, and the second is the Itô integral. In most formulations below, equation (1) is assumed to be defined on a finite interval $[0,T]$, but in fact, all the results are also true for the semiaxis $t \geq 0$.

Equation (1) covers many important classes of stochastic fractional differential and integral equations. To see how a stochastically perturbed deterministic equation with fractional derivatives can be converted into (1), consider the deterministic equation

$$({}^C D^{\alpha}_{0+} x)(t) = f(t, x(t)) \quad (\alpha > 0),$$

with the fractional Caputo derivative, see e.g. the monograph [6]. If this equation is perturbed by the white noise $\dot{B}(t)$, then we obtain a formally written equation

$$({}^C D^{\alpha}_{0+} x)(t) = f(t, x(t)) + g(t, x(t))\dot{B}(t)$$

or

$$d^{\alpha} x(t) = f(t, x(t))\,dt + g(t, x(t))\,dB(t), \tag{2}$$

where d^α is the fractional Caputo differential. In this case, the transition from (2) to a well-defined integral equation (1) is based on the fractional integration formula

$$({}^C D^\alpha_{0+} x)(t) = f(t) \implies x(t) = \sum_{k=0}^{l-1} \frac{x^{(k)}(0)t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

where $\Gamma(\alpha) = \int\limits_0^\infty s^{\alpha-1} e^{-s}\,ds$ – the Gamma function, and $l = \alpha$ if $\alpha \in N$, and $l = [\alpha] + 1$ if $\alpha \notin N$. This formula allows us to move from the differential form (2) to the integral one:

$$x(t) = \sum_{k=0}^{l-1} \frac{x^{(k)}(0)t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x(s))\,dB(s),$$

and then a solution of equation (2) is by definition understood as a stochastic process $x(t)$ satisfying this integral equation.

Another example of (1) are equations with fractional Wiener processes describing a popular class of models primarily developed in connection with their applications in financial mathematics, see, for example, [3], as well as numerous references cited in this monograph. An example is an equation of the form

$$dx(t) = f(t,x(t))\,dt + g(t,x(t))\,dB^\beta(t), \tag{3}$$

where B^β is a fractional Wiener process with the Hurst parameter β $(0.5 < \beta < 1)$. Note that without loss of generality we can assume that B^β is written in the Riemann–Liouville form, since this form differs from the standard one by a progressively measurable stochastic process with absolutely continuous trajectories, which can therefore be included in the first term on the right-hand side of the equation (3). This observation makes it possible to write the equation (3) as an integral equation (1) using the well-known formula [3]

$$\int_0^t \xi(t)\,dB^\beta(t) = \frac{1}{\Gamma(\beta+1/2)} \int_0^t \xi(s)(t-s)^{\beta-1/2}\,dB(t).$$

Then equation (3) can be rewritten in the integral form

$$x(t) = x(0) + \int_0^t f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x(s))\,dB(s),$$

where $\alpha = \beta + 1/2$. By a solution of equation (3) we mean a stochastic process $x(t)$ satisfying this integral equation in order to avoid technical difficulties associated with integration over fractional Wiener process [3].

The third important class of stochastic equations included in the general form (1) are equations with multiple time scales

$$dx(t) = \sum_{i=1}^m f_i(t,x(t))\,(dt)^{\alpha_i} + g(t,x(t))\,dB(t) \ \ (0 < \alpha_i < 1), \tag{4}$$

which were introduced in [9]. Here $(dt)^{\alpha_i}$ are Jumarie-type differentials defining independent time scales $T_i(t) = t^{\alpha_i}$ (see [9] for a more detailed description of these time scales). The transition from

(4) to the integral equation (1) is based on the formula

$$\int_{t_0}^{t} \xi(t)(dt)^{\alpha} = \alpha \int_{t_0}^{t} \xi(s)(t-s)^{\alpha-1}\, dt,$$

developed in [9], which again gives a special case of the equation (1):

$$x(t) = x(0) + \sum_{i=1}^{m} \alpha_i \int_{0}^{t} (t-s)^{\alpha_1 - 1} f_i(s, x(s))\, ds + \int_{0}^{t} g(s, x(s))\, dB(s).$$

By combining all these special cases, one can also obtain various mixed integral equations (1) with singular kernels K_i, K_{ij} of the form $const(t-s)^{\alpha-1}$ $(0 < \alpha < 1)$, and some further examples can be found in Corollaries 1–6 below.

In what follows, we use the following constants that remain fixed:

- $n \in N$ is the dimension of the phase space of the equation, i.e. the size of the solution vector of the equation.
- $m, m_i \in N$.
- i is the index satisfying the conditions $1 \leq i \leq m$.
- j is the index satisfying the conditions $1 \leq j \leq m_i$.
- $T > 0$, $p \geq 2$, $q \geq 1$, $q_i \geq 1$, $q_{ij} \geq 1$, $\alpha_i > 0$, $\alpha_{ij} > 1/2$ – real numbers.

The following notations will also be used:

- $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$.
- $|\cdot|$ – fixed norm in $\mathbb{R}^n$ and $\|\cdot\|$ – matrix norm consistent with the norm $|\cdot|$.
- I_A – indicator (characteristic function) of the set A.
- $\text{Bor}(M)$ – σ-algebra of all Borel subsets of the metric space M.
- L_q^n – Lebesgue space of equivalence classes of n-dimensional functions on the interval $[0, T]$.
- $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$ is a stochastic basis, where Ω is the set of elementary events, $\mathcal{F}$ is the σ-algebra of events on Ω, $(\mathcal{F})_{t\geq 0}$ is a right-continuous non-decreasing flow of σ-subalgebras of $\mathcal{F}$, P is a probability measure on $\mathcal{F}$, and all σ-algebras are complete with respect to this measure.
- E is the mathematical expectation constructed with respect to the measure P.
- $B(t)$ $(t \in \mathbb{R}_+)$ – scalar standard Wiener process.
- $B_i(t)$ $(t \in \mathbb{R}_+)$ – scalar standard and jointly independent Wiener processes.
- k_p^n – linear space of n-dimensional $\mathcal{F}_0$-measurable random variables χ satisfying the condition $E|\chi|^p < \infty$; the norm in k_p^n is the p-th root of this variable.
- $\mathcal{D}_p^n$ is the linear normed space of all n-dimensional progressively measurable stochastic processes $x(\cdot)$ on the interval $[0, T]$ satisfying the condition $\sup\limits_{0\leq t\leq T} E|x(t)|^p < \infty$; the norm in $\mathcal{D}_p^n$ is the p-th root of this quantity.

Let J be the interval $[0,T]$ or the semiaxis $\mathbb{R}_+$. Recall that a stochastic process $x(t,\omega)$ ($t\in J$, $\omega\in\Omega$) whose restriction to the set $[0,v]\times\Omega$ is $\mathrm{Bor}([0,v])\otimes\mathcal{F}_v$-measurable for any $v\in J$, is called progressively measurable (with respect to the stochastic basis $\mathcal{B}$).

Definition 1. Let a Volterra operator V map stochastic processes from $\mathcal{D}_p^n$ to progressively measurable processes and let there exist a linear bounded operator $Q:\mathcal{D}_p^n\to\mathcal{D}_p^n$ and a measurable deterministic function $\Psi(t)\geq 0$, $t\in[0,T]$, such that the inequality

$$\big|(Vx)(t)-(Vy)(t)\big|\leq\Psi(t)|(Q(x-y))(t)|$$

for all $x,y\in\mathcal{D}_p^n$ and μ-almost all $0\leq t\leq T$. Then we will say that the operator V satisfies *the generalized Lipschitz condition with the operator Q and the function Ψ.*

The theorem below describes conditions of existence and uniqueness of the main equation (1).

Theorem 1. *Let the following conditions be satisfied for the equation* (1) *on the interval* $[0,T]$*:*

(1) $\kappa\in\mathcal{D}_p^n$.

(2) *The operators F_i,G_{ij} satisfy the generalized Lipschitz conditions with linear bounded operators $Q_i,Q_{ij}:\mathcal{D}_p^n\to\mathcal{D}_p^n$ and functions $\Psi_i\in L^1_{q_i}$, $\Psi_{ij}\in L^1_{2q_{ij}}$, respectively.*

(3) $F_i\widehat{0}\in\mathcal{D}_p^n$, $G_{ij}\widehat{0}\in\mathcal{D}_p^n$, *where $\widehat{0}$ is the zero element of $\mathcal{D}_p^n$.*

(4) $C_i:=\sup\limits_{0\leq t\leq T}\int\limits_0^t\|K_i(t,s)\|^{\frac{q_i}{q_i-1}}\,ds<\infty,\quad C_{ij}:=\sup\limits_{0\leq t\leq T}\int\limits_0^t\|K_{ij}(t,s)\|^{\frac{2q_{ij}}{q_{ij}-1}}\,ds<\infty.$

Then this equation has a unique solution, belonging to the space $\mathcal{D}_p^n$.

In what follows we apply Theorem 1 to several specific classes of stochastic fractional equations. The interval on which the existence of solutions is proved is always assumed to be finite and equal to $[0,T]$, and the solution on this interval belongs to the space $\mathcal{D}_p^n$, but all the results below remain valid for the semi-axis with obvious changes in the formulations.

Corollary 1. *Let in the equation* (1) $K_i(t,s)=(t-s)^{\alpha_i-1}$, $K_{ij}(t,s)=(t-s)^{\alpha_{ij}-1}$, *and the operators F_i, G_{ij} satisfy conditions* (2), (3) *of Theorem 1, where*

$$q_i>\max\{\alpha_i^{-1};1\},\quad q_{ij}>\max\{(2\alpha_{ij}-1)^{-1};1\}.$$

Then for any $\kappa\in\mathcal{D}_p^n$ the equation (1) *has a unique solution.*

Corollary 1 is a far-reaching generalization of the corresponding results on fractional equations with Caputo derivatives from [8] (for the finite-dimensional case) and [4]. In particular, it includes random right-hand sides and random delays.

The two corollaries below deal with the initial value problem for equations with distributed and random delays, respectively. Both types of initial value problems are special cases of the equation (1), since, as shown below, they are reduced to this equation using the technique described in the monograph [2].

Consider the equation

$$\begin{aligned}x(t)=x(0)&+\sum_{i=1}^m\int\limits_0^t(t-s)^{\alpha_i-1}f_i(s,(H_ix)(s))\,ds\\&+\sum_{i=1}^m\sum_{j=1}^{m_i}\int\limits_0^t(t-s)^{\alpha_{ij}-1}g_{ij}(s,(H_{ij}x)(s))\,dB_i(s)\ \ (t\in[0,T]),\end{aligned}\tag{5}$$

where $f_i(t,\omega,v)$, $g_{ij}(t,\omega,v)$ – n-dimensional random functions that for each $v \in \mathbb{R}^{nl}$ are progressively measurable in variables $(t,\omega) \in [0,T] \times \Omega$, and for $P \otimes \mu$-almost all (t,ω) are continuous in v. The initial condition for (5) is defined by

$$x(s) = \varphi(s) \ \ (s \in \mathbb{R}_-), \tag{6}$$

where φ is a given stochastic process on $\mathbb{R}_-$. By a solution of the problem (5), (6) $x(t)$ $(t \le T)$ we mean an n-dimensional stochastic process whose restriction to the interval $[0,T]$ belongs to the space $\mathcal{D}_p^n$ and which satisfies the initial condition (6).

Let us start with an equation that includes distributed delay.

Corollary 2. *Let the following conditions be satisfied:*

(1) φ – $(\mathrm{Bor}(\mathbb{R}_-) \otimes \mathcal{F}_0)$*-measurable n-dimensional stochastic process.*

(2) *There exist measurable non-negative functions* $\Psi_i(t)$, $\Psi_{ij}(t)$ $(t \in [0,T])$, $\Psi_i \in L^1_{q_i}$, $\Psi_{ij} \in L^1_{2q_{ij}}$, *where* $q_i > \max\{\alpha_i^{-1}; 1\}$, $q_{ij} > \max\{(2\alpha_{ij}-1)^{-1}; 1\}$, *for which* $P \otimes \mu$*-the inequalities*

$$|f_i(t,u) - f_i(t,v))| \le \Psi_i(t)|u-v| \ \ \textit{and} \ \ |g_{ij}(t,u) - g_{ij}(t,v))| \le \Psi_{ij}(t)|u-v|$$

for any $u, v \in \mathbb{R}^{nl}$ *and* $t \in [0,T]$.

(3)

$$(H_i z)(t) = \int\limits_{-\infty}^{t} d_s\mathcal{R}_i(t,s)z(s), \quad (H_{ij} z)(t) = \int\limits_{-\infty}^{t} d_s\mathcal{R}_{ij}(t,s)z(s),$$

where Borel functions $\mathcal{R}_i$, $\mathcal{R}_{ij}$, *defined on the set* $[0,T] \times (-\infty,t]$ *and taking values in the space of* $(nl) \times n$*-matrices, satisfy conditions*

$$\sup_{0\le t\le T} \mathrm{Var}_0^t\, \mathcal{R}_i(t,\,\cdot\,) < \infty, \quad \sup_{0\le t\le T} \mathrm{Var}_0^t\, \mathcal{R}_{ij}(t,\,\cdot\,) < \infty,$$

$$\sup_{0\le t\le T} E\left|f_i\left(t, \int\limits_{-\infty}^{0} d_s\mathcal{R}_i(t,s)\varphi(s)\right)\right|^p < \infty, \quad \sup_{0\le t\le T} E\left|g_{ij}\left(t, \int\limits_{-\infty}^{0} d_s\mathcal{R}_{ij}(t,s)\varphi(s)\right)\right|^p < \infty.$$

Then for any $x(0) \in k_p^n$ *the problem* (5), (6) *has a unique solution.*

The following corollary considers the initial value problem (5), (6) with random delays.

Corollary 3. *Let conditions* (1), (2) *of Corollary 2 be satisfied, and let condition* (3) *be replaced by condition*

3A. $(H_i z)(t) = (x(h_i^1(t)), \dots, x(h_i^l(t)))$, $(H_{ij} z)(t) = (x(h_{ij}^1(t)), \dots, x(h_{ij}^l(t)))$, *where scalar stochastic processes* $h_i^k(t)$, $h_{ij}^k(t)$ $(k = 1, \dots, l)$ *satisfy the conditions* $h(t) \le t$ *a.s.* $0 \le t \le T$, $h^{-1}(B) \in \mathrm{Bor}([0,T]) \otimes \mathcal{F}_v$ *for any* $v \in [0,T]$ *and any Borel set* $B \subset (-\infty, v]$ *and*

$$\sup_{0\le t\le T} E\left|f_i\left(t, \varphi(h_i^1(t))I_{\{h_i^1(t)<0\}}, \dots, \varphi(h_i^l(t))I_{\{h_i^l(t)<0\}}\right)\right|^p < \infty,$$

$$\sup_{0\le t\le T} E\left|g_{ij}\left(t, \varphi(h_{ij}^1(t))I_{\{h_{ij}^1(t)<0\}}, \dots, \varphi(h_{ij}^l(t))I_{\{h_{ij}^l(t)<0\}}\right)\right|^p < \infty.$$

Then for any $x(0)\in k_p^n$ equation with random delays

$$x(t)=x(0)+\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}f_i\big(s,x(h_i^1(t)),\dots,x(h_i^l(t))\big)\,ds$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{m_i}\frac{1}{\Gamma(\alpha_{ij})}\int_0^t(t-s)^{\alpha_{ij}-1}g_{ij}\big(s,x(h_{ij}^1(t)),\dots,x(h_{ij}^l(t))\big)\,dB_i(s)\ \ (t\in[0,T])$$

has only one solution satisfying the equality (6)*.*

Consider now equations with arbitrary homogeneous singular kernels.

Let

$$x(t)=\kappa(t)+\sum_{i=1}^{m}\int_0^t K_i(t-s)(F_ix)(s)\,ds+\sum_{i=1}^{m}\sum_{j=1}^{m_i}\int_0^t K_{ij}(t-s)(G_{ij}x)(s)\,dB_i(s)\ \ (t\in[0,T]).\quad (7)$$

Corollary 4. *Let conditions* (1)–(3) *of Theorem 1 be satisfied, and condition* (4) *be replaced by*

4A. *The columns of the matrices K_i and K_{ij} belong to the spaces $L^n_{q_i(q_i-1)^{-1}}$ and $L^n_{2q_{ij}(q_{ij}-1)^{-1}}$, respectively.*

Then equation (7) *has a unique solution belonging to the space $\mathcal{D}_p^n$.*

Corollary 4 generalizes the main result of the paper [4].

Consider now equations including generalized fractional derivatives. They are represented by (1) on the interval $[0,T]$, where

$$K_i(t,s)=\psi_i'(s)(\psi_i(t)-\psi_i(s))^{\alpha_i-1}\ \text{ and }\ K_{ij}(t,s)=\psi_{ij}'(s)(\psi_{ij}(t)-\psi_{ij}(s))^{\alpha_{ij}-1},\quad (8)$$

the functions ψ_i and ψ_{ij} have continuous derivatives on $[0,T]$, and $\psi_i'(t)>0$, $\psi_{ij}'(t)>0$, $t\in[0,T]$. Obviously, this equation is a stochastic generalization of equations with Caputo derivatives.

Corollary 5. *Let the operators F_i, G_{ij} satisfy conditions* (2), (3) *of Theorem 1, where*

$$q_i>\max\{\alpha_i^{-1};1\},\quad q_{ij}>\max\big\{(2\alpha_{ij}-1)^{-1};1\big\}.$$

Then for any $\kappa\in\mathcal{D}_p^n$ the equation (1)*, where K_i and K_{ij} are defined by the formulas* (8)*, has a unique solution.*

Corollary 5 generalizes the existence and uniqueness theorem from [1].

Finally, we consider equations including multifractional Wiener processes described by (1) on $[0,T]$, where

$$K_i(t,s)=\frac{1}{\Gamma(\theta_i(t))}\,(t-s)^{\theta_i(t)-1}\ \text{ and }\ K_{ij}(t,s)=c_{ij}(\theta_{ij}(t))(t-s)^{\theta_{ij}(t)-1/2}.\quad (9)$$

Corollary 6. *Let $c_{ij}(u)$ $(u>0)$, $\theta_i(t)$, $\theta_{ij}(t)$ $(t\in[0,T])$ be Borel, bounded scalar functions, where $\theta_i(t)\ge\alpha_i$, $\theta_{ij}(t)\ge\delta_{ij}>0$ for all $t\in[0,T]$. Let, further, the operators F_i, G_{ij} satisfy conditions* (2), (3) *of Theorem 1, where*

$$q_i>\max\{\alpha_i^{-1};1\},\quad q_{ij}>\max\big\{(2\delta_{ij})^{-1};1\big\}.$$

Then for any $\kappa\in\mathcal{D}_p^n$ the equation (1)*, where K_i and K_{ij} are defined by the formulas* (9)*, has a unique solution.*

Such equations were considered in [5]. Corollary 6 does not formally generalize the result on the existence of weak solutions for the equations offered in [5], but it does extend the existence and uniqueness theorem to equations of a much more general form.

The proofs of the above results can be found in the paper [7].

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An Efficient Numerical Method For Solving Problem for Impulsive Differential Equations with Loadings Subject to Multipoint Conditions

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1 Introduction

Impulsive differential equations are a significant area of mathematical research, driven by their ability to model real-world phenomena exhibiting sudden changes at specific moments. Such systems arise in various fields, including physics, biology, engineering, and economics, where abrupt transitions, discontinuities, or shocks are inherent. These equations offer a robust framework to capture behaviors like population explosions, mechanical shocks, or instantaneous changes in electrical circuits [1, 6, 8].

The concept of "loadings" in impulsive differential equations introduces an additional layer of complexity and applicability. Loadings can represent external influences or internal accumulations that act on the system during the impulse events [7]. This perspective extends the classical theory, enabling more comprehensive modeling of systems with cumulative or distributed effects accompanying the impulses.

The study of impulsive differential equations with loadings bridges the gap between theoretical advancements and practical applications. It explores existence, uniqueness, stability, and qualitative behavior of solutions while accounting for the dynamic interplay between impulses and loadings. Such investigations are critical in optimizing real-world systems, predicting outcomes, and controlling processes influenced by sudden changes and distributed forces [2, 5].

This paper focuses on developing numerical method for solving problem for impulsive differential equations with loadings subject to multipoint conditions. The objective is to provide numerical algorithm for solving problem for impulsive differential equations with loadings subject to multipoint conditions. By doing so, it contributes to the growing body of knowledge that supports both the theoretical understanding and practical use of impulsive systems in diverse scientific and engineering domains.

2 Setting of the problem and the main results

In this paper, by means of the Dzhumabaev parameterization method [3], we investigate the following problem for impulsive differential equations with loadings subject to multipoint conditions

$$\frac{dx}{dt} = A_0(t)x + \sum_{i=1}^{m} A_i(t) \lim_{t\to\theta_i+0} x(t) + f(t), \ \ x \in \mathbb{R}^n, \ \ t \in (0,T), \tag{2.1}$$

$$B_i \lim_{t\to\theta_i-0} x(t) - C_i \lim_{t\to\theta_i+0} x(t)$$

$$= \varphi_i + \sum_{k=1}^{i-1} D_k \lim_{t\to\theta_k-0} x(t) + \sum_{k=1}^{i-1} E_k \lim_{t\to\theta_k+0} x(t), \quad \varphi_i \in \mathbb{R}^n, \quad i = \overline{1,m}, \quad (2.2)$$

$$G_0 x(0) + G_1 \lim_{t\to\theta_1+0} x(t) + G_2 x(T) = d, \quad d \in \mathbb{R}^n. \quad (2.3)$$

Here $(n\times n)$-matrices $A_i(t)$ $(i = \overline{0,m})$ and n-vector-function $f(t)$ are piecewise continuous on $[0,T]$ with possible discontinuities of the first kind at the points $t = \theta_i$ $(i = \overline{1,m})$. B_i, C_i $(i = \overline{1,m})$, G_j $(j = \overline{0,2})$, D_k and E_k $(k = \overline{1,m-1})$ are constant $(n\times n)$-matrices, and φ_i $(i = \overline{1,m})$ and d are constant n vectors, $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = T$.

Let $PC([0,T],\theta,\mathbb{R}^n)$ denote the space of piecewise continuous functions $x(t)$ with the norm

$$\|x\|_1 = \max_{i=\overline{0,m}} \sup_{t\in[\theta_i,\theta_{i+1})} \|x(t)\|.$$

A solution to problem (2.1)–(2.3) is a piecewise continuously differentiable vector function $x(t)$ on $[0,T]$, which satisfies the system of the differential equations with loadings (2.1) on $[0,T]$ except the points $t = \theta_i$ $(i = \overline{1,m})$, the conditions of impulse effects at the fixed time points (2.2) and the condition (2.3).

Definition. Problem (2.1)–(2.3) is called uniquely solvable, if for any function $f(t) \in PC([0,T],\theta,\mathbb{R}^n)$ and vectors $d \in \mathbb{R}^n$, $\varphi_i \in \mathbb{R}^n$ $(i = \overline{1,m})$, it has a unique solution.

In this paper, we use the approach offered in [4] to solve the boundary value problem for impulsive differential equations with loadings subject to the multipoint conditions (2.1)–(2.3).

The interval $[0,T]$ is divided into subintervals by points:

$$[0,T) = \bigcup_{r=1}^{m+1} [\theta_{r-1},\theta_r).$$

Define the space $C([0,T],\theta,\mathbb{R}^{n(m+1)})$ of systems functions $x[t] = (x_1(t), x_2(t), \ldots, x_{m+1}(t))$, where $x_r : [\theta_{r-1},\theta_r) \to \mathbb{R}^n$ are continuous on $[\theta_{r-1},\theta_r)$ and have finite left-sided limits $\lim_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1,m+1}$, with the norm

$$\|x[\cdot]\|_2 = \max_{r=\overline{1,m+1}} \sup_{t\in[\theta_{r-1},\theta_r)} \|x_r(t)\|.$$

Denote by $x_r(t)$ the restriction of the function $x(t)$ to the $r-$th interval $[\theta_{r-1},\theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, and introducing the parameters

$$\lambda_r = \lim_{t\to\theta_{r-1}+0} x_r(t), \quad r = \overline{1,m+1},$$

and performing a replacement of the function $u_r(t) = x_r(t) - \lambda_r$ on each interval $[\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, we obtain the boundary value problem with parameters λ_r, $r = \overline{1,m+1}$:

$$\frac{du_r}{dt} = A_0(t)(u_r + \lambda_r) + \sum_{i=1}^{m} A_i(t)\lambda_{i+1} + f(t), \quad t \in [\theta_{r-1},\theta_r), \quad r = \overline{1,m+1}, \quad (2.4)$$

$$u_r(\theta_{r-1}) = 0, \quad r = \overline{1,m+1}, \quad (2.5)$$

$$B_i \lim_{t\to\theta_i-0} u_i(t) + B_i\lambda_i - C_i\lambda_{i+1} = \varphi_i + \sum_{k=1}^{i-1} D_k \lim_{t\to\theta_k-0} \big[u_k(t) + \lambda_k\big] + \sum_{k=1}^{i-1} E_k\lambda_{k+1}, \quad i = \overline{1,m}, \quad (2.6)$$

$$G_0\lambda_1 + G_1\lambda_2 + G_2\lambda_{m+1} + G_2 \lim_{t\to T-0} u_{m+1}(t) = d. \tag{2.7}$$

A solution to problem (2.4)–(2.7) is a pair $(\lambda^*, u^*[t])$, with elements

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{m+1}^*) \in \mathbb{R}^{n(m+1)},$$
$$u^*[t] = \big(u_1^*(t), u_2^*(t), \dots, u_{m+1}^*(t)\big) \in C([0,T], \theta, \mathbb{R}^{n(m+1)}),$$

where $u_r^*(t)$ are continuously differentiable on $[\theta_{r-1}, \theta_r)$, $r = \overline{1, m+1}$, and satisfying the system of ordinary differential equations (2.4) and conditions (2.5)–(2.7) at $\lambda_r = \lambda_r^*$, $j = \overline{1, m+1}$.

Problem (2.1)–(2.3) is equivalent to problem (2.4)–(2.7). If the function $x^*(t)$ is a solution to problem (2.1)–(2.3), then the triple $(\lambda^*, u^*[t])$, where

$$\lambda^* = \big(x^*(\theta_0), x^*(\theta_1), \dots, x^*(\theta_m)\big)$$

and

$$u^*[t] = \big(x^*(t) - x^*(\theta_0), x^*(t) - x^*(\theta_1), \dots, x^*(t) - x^*(\theta_m)\big),$$

is a solution to problem (2.4)–(2.7). Conversely, if the triple $(\widetilde{\lambda}, \widetilde{u}[t])$, with elements

$$\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_{m+1}), \quad \widetilde{u}[t] = \big(\widetilde{u}_1(t), \widetilde{u}_2(t), \dots, \widetilde{u}_{m+1}(t)\big),$$

is a solution to problem (2.4)–(2.7), then the function $\widetilde{x}(t)$ defined by the equalities

$$\widetilde{x}(t) = \widetilde{u}_r(t) + \widetilde{\lambda}_r, \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, m+1}$$

and

$$\widetilde{x}(T) = \widetilde{\lambda}_{m+1} + \lim_{t\to T-0} \widetilde{u}_{m+1}(t),$$

will be the solution of the original problem (2.1)–(2.3).

Let $\Phi_r(t)$ be a fundamental matrix to the differential equation

$$\frac{dx}{dt} = A(t)x \ \text{ on } [\theta_{r-1}, \theta_r], \ \ r = \overline{1, m+1}.$$

Then, the solution to the Cauchy problem (2.5), (2.6) can be written as follows

$$u_r(t) = \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) A_0(\tau)\, d\tau\, \lambda_r + \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) \sum_{i=1}^{m} A_i(\tau)\, d\tau\, \lambda_{i+1}$$
$$+ \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) f(\tau)\, d\tau, \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, m+1}. \tag{2.8}$$

Substituting the right-hand side of (2.8) into the impulse conditions (2.6) and condition (2.7) at the corresponding limit values, we obtain the following system of linear algebraic equations with respect to parameters λ_r, $r = \overline{1, m+1}$:

$$B_i \Phi_i(\theta_i) \int\limits_{\theta_{i-1}}^{\theta_i} \Phi_i^{-1}(\tau) \Big\{ A_0(\tau)\lambda_i + \sum_{j=1}^{m} A_j(\tau)\lambda_{j+1} \Big\}\, d\tau + B_i\lambda_i - C_i\lambda_{i+1}$$

$$
-\sum_{k=1}^{i-1} D_k\lambda_k - \sum_{k=1}^{i-1} E_k\lambda_{k+1} - \sum_{k=1}^{i-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)\Big\{A_0(\tau)\lambda_k + \sum_{j=1}^{m} A_j(\tau)\lambda_{j+1}\Big\}\,d\tau
$$

$$
= \varphi_i - B_i\Phi_i(\theta_i)\int\limits_{\theta_{i-1}}^{\theta_i} \Phi_i^{-1}(\tau)f(\tau)\,d\tau + \sum_{k=1}^{i-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)f(\tau)\,d\tau, \quad i=\overline{1,m}, \qquad (2.9)
$$

$$
G_0\lambda_1 + G_1\lambda_2 + G_2\Big[I + \Phi_{m+1}(T)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)A_0(\tau)\,d\tau\Big]\lambda_{m+1}
$$

$$
+ G_2\Phi_{m+1}(T)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)\sum_{j=1}^{m} A_j(\tau)\lambda_{j+1}\,d\tau = d - G_2\Phi_{m+1}(t)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)f(\tau)\,d\tau. \qquad (2.10)
$$

We denote the matrix corresponding to the left side of the system of equations (2.9), (2.10) by $Q_*(\theta)$ and write the system in the form

$$
Q_*(\theta)\lambda = F_*(\theta), \quad \lambda \in \mathbb{R}^{n(m+1)}, \qquad (2.11)
$$

where

$$
F_*(\theta) = \begin{pmatrix}
\varphi_1 - B_1\Phi_1(\theta_1)\int\limits_{\theta_0}^{\theta_1} \Phi_1^{-1}(\tau)f(\tau)\,d\tau \\
\varphi_2 - B_2\Phi_2(\theta_2)\int\limits_{\theta_1}^{\theta_2} \Phi_2^{-1}(\tau)f(\tau)\,d\tau + D_1\Phi_1(\theta_1)\int\limits_{\theta_0}^{\theta_1} \Phi_1^{-1}(\tau)f(\tau)\,d\tau \\
\vdots \\
\varphi_m - B_m\Phi_m(\theta_m)\int\limits_{\theta_{m-1}}^{\theta_m} \Phi_m^{-1}(\tau)f(\tau)\,d\tau + \sum_{k=1}^{m-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)f(\tau)\,d\tau \\
d - \Phi_{m+1}(t)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)f(\tau)\,d\tau
\end{pmatrix}.
$$

Theorem. *Let the matrix $Q_*(\theta) : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n(m+1)}$ be invertible. Then the boundary value problem (2.1)–(2.3) has a unique solution $x^*(t)$ for any $f(t) \in PC([0,T],\theta,\mathbb{R}^n)$, $d \in \mathbb{R}^n$, and $\varphi_i \in \mathbb{R}^n$, $i = \overline{1,m}$.*

Solvability of the boundary value problem (2.1)–(2.3) is equivalent to the solvability of system (2.11). The solution to system (2.11) is a vector λ^*, consisting of the values of solutions to problem (2.1)–(2.3) at the initial points of subintervals, i.e., $\lambda_r^* = x^*(\theta_{r-1})$, $r = \overline{1,m+1}$.

If $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{m+1}^*)$ solution to system (2.11) is known, then a solution to problem (2.1)–(2.3) is determined by the equalities:

$$
x^*(t) = \Phi_r(t)\Phi_r^{-1}(\theta_{r-1})\lambda_r^* + \Phi_r(t)\int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau)\sum_{j=1}^{m} A_j(\tau)\,d\tau\,\lambda_{j+1}^*
$$

$$+ \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) f(\tau)\, d\tau, \;\; t \in [\theta_{r-1}, \theta_r), \;\; r = \overline{1, m+1}, \tag{2.12}$$

$$x^*(T) = \Phi_{m+1}(t)\Phi_{m+1}^{-1}(\theta_m)\lambda_{m+1}^* + \Phi_{m+1}(t) \int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau) \sum_{j=1}^{m} A_j(\tau)\, d\tau\, \lambda_{j+1}^*$$
$$+ \Phi_{m+1}(t) \int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau) f(\tau)\, d\tau. \tag{2.13}$$

Expressions (2.12) and (2.13) give the analytical form of solution to problem (2.1)–(2.3).

We offer the following algorithm for numerical solving of linear boundary value problem for impulsive differential equations with loadings subject to the multipoint conditions (2.1)–(2.3).

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = T$. Divide each rth interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, m+1}$, into N_r parts.

2. Solve the following Cauchy problem for ordinary differential equations

$$\frac{dz}{dt} = A_0(t)z + A_j(t), \;\; z(\theta_{r-1}) = 0, \;\; t \in [\theta_{r-1}, \theta_r], \;\; j = \overline{0, m}, \;\; r = \overline{1, m+1},$$
$$\frac{dz}{dt} = A_0(t)z + f(t), \;\; z(\theta_{r-1}) = 0, \;\; t \in [\theta_{r-1}, \theta_r], \;\; r = \overline{1, m+1}.$$

3. Construct the system of linear algebraic equations in parameters

$$Q_*^{\widetilde{h}}(\theta)\lambda = F_*^{\widetilde{h}}(\theta), \;\; \lambda \in \mathbb{R}^{n(m+1)},$$

and find its solution $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, \dots, \lambda_{m+1}^{\widetilde{h}})$ are the values of an approximate solution to problem (2.1)–(2.3) at the left end-points of the subintervals: $x^{\widetilde{h}_r}(\theta_{r-1}) = \lambda_r^{\widetilde{h}}$, $r = \overline{1, m+1}$.

4. To define the values of an approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, m+1}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{m} A_j(t)\lambda_{j+1}^{\widetilde{h}} + f(t), \quad x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \;\; t \in [\theta_{r-1}, \theta_r], \;\; r = \overline{1, m+1}.$$

Thus, this algorithm allows us to find the numerical solution to problem (2.1)–(2.3).

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Approximate Solution of the Optimal Control Problem for a Parabolic Differential Inclusion with Fast-Oscillating Coefficients on Infinite Interval

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1 Introduction

The averaging method is a powerful tool for analyzing and solving optimal control problems, in particular for systems described by differential equations and inclusions with rapidly oscillating coefficients. It was originally developed and rigorously justified by Krylov and Bogolyubov for the approximate analysis of oscillating processes in non-linear mechanics, and then further refined for the control-related problems, see, e.g. a monograph by Plotnikov [10]. Motivated by the modern control engineering applications, the averaging method has been recenlty applied to the solution of optimal control problems for linear by control systems with rapidly oscillating coefficients on a finite interval [9], and on the semi-axis [8]. The approximate solutions of the optimal control problems for non-linear systems of differential inclusions with fast-oscillating parameters were investigated in [11] and [3], for the cases of a finite interval and on the semi-axis, respectively. The optimal control problem on the semi-axis for the Poisson equation with nonlocal boundary conditions was studied in [4]. Further applications of the averaging method for parabolic systems with fast-oscillating coefficients were considered in [5–7].

In the present paper, we use the averaging method for the investigation of the optimal control problem for nonlinear parabolic differential inclusion with fast-oscillating (w.r.t. time variable) coefficients on an infinite time interval. With this, we prove that the optimal control for the problem with averaged coefficients can be considered as "approximately" optimal for the original system.

2 Setting of the problem and the main results

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. In a cylinder $Q=(0,+\infty)\times\Omega$, we consider an initial boundary-value problem for a parabolic inclusion

$$\begin{cases}\dfrac{\partial y}{\partial t}\in Ay+f\Big(\dfrac{t}{\varepsilon},y(t,x)\Big)+g(y)u, \quad (t,x)\in Q,\\ y\big|_{\partial\Omega}=0,\\ y\big|_{t=0}=y_0(x).\end{cases} \tag{2.1}$$

Here $\varepsilon>0$ is a small parameter, $f:\mathbb{R}_+\times\mathbb{R}_+\to\operatorname{conv}(\mathbb{R})$ is a given multivalued mapping, $g:\mathbb{R}\to\mathbb{R}$, $q:\Omega\times\mathbb{R}\to\mathbb{R}$ are given real-valued mappings, A is an elliptic operator which can be defined by the rule:

$$Ay=\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(\frac{\partial y}{\partial x_i}\Big),$$

y is an unknown state function, u is an unknown control function, which are determined by requirements

$$u \in U \subseteq L^2(Q), \tag{2.2}$$

$$J(y,u) = \int_Q e^{-\gamma t} q(x, y(t,x))\, dt\, dx + \alpha \int_Q u^2(t,x)\, dt\, dx \longrightarrow \inf, \tag{2.3}$$

where γ, α are positive constants.

We consider the problem of finding an approximate solution of (2.1)–(2.3) by transition to the averaged coefficients. For this purpose, we assume that there exists $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that uniformly w.r.t. $y \in \mathbb{R}$ there exists

$$\operatorname{dist}_H \Big(\overline{f}(y), \frac{1}{T} \int_0^T f(s,y)\, ds \Big) \longrightarrow 0, \;\; T \to \infty, \tag{2.4}$$

where $\operatorname{dist}_H(A,B)$ is Hausdorff metric between sets A and B, and integral of multivalued map we consider in the sense of Aumann [1].

Let us consider the following optimal control problem

$$\begin{cases} \dfrac{\partial y}{\partial t} \in Ay + \overline{f}(y) + g(y)u, & (t,x) \in Q, \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0(x), \end{cases} \tag{2.5}$$

$$u \in U \subseteq L^2(Q), \tag{2.6}$$

$$J(y,u) = \int_Q e^{-\gamma t} q(x, y(t,x))\, dt\, dx + \alpha \int_Q u^2(t,x)\, dt\, dx \longrightarrow \inf\,. \tag{2.7}$$

Under the natural assumptions on f, g, u, q we prove that the optimal control problem (2.1)–(2.3) has a solution $\{\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}\}$, i.e. for every $u \in U$ and for any solution y^{ε} of (2.1) with control u we have

$$J(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \le J(y^{\varepsilon}, u).$$

Note that we can apply similar suggestions to problem (2.5)–(2.7).

Assume that $\{\overline{y}, \overline{u}\}$ is a solution of (2.5)–(2.7). The main goal of the paper is to prove the convergence

$$J(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \longrightarrow J(\overline{y}, \overline{u}), \;\; \varepsilon \to 0.$$

We suggest that the next assumptions for parameters of problem (2.1)–(2.3) are fulfilled.

Condition 2.1. Multi-valued function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \operatorname{conv}(\mathbb{R})$ is continuous and there exist $C, C_1 > 0$ such that

$$\forall\, t \ge 0 \;\; \forall\, y \in \mathbb{R} \quad \|f(t,y)\|_+ := \sup_{\xi \in f(t,x)} \|\xi\|_{\mathbb{R}} \le C + C_1 \|y\|_{\mathbb{R}},$$

where $\|\xi\|_{\mathbb{R}}$ denotes the Euclidian norm of $\xi \in \mathbb{R}^n$.

Condition 2.2. Function $g : \mathbb{R} \to \mathbb{R}$ is continuous function and there exists $C_2 > 0$ such that

$$\forall\, y \in \mathbb{R} \quad \|g(y)\|_{\mathbb{R}} \le C_2.$$

Condition 2.3. Function $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $C_3 > 0$ and functions $K_1 \in L^2(\Omega)$, $K_2 \in L^1(\Omega)$ such that

$$\|q(x,\xi)\|_{\mathbb{R}} \le C_3\|\xi\|_{\mathbb{R}}^2 + K_1(\Omega), \quad q(x,\xi) \le K_2(x).$$

Condition 2.4. $U \subseteq L_2(Q)$ is closed and convex, $0 \in U$.

Condition 2.5. $\gamma > 2C_1^2 + 1 + C_2$.

Condition 2.6. Uniformly w.r.t. $y \in \mathbb{R}$ there exists the limit (2.4).

For $u \in U$ and $y_0 \in L^2(\Omega)$ we understand solution of (2.1) as a mild solution on every finite time interval, i.e. y is a solution of (2.1) if $y \in L^2_{loc}(0,+\infty; H_0^1(\Omega)) \cap L^\infty_{loc}(0,+\infty; L^2(\Omega))$ such that $\forall\, T > 0$, $\forall\, \varphi \in H_0^1(\Omega)$, $\forall\, \eta \in C_0^\infty(0,T)$ the following equality holds:

$$-\int_0^T (y,\varphi)_H \cdot \eta'\,dt + \int_0^T (\nabla y, \nabla\varphi)_H \cdot \eta\,dt = \int_0^T (f(t),\varphi)_H \cdot \eta\,dt + \int_0^T (g(y)u,\varphi)_H \cdot \eta\,dt, \quad f(t) \in f\Big(\frac{t}{\varepsilon}, y\Big)$$

and $f \in L^2_{loc}(0,+\infty; L^2(\Omega))$.

Here and after we denote by $\|\cdot\|_H$ and $(\cdot\,,\cdot)_H$ the classical norm and scalar product in $H = L^2(\Omega)$, by $\|\cdot\|_V$ the classical norm in $V := H_0^1(\Omega)$, by V^* the dual space to V.

Note that due to Conditions 2.1, 2.2 and properties of the operator A for y from definition of mild solution we have

$$\frac{\partial y}{\partial t} \in L^2_{loc}(0,+\infty; V^*).$$

In the sequel we denote by $\mathcal{F}^\varepsilon$ (or $\overline{\mathcal{F}}$) a set of all pairs $\{y,u\}$, where y is a solution of (2.1) (or (2.5)) with control u.

The following Lemma gives us a result on solvability of the optimal control problem (2.1)–(2.3).

Lemma. *Under Conditions 2.1–2.5 for every $\varepsilon > 0$ problem* (2.1)–(2.3) *has a solution $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$, that is*

$$J(\overline{y}^\varepsilon, \overline{u}^\varepsilon) \le J(y,u) \quad \forall\,\{y,u\} \in \mathcal{F}^\varepsilon.$$

Note that the existence of a solution $\{\overline{y}, \overline{u}\}$ of (2.5)–(2.7) can be proved following similar arguments to the proof of the existence of $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$ for problem (2.1)–(2.3).

Theorem. *Suppose that Conditions 2.1-2.6 hold and, moreover, problem* (2.5) *has a unique solution for every $u \in U$.*

We assume additionally that $\forall\,\eta > 0$ $\exists\,\delta > 0$ $\forall\,t \ge 0$ $\forall\,y,z \in \mathbb{R}$

$$\|y - z\|_{\mathbb{R}} < \delta \implies \operatorname{dist}(f(t,y), f(t,z)) < \eta.$$

Let $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$ be a solution of (2.1)–(2.3)*. Then*

$$J(\overline{y}^\varepsilon, \overline{u}^\varepsilon) \longrightarrow J(\overline{y}, \overline{u}), \quad \varepsilon \to 0,$$

and up to subsequence

$$\overline{y}^\varepsilon \to \overline{y} \ \text{ in } L^2(0,+\infty; H),$$
$$\overline{u}^\varepsilon \to \overline{u} \ \text{ weakly in } L^2(0,+\infty; H),$$

where $\{\overline{y}, \overline{u}\}$ is a solution of (2.5)–(2.7).

These results are substantiated in [2].

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Antiperiodic in Time Boundary Value Problem for One Class of Nonlinear High-Order Partial Differential Equations

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In the plane of variables x and t consider a nonlinear high−order partial differential equation of the form

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^{4k} u}{\partial x^{4k}} + f(u) = F, \tag{1}$$

where f, F are given, while u is an unknown functions, k is a natural number.

For the equation (1) we consider the following antiperiodic in time problem: find in the domain $D_T:\ 0<x<l,\ 0<t<T$ a solution $u=u(x,t)$ of the equation (1) according to the boundary conditions

$$u(x,0) = -u(x,T), \quad u_t(x,0) = -u_t(x,T), \ \ 0 \le x \le l, \tag{2}$$

$$\frac{\partial^i u}{\partial x^i}\,(0,t) = 0, \quad \frac{\partial^i u}{\partial x^i}\,(l,t) = 0, \ \ 0 \le t \le T, \ \ i = 0, \dots, 2k-1. \tag{3}$$

Note that to the study of antiperiodic and periodic problems for nonlinear partial differential equations, having a structure different from (1), is devoted numerous literature (see, for example, [1, 2, 4–8] and the literature cited therein). For the equation (1) with $k=1$, antiperiodic problem, both in terms of time and space variables, is considered in the work [3].

Denote by $C^{2,4k}(\overline{D}_T)$ the space of functions continuous in $\overline{D}_T$, having in $\overline{D}_T$ continuous partial derivatives $\frac{\partial^i u}{\partial t^i}$, $i=1,2$, $\frac{\partial^j u}{\partial x^j}$, $j=1,\dots,4k$. Let

$$\begin{aligned} C_0^{2,4k}(\overline{D}_T) := \Big\{ u \in C^{2,4k}(\overline{D}_T): \ \ & \frac{\partial^i u}{\partial t^i}\,(x,0) = -\frac{\partial^i u}{\partial t^i}\,(x,T), \ \ 0 \le x \le l, \ \ i = 0,1; \\ & \frac{\partial^j u}{\partial x^j}\,(0,t) = 0, \ \ \frac{\partial^j u}{\partial x^j}(l,t) = 0, \ \ 0 \le t \le T, \ \ j = 0, \dots, 2k-1 \Big\}. \end{aligned}$$

Consider the Hilbert space $W_0^{1,2k}(D_T)$ as a completion of the classical space $C_0^{2,4k}(\overline{D}_T)$ with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{2k} \Big(\frac{\partial^i u}{\partial x^i}\Big)^2 \bigg]\, dx\, dt. \tag{4}$$

It follows from (4) that if $u \in W_0^{1,2k}(D_T)$, then $u \in W_2^1(D_T)$ and $\frac{\partial^i u}{\partial x^i} \in L_2(D_T)$, $i=2,\dots,2k$. Here $W_2^1(D_T)$ is the well-known Sobolev space consisting of the elements $L_2(D_T)$, having up to the first order generalized derivatives from $L_2(D_T)$.

Remark 1. Let $u \in C_0^{2,4k}(\overline{D}_T)$ be a classical solution of the problem (1)–(3). Multiplying the both sides of the equation (1) by an arbitrary function $\varphi \in C_0^{2,4k}(\overline{D}_T)$ and integrating obtained equality over the domain D_T with taking into account that the functions from the space $C_0^{2,4k}(\overline{D}_T)$ satisfy the boundary conditions (2) and (3), we get

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^{2k} u}{\partial x^{2k}}\,\frac{\partial^{2k} \varphi}{\partial x^{2k}}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in C_0^{2,4k}(\overline{D}_T). \tag{5}$$

We take the equality (5) as a basis of definition of a weak generalized solution of the problem (1)–(3) in the space $W_0^{1,2k}(D_T)$. But for this, certain restrictions must be imposed on the function f so that the integral

$$\int\limits_{D_T} f(u)\varphi\,dx\,dt \tag{6}$$

exists.

Remark 2. Below, from function f in the equation (1) we require that

$$f \in C(\mathbb{R}), \ \ |f(u)| \le M_1 + M_2|u|^{\alpha}, \ \ \alpha = const > 1, \ \ u \in \mathbb{R}, \tag{7}$$

where $M_i = const \ge 0$, $i = 1, 2$. As it is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ equals two, the embedding operator

$$I : W_2^1(D_T) \to L_q(D_T)$$

is linear and compact operator for any fixed $q = const > 1$. At the same time the Nemitskii operator $N : L_q(D_T) \to L_2(D_T)$, acting by formula $Nu = f(u)$, where $u \in L_q(D_T)$, and function f satisfies the condition (7) is bounded and continuous, when $q \ge 2\alpha$. Therefore, if we take $q = 2\alpha$, then the operator

$$N_0 = NI : W_2^1(D_T) \to L_2(D_T)$$

will be continuous and compact. Hence, in particular, we have that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$ and from $u_n \to u$ in the space $W_2^1(D_T)$ it follows $f(u_n) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the condition (7) and $F \in L_2(D_T)$. A function $u \in W_0^{1,2k}(D_T)$ is named a weak generalized solution of the problem (1)–(3) if the integral equality (5) holds for any function $\varphi \in W_0^{1,2k}(D_T)$, i.e.,

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^{2k} u}{\partial x^{2k}}\,\frac{\partial^{2k} \varphi}{\partial x^{2k}}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in W_0^{1,2k}(D_T). \tag{8}$$

Note that due to Remark 2 the integral (6) in the left-hand side of the equality (8) is defined correctly since from $u \in W_0^{1,2k}(D_T)$ it follows that $f(u) \in L_2(D_T)$, and since $\varphi \in L_2(D_T)$, then $f(u)\varphi \in L_1(D_T)$.

It is easy to see that if a weak generalized solution u of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_0^{2,4k}(\overline{D}_T)$, then it is a classical solution to this problem.

In the space $C_0^{2,4k}(\overline{D}_T)$ together with the scalar product

$$(u,v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \sum_{i=1}^{2k} \frac{\partial^i u}{\partial x^i}\,\frac{\partial^i v}{\partial x^i}\Big]\,dx\,dt \tag{9}$$

with the norm $\|\cdot\|_0=\|\cdot\|_{W_0^{1,2k}(D_T)}$, defined by the right−hand side of the equality (4), let us consider the following scalar product

$$(u,v)_1=\int\limits_{D_T}\Big[\frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t}+\frac{\partial^{2k}u}{\partial x^{2k}}\,\frac{\partial^{2k}v}{\partial x^{2k}}\Big]\,dx\,dt \tag{10}$$

with the norm

$$\|u\|_1^2=\int\limits_{D_T}\bigg[\Big(\frac{\partial u}{\partial t}\Big)^2+\Big(\frac{\partial^{2k}u}{\partial x^{2k}}\Big)^2\bigg]\,dx\,dt, \tag{11}$$

where $u,v\in C_0^{2,4k}(\overline{D}_T)$.

The following inequalities

$$c_1\|u\|_0\le\|u\|_1\le c_2\|u\|_0\ \ \forall\,u\in C_0^{2,4k}(\overline{D}_T)$$

with positive constants c_1 and c_2, not dependent on u, are valid. Hence due to (9)–(11) it follows that if we complete the space $C_0^{2,4k}(\overline{D}_T)$ with respect to the norm (11), then we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with the equivalent scalar products (9) and (10). Using this circumstance, one can prove the unique solvability of the linear problem corresponding to (1)–(3), when $f=0$, i.e. for any $F\in L_2(D_T)$ there exists a unique solution $u=L_0^{-1}F\in W_0^{1,2k}(D_T)$ to this problem, where the linear operator

$$L_0^{-1}:L_2(D_T)\to W_0^{1,2k}(D_T)$$

is continuous.

Remark 3. From the above reasoning it follows that when the conditions (7) are fulfilled, the nonlinear problem (1)–(3) is equivalently reduced to the functional equation

$$u=L_0^{-1}\,[f(u)-F] \tag{12}$$

in the Hilbert space $W_0^{1,2k}(D_T)$.

As noted below, if the nonlinear function f is not required to fulfill other conditions in addition to (7), then the problem (1)–(3) may not have a solution. At the same time, if the additional condition

$$\lim_{|u|\to\infty}\sup\frac{f(u)}{u}\le 0 \tag{13}$$

is satisfied, an a priori estimate is proved for the solution of the functional equation (12) in the space $W_0^{1,2k}(D_T)$, from which, taking into account Remarks 2 and 3, the existence of a solution to the equation (12) follows, and, consequently, of the problem (1)–(3) in the space $W_0^{1,2k}(D_T)$ in the sense of Definition 1. Thus, the following theorem holds.

Theorem 1. *Let the conditions* (7) *and* (13) *be fulfilled. Then for any $F\in L_2(D_T)$ the problem* (1)–(3) *has at least one weak generalized solution u in the space $W_0^{1,2k}(D_T)$ in the sense of Definition 1.*

In turns out that in the case of the problem (1)–(3), the monotonicity of the function f one can ensure uniqueness of its solution.

Theorem 2. *If the condition* (7) *is fulfilled and f is a non-strictly decreasing function, i.e.*

$$(f(y)-f(z))(y-z)\le 0\ \ \forall\,y,z\in\mathbb{R}, \tag{14}$$

then for any $F\in L_2(D_T)$ the problem (1)–(3) *can not have more than one weak generalized solution in the space $W_0^{1,2k}(D_T)$ in the sense of Definition 1.*

From these theorems it follows the following theorem.

Theorem 3. *Let the conditions* (7), (13) *and* (14) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has a unique weak generalized solution* u *in the space* $\overset{\circ}{W}{}_0^{1,2k}(D_T)$ *in the sense of Definition 1.*

As noted above, if no other conditions are imposed on the nonlinear function f in addition to the condition (7), then the problem (1)–(3) may not have a solution. Indeed, the following theorem holds.

Theorem 4. *Let the function* f *satisfy the conditions* (7) *and*

$$f(u) \leq -|u|^{\gamma} \;\; \forall\, u \in \mathbb{R}, \;\; \gamma = const > 1, \tag{15}$$

and the function $F = \beta F_0$, *where* $F_0 \in L_2(D_T)$, $F_0 > 0$ *in the domain* D_T, $\beta = const > 0$. *Then there exists a number* $\beta_0 = \beta_0(F_0, \gamma)$ *such that for* $\beta > \beta_0$ *the problem* (1)–(3) *can not have a weak generalized solution in the space* $W_0^{1,2k}(D_T)$ *in the sense of Definition 1.*

It is easy to see that when the condition (15) is fulfilled, then the condition (13) is violated.

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On Two-Point Boundary Value Problems for Higher Order Singular Advanced Differential Equations

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In the present report, on the finite interval $]a,b[$ we consider the n-th order advanced ordinary differential equation

$$u^{(n)}(t)=f\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big) \tag{1}$$

under the two-point nonlinear boundary conditions

$$\varphi\big(u(a),\dots,u^{(m-1)}(a)\big)=c_0,\quad u^{(i-1)}(b)=\varphi_i(u^{(n-1)}(b))\ \ (i=1,\dots,n-1). \tag{2}$$

Here $n\geq 2$, $m\in\{1,\dots,n\}$, c_0 is a positive constant, and $f:]a,b]\times\mathbb{R}^n\to\mathbb{R}$, $\tau_i:[a,b]\to[a,b]$ $(i=1,\dots,n)$, $\varphi:\mathbb{R}^m\to\mathbb{R}$, $\varphi_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n-1)$ are continuous functions, $\mathbb{R}=\,]-\infty,+\infty[\,$. Moreover,

$$a\leq t<\tau_i(t)\leq b \ \text{ for } \ a\leq t<b \ \ (i=1,\dots,n), \tag{3}$$

$$\varphi(0,\dots,0)=0,\ \ \varphi(x_1,\dots,x_m)\to+\infty \ \text{ as } \ (-1)^{i-1}x_i\to+\infty \ \ (i=1,\dots,m), \tag{4}$$

$$(-1)^{n-i}\varphi_i(x)x\geq 0 \ \text{ for } \ x\in\mathbb{R} \ \ (i=1,\dots,n-1). \tag{5}$$

A solution of equation (1) is sought in the space of n-times continuously differentiable functions defined in the interval $]a,b[\,$.

By $u(a)$ and $u(b)$ (by $u^{(i)}(a)$ and $u^{(i)}(b)$) we denote, respectively, the right and the left limits of the solution u (of the i-th derivative of u) at the points a and b.

A solution u of equation (1) is said to be **a solution of problem** (1)**,** (2) if there exist one-sided limits $u^{(i-1)}(a)$ $(i=1,\dots,m)$, $u^{(k-1)}(b)$ $(k=1,\dots,n)$, and equalities (2) are satisfied.

A solution u is said to be a Kneser solution if

$$(-1)^i u^{(i)}(t)u(t)\geq 0 \ \text{ for } \ a<t<b \ \ (i=1,\dots,n-1).$$

In the case, where $\tau(t)\equiv t$, two-point boundary value problems for equation (1) have long attracted the attention of specialists, and most of them, namely, some problems with boundary conditions of type (2), have been studied in sufficient detail (see [1–7] and the references therein). As for the case of advance, i.e. when inequalities (3) hold, two-point boundary value problems for equation (1), as far as we know, remains still unstudied.

The results below fill the above mentioned gap to some extent. They contain unimprovable in a certain sense conditions guaranteeing, respectively, the solvability and unique solvability of problem (1), (2) in the space of Kneser type functions. It should be noted that these conditions do not restrict the growth order of the function f in the phase variables at infinity, and contain the case where the function f has a nonintegrable singularity in the time variable at the point $t=0$, more precisely, the case, where

$$\int\limits_a^b |f(t,x_1,\dots,x_n)|\,dt=+\infty \ \text{ for } \ x_i\neq 0 \ \ (i=1,\dots,n).$$

To formulate the main results, we need to introduce the following notations.

$$f^*(t;r)=\max\Big\{|f(t,x_1,\dots,x_n)|:\ \ 0\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}\ \text{ for }\ a<t\le b,\ \ r>0,$$
$$f_*(t;\delta,r)=\min\Big\{|f(t,x_1,\dots,x_n)|:\ \ \delta\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}\ \text{ for }\ a<t\le b,\ \ r>\delta>0,$$
$$\varphi_r(x)=\min\Big\{\varphi(x_1,\dots,x_{m-1},x):\ \ 0\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,m-1)\Big\}$$
$$\text{for }\ m>1,\ \ r>0,\ \ x\in\mathbb{R},$$
$$\varphi_r(x)=\varphi(x)\ \text{ for }\ m=1,\ \ r>0,\ \ x\in\mathbb{R}.$$

Theorem 1. *If along with* (3)–(5) *the conditions*

$$f(t,0,\dots,0)=0,\quad (-1)^n f(t,x_1,\dots,x_n)\ge 0\ \textit{ for }\ a<t<b,\ \ (-1)^{i-1}x_i\ge 0\ \ (i=1,\dots,n),\quad (6)$$
$$\int_a^b (t-a)^{n-m} f^*(t;r)\,dt<+\infty\ \textit{ for }\ r>0 \quad (7)$$

hold, then problem (1), (2) *has at least one nonnegative Kneser solution.*

Theorem 2. *If along with* (3), (5), (6) *the conditions*

$$\varphi(0,\dots,0)=0,\quad \varphi_r(x)\to+\infty\ \textit{ for }\ r>0,\ \ (-1)^{n-1}x\to+\infty, \quad (4')$$
$$\int_a^b (t-a)^{n-m} f_*(t;\delta,r)\,dt=+\infty\ \textit{ for }\ r>\delta>0$$

hold, then problem (1), (2) *has no nonnegative Kneser solution.*

From the above formulated theorems it follows

Corollary 1. *Let conditions* (3), (4′), (5), (6) *hold and let for every constants* $r>0$ *and* $\delta\in\,]0,r[$ *there exist a positive number* $\rho(r,\delta)$ *such that*

$$f^*(t;r)\le\rho(r,\delta)f_*(t;\delta,r)\ \textit{ for }\ a<t<b. \quad (8)$$

Then for problem (1), (2) *to have at least one nonnegative Kneser solution, it is necessary and sufficient that condition* (7) *to satisfied.*

Remark 1. Conditions (6) and (8) are satisfied, for example, in the case, where

$$f(t,x_1,\dots,x_n)=\sum_{j=1}^k p_j(t)f_j(x_1,\dots,x_n),$$

where k is any natural number, $p_j:\,]a,b]\to\mathbb{R}$, $f_j:\mathbb{R}^n\to\mathbb{R}$ $(j=1,\dots,k)$ are continuous functions such that

$$(-1)^n p_j(t)\ge 0\ \text{ for }\ a<t<b\ \ (j=1,\dots,k),$$
$$f_j(0,\dots,0)=0\ \ (j=1,\dots,k),$$
$$\min\Big\{f_j(x_1,\dots,x_n):\ \ \delta\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}>0\ \text{ for }\ r>\delta>0\ \ (j=1,\dots,k).$$

Example 1. Consider the equation

$$u^{(n)}(t) = p(t)h\big(|u^{(n-1)}(\tau(t))|\big) \tag{9}$$

with the boundary conditions (2), where $m \le n-1$, while $p :]a,b] \to \mathbb{R}$, $h : [0,+\infty[\to \mathbb{R}$, and $\tau : [a,b] \to [a,b]$ are continuous functions, and

$$(-1)^n p(t) \ge 0 \ \text{ for } \ a < t < b, \quad \int\limits_a^b (t-a)^{n-m}|p(t)|\,dt < +\infty,$$

$$\int\limits_a^b |p(t)|\,dt = +\infty, \tag{10}$$

$$h(0) = 0, \ \ h(x) > 0 \ \text{ for } \ x > 0,$$

$$\int\limits_\delta^{+\infty} \frac{ds}{h(s)} < +\infty \ \text{ for } \ \delta > 0. \tag{11}$$

If along with (4), (5) the condition

$$a \le t < \tau(t) \le b \ \text{ for } \ a \le t < b \tag{12}$$

is satisfied, then according to Theorem 1 problem (9), (2) has at least one nonnegative Kneser solution. Assume now that all the above conditions are satisfied except of (12) instead of which we have

$$\tau(t) = t \ \text{ for } \ a \le t \le a_0, \quad t < \tau(t) \le b \ \text{ for } \ a_0 < t < b,$$

where $a_0 \in]a,b[$. Show that in this case problem (9), (2) has no nonnegative Kneser solution. Assume the contrary that there exists such a solution u. Then there are $\delta > 0$ and $t_0 \in]a,a_0]$ such that

$$0 < \delta \le (-1)^{n-1}u^{(n-1)}(t) < +\infty \ \text{ for } \ a < t \le t_0.$$

On the other hand,

$$|u^{(n-1)}(t)|' = -|p(t)|h\big(|u^{(n-1)}(t)|\big) \ \text{ for } \ a < t \le t_0.$$

Therefore,

$$\int\limits_\delta^{|u^{(n-1)}(t)|} \frac{dx}{h(x)} = \int\limits_t^{t_0} |p(t)|\,dt \ \text{ for } \ a < t \le t_0,$$

which contradicts conditions (10) and (11).

The above constructed example shows that if instead of (3) for some $a_0 \in]a,b[$ the conditions

$$a \le t < \tau_i(t) \le b \ \text{ for } \ a \le t < b \ \ (i = 1,\dots,n-1), \quad \tau_n(t) = t \ \text{ for } \ a \le t \le a_0,$$
$$t < \tau_n(t) \le b \ \text{ for } \ a_0 < t < b$$

hold, then conditions (4)–(7) do not guarantee the existence of a nonnegative Kneser solution of problem (1), (2).

To simplify the presentation, we will consider the question on the uniqueness of a solution of problem (1), (2) in the case where the boundary conditions (2) have the form

$$\begin{gathered} \sum_{i=1}^m \alpha_i |u^{(i-1)}(a)|^{\mu_i} \operatorname{sgn}(u^{(i-1)}(a)) = c_0, \\ u^{(j-1)}(b) = \beta_j |u^{(n-1)}(b)|^{\nu_j} \operatorname{sgn}(u^{(n-1)}(b)) \ \ (j = 1,\dots,n-1), \end{gathered} \tag{2'}$$

where

$$(-1)^{i-1}\alpha_i \geq 0, \ \ \mu_i > 0 \ \ (i=1,\dots,m), \ \ \alpha_m \neq 0,$$
$$(-1)^{n-j}\beta_j \geq 0, \ \ \nu_j > 0 \ \ (j=1,\dots,n-1).$$

Evidently, in this case the function

$$\varphi(x_1,\dots,x_m) \equiv \sum_{i=1}^{m} \alpha_i |x_i|^{\mu_i} \operatorname{sgn}(x_i)$$

satisfies conditions (4′), and the functions $\varphi_j(x) = \beta_j |x|^{\nu_j} \operatorname{sgn}(x)$ $(j=1,\dots,n-1)$ – conditions (5).

We will say that the function f is **locally Lipschitzian in the phase variables on the set** $[a_0,b]\times\mathbb{R}^n$ if for every $r>0$ there exists $\ell(r)>0$ such that

$$\big|f(t,x_1,\dots,x_n) - f(t,y_1,\dots,y_n)\big| \leq \ell(r)\sum_{i=1}^{n}|x_i-y_i| \ \text{ for } \ a_0 \leq t \leq b, \ \ \sum_{i=1}^{n}\big(|x_i|+|y_i|\big) \leq r.$$

Theorem 3. *Let along with* (3), (7) *the condition*

$$f(t,0,\dots,0)=0, \quad (-1)^n f(t,x_1,\dots,x_n) \geq (-1)^n f(t,y_1,\dots,y_n) \geq 0$$
$$\text{for } \ a<t<b, \ \ (-1)^{i-1}x_i \geq (-1)^{i-1}y_i \geq 0 \ \ (i=1,\dots,n)$$

hold. Let, moreover, there exist $a_0 \in]a,b[$ such that the function f is locally Lipschitzian in the phase variables on the set $[a_0,b]\times\mathbb{R}^n$. Then problem (1), (2′) *has a unique nonnegative Kneser solution.*

Finally, we consider two nontrivial particular cases of equation (1):

$$u^{(n)}(t) = \sum_{i=1}^{n} p_i(t) f_i\big(u^{(i-1)}(\tau_i(t))\big), \tag{13}$$

$$u^{(n)}(t) = \sum_{i=1}^{n} p_i(t)\big|u^{(i-1)}(\tau_i(t))\big|^{\lambda_i} \operatorname{sgn}\big(u^{(i-1)}(\tau_i(t))\big), \tag{14}$$

where $p_i :]a,b] \to \mathbb{R}$, $f_i : \mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ are continuous functions, while λ_i $(i=1,\dots,n)$ are constants.

Corollary 1 and Theorem 3 yield the following results.

Corollary 2. *Let the functions τ_i $(i=1,\dots,n)$ satisfy inequalities* (3), *and let the functions p_i and f_i $(i=1,\dots,n)$ be such that*

$$(-1)^{n+i-1}p_i(t) \geq 0 \ \text{ for } \ a<t<b \ \ (i=1,\dots,n), \tag{15}$$
$$f_i(0)=0, \ \ (-1)^{i-1}f_i(x) > 0 \ \text{ for } \ (-1)^{i-1}x>0 \ \ (i=1,\dots,n).$$

Then for problem (13), (2′) *to have at least one nonnegative Kneser solution, it is necessary and sufficient that the conditions*

$$\int_a^b (t-a)^{n-m}|p_i(t)|\,dt < +\infty \ \ (i=1,\dots,n) \tag{16}$$

to satisfied.

Corollary 3. *Let the functions τ_i and p_i $(i=1,\dots,n)$ satisfy inequalities* (3) *and* (15)*, and let f_i $(i=1,\dots,n)$ be locally Lipschitzian functions such that*

$$f_i(0)=0,\ \ (-1)^{i-1}f_i(x)\geq(-1)^{i-1}f_i(y)>0\ \ for\ \ (-1)^{i-1}x\geq(-1)^{i-1}y>0\ \ (i=1,\dots,n).$$

Then for problem $(13),(2')$ *to have a unique nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

From Corollaries 2 and 3 it follow Corollaries 4 and 5, respectively.

Corollary 4. *Let*

$$\lambda_i>0\ \ (i=1,\dots,n),$$

and let the functions τ_i, p_i $(i=1,\dots,n)$ satisfy inequalities (3) *and* (15)*. Then for problem* $(14),(2')$ *to have at least one nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

Corollary 5. *Let*

$$\lambda_i\geq 1\ \ (i=1,\dots,n),$$

and let the functions τ_i, p_i $(i=1,\dots,n)$ satisfy inequalities (3) *and* (15)*. Then for problem* $(14),(2')$ *to have a unique nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

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Ill-Posed Initial-Boundary Value Problems for Linear Hyperbolic Systems of Second Order

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In the rectangle $\Omega=[0,\omega_1]\times[0,\omega_2]$ consider the initial-boundary value problem

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{1}$$

$$u(0,y)=\varphi(y), \quad h(u_x(x,\,\cdot\,))(x)=\psi(x), \tag{2}$$

where $P_j\in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q\in C(\Omega;\mathbb{R}^n)$, $\varphi\in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi\in C([0,\omega_1];\mathbb{R}^n)$, and $h:C([0,\omega_2];\mathbb{R}^n)\to C([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator.

We make use of the following notations:

- I_m is $m\times m$ identity matrix, O_m is the $m\times m$ zero matrix, $O_{m,k}$ is $m\times k$ zero matrix.
- If $A=(a_{ij}))_{i,j=1}^n\in\mathbb{R}^{n\times n}$, then by $[A]_{m,m}$ we denote its principal $m\times m$ submatrix, i.e.
$$[A]_{m,m}=(a_{ij})_{i,j=1}^m.$$
- If $z=(z_i)_{i=1}^n\in\mathbb{R}^n$, then $[z]^m=(z_i)_{i=1}^m$ and $[z]_m=(z_i)_{i=m+1}^n$.
- $C^{m,k}(\Omega;\mathbb{R}^n)$ is the Banach space of functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives $u^{(i,j))}$ $(i=0,\dots,m;\ j=0,\dots k)$, endowed with the norm
$$\|u\|_{C^{\mathbf{m}}(\Omega)}=\sum_{i=0}^m\sum_{j=0}^k\Big\|\frac{\partial^{i+j}u}{\partial x^i\partial y^j}\Big\|_{C(\Omega)}.$$

By a solution of problem $(1),(2)$ we understand a *classical* solution, i.e., a function $u\in C^{1,1}(\Omega)$ satisfying equation (1) and the boundary conditions (2) everywhere in Ω.

Along with problem $(1),(2)$ consider the problem

$$v'=P_1(x^*,y)v, \tag{3}$$

$$h(v)(x^*)=0. \tag{4}$$

Problem $(3),(4)$ is called ***associated problem*** of problem $(1),(2)$. Notice that problem $(3),(4)$ is a boundary value problem for a linear ordinary differential equation depending on the parameter x^*.

The associate problem $(3),(4)$ plays a decisive in the study of problem $(1),(2)$. Theorems 4.1 and 4.1′ from [1] state that if for every $x^*\in[0,\omega_1]$ problem $(3),(4)$ has only the trivial solution, then problem $(1),(2)$ is well-posed, i.e., it is uniquely solvable for arbitrary $\varphi\in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi\in C([0,\omega_1];\mathbb{R}^n)$ and $q\in C(\Omega;\mathbb{R}^n)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)}\leq M\Big(\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}+\|q\|_{C(\Omega)}\Big), \tag{5}$$

where M is a positive constant independent of φ, ψ and q.

In [3] the inverse statement is proved: if problem (1), (2) is well-posed, i.e., if a solution of problem (1), (2) admits estimate (5), then the associate problem (3), (4) has only the trivial solution for every $x^* \in [0,\omega_1]$.

Well-posed initial-boundary value problems and nonlocal boundary value problems for linear hyperbolic systems were studied in [1] and [3]. Well-posed initial-boundary value problems and nonlocal boundary value problems for higher order linear hyperbolic equations were studied in [4] and [2]. Ill-posed initial-boundary value problems for higher order linear hyperbolic equations were studied in [5].

Let $Y(x^*,y)$ be the fundamental matrix of differential system (3) such that $Y(x^*,0)=I$.

If $h : C([0,\omega_2];\mathbb{R}^n) \to C([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator, then according to Lemmas 2.1_1 and 2.3_1 from [1],

$$h(z)(x) = H(x)z(0) + \int_0^{\omega_2} K(x,t)v'(t)\,dt,$$

$$h(Y(x,\,\cdot\,)z(\,\cdot\,))(x) = M_0(x)z(0) + \int_0^{\omega_2} M(x,t)v'(t)\,dt,$$

where

$$H(x) \in C([0,\omega_1];\mathbb{R}^{n\times n}), \quad K \in L^\infty(\Omega;\mathbb{R}^{n\times n}),$$

$$M_0(x) = H(x) + \int_0^{\omega_2} M(x,t)Y_t(x,t)\,dt,$$

$$M(x,y) = K(x,y)Y(x,y) + \int_y^{\omega_2} K(x,t)Y_t(x,t)\,dt.$$

If $h : C([0,\omega_2];\mathbb{R}^n) \to C^1([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator, then

$$H(x) \in C^1([0,\omega_1];\mathbb{R}^{n\times n}),$$

$$M_0(x) \in C^1([0,\omega_1];\mathbb{R}^{n\times n}), \qquad \int_0^y M(x,t)\,dt \in C^{1,1}(\Omega;\mathbb{R}^{n\times n}).$$

In terms of the matrix H, problem (1), (2) is well-posed if and only if $\operatorname{rank} H(x) = n$ for every $x \in [0,\omega_1]$. The ill-posed problem (1), (2) in the case where $\operatorname{rank} H(x) \equiv 0$ was studied in [1].

In this paper, we study the ill-posed problem (1), (2) in case where $\operatorname{rank} H(x) = n-m$ for some $m \in \{1,\dots,n\}$. For the sake of technical simplicity we will assume that

$$H(x) = \begin{pmatrix} O_m & O_{m,n-m} \\ O_{n-m,m} & H_0(x) \end{pmatrix}, \quad \operatorname{rank} H_0(x) = n-m \ \text{ for } \ x \in [0,\omega_1]. \tag{6}$$

Theorem 1. *Let* (6) *hold, let* $P_1 \in C^{1,0}(\Omega;\mathbb{R}^n)$ *and let*

$$\det\bigg[\int_0^{\omega_2} M(x,t)Z^{-1}(x,t)\big(P_0(x,t) + P_2(x,t)P_1(x,t)\big)Z(x,t)\,dt\bigg]_{mm} \neq 0 \ \ \textit{for} \ \ x \in [0,\omega_1]. \tag{7}$$

Then problem (1), (2) *is solvable in the weak sense if and only if the equality*

$$\Big[\int_0^{\omega_2} M(0,t)Z^{-1}(0,t)\big(P_0(0,t)\varphi(t)+P_2(0,t)\varphi'(t)+q(0,t)\big)\,dt\Big]_m=[\psi(0)]_m \tag{8}$$

holds. Moreover, if equality (8) *holds, then problem* (1), (2) *has a unique weak solution* $u\in C^{0,1}(\Omega;\mathbb{R}^n)$ *admitting the estimate*

$$\big\|[u]^m\big\|_{C^{0,1}(\Omega)}\le M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}\Big), \tag{9}$$

$$\big\|[u]_m\big\|_{C^{1,1}(\Omega)}\le M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}\Big), \tag{10}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Theorem 2. *Let* $P_j\in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q\in C^{1,0}(\Omega;\mathbb{R}^n)$, $\psi\in C^1([0,\omega_1];\mathbb{R}^n)$, *let* $h:C([0,\omega_2];\mathbb{R}^n)\to C^1([0,\omega_1];\mathbb{R}^n)$ *be a bounded linear operator, and let conditions* (6) *and* (7) *hold. Then problem* (1), (2) *is solvable in the classical sense if and only if equality* (8) *holds. Moreover, if equality* (8) *holds, then problem* (1), (2) *has a unique classical solution* u *admitting the estimate*

$$\big\|[u]^m\big\|_{C^{1,1}(\Omega)}\le M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C^1([0,\omega_1])}\Big), \tag{11}$$

$$\big\|[u]_m\big\|_{C^{2,1}(\Omega)}\le M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C^1([0,\omega_1])}\Big), \tag{12}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Remark 1. Estimates (9), (10), (11) and (12) are sharp and cannot be relaxed. Indeed, let $n=2m$, $u=(v,w)$, $v,w\in\mathbb{R}^m$. Consider the problem

$$\begin{aligned} v_{xy}&=P(x)v+q_1(x),\\ w_{xy}&=Q_1(x,y)v+Q_2(x,y)w+q_2(x,y), \end{aligned} \tag{13}$$

$$v(0,y)=c,\quad w(0,y)=\varphi_2(y),\quad v_x(x,0)=v_x(x,\omega_2),\quad w_x(x,0)=0. \tag{14}$$

For problem (13), (14)

$$H(x)=\begin{pmatrix} O_m & O_m\\ O_m & I_m\end{pmatrix}.$$

Assume that problem (13), (14) has a unique weak solution $(v(x,y),w(x,y))\in C^{0,1}(\Omega;\mathbb{R}^{2m})$. Then $w(x,y)$ is a solution of the integral equation

$$w(x,y)=\varphi_2(y)+\int_0^x\int_0^y\big(Q_2(s,t)w(s,t)+Q_1(s,t)v(s,t)+q_2(s,t)\big)\,dt\,ds. \tag{15}$$

The integral equation (15) is uniquely solvable for every $v(x,y)\in C(\Omega;\mathbb{R}^m)$, and its solution belongs to $C^{1,1}(\Omega;\mathbb{R}^m)$.

Consequently, the unique solvability of problem (13), (14) is equivalent to the unique solvability of the problem

$$v_{xy}=P(x)v+q_1(x), \tag{16}$$

$$v(0,y)=c,\quad v_x(x,0)=v_x(x,\omega_2). \tag{17}$$

Let $v \in C^{0,1}(\Omega;\mathbb{R}^m)$ be a weak solution of problem (16), (17). It admits a continuous ω_2-periodic continuation with respect to y. It is clear that $v(x,y+r)$ is a solution of problem (16), (17) for every $r \in [0,\omega_1]$. Therefore, the unique solvability of problem (16), (17) implies that

$$v(x,y) \equiv v(x).$$

Consequently, v is a solution of the linear algebraic system

$$P(x)v(x) + q_1(x) = 0.$$

The latter system is uniquely solvable for an arbitrary $q_1(x)$ if and only if

$$\det P(x) \neq 0 \ \text{ for } \ x \in [0,\omega_1],$$

i.e., inequality (7) holds for problem (13), (14). Thus problem (16), (17) has the unique weak solution

$$v(x) = -P^{-1}(x)q_1(x)$$

if and only if

$$c = -P^{-1}(0)q(0).$$

This solution is classical if $P \in C^1([0,\omega_1];\mathbb{R}^{m\times m})$ and $q_1 \in C^1([0,\omega_1];\mathbb{R}^m)$. This confirms the sharpness of estimates (9), (10), (11) and (12).

If $\det P(x^*) = 0$ for some $x^* \in [0,\omega_1]$, then problem (16), (17) may not have a weak solution even if $P \in C^\infty(\omega;\mathbb{R}^{m\times m})$ and $q \in C^\infty(\omega;\mathbb{R}^m)$.

Finally, notice that if $\det P(0) \neq 0$, $P(x) = O_m$ for $x \in [a,b]$ for some interval $[a,b] \subset (0,\omega_1)$, and $q(x) = P(x)\overline{q}(x)$, then problem (16), (17) and, consequently, problem (13), (14), have infinite-dimensional sets of solutions.

For the system

$$u_{xy} = P_0(x,y)u + P_2(x,y)u_y + q(x,y), \tag{18}$$

consider the initial-boundary conditions

$$u(0,y) = \varphi(y), \quad \big[u_x(x,\gamma_2(x)) - u_x(x,\gamma_1(x))\big]^m = [\psi(x)]^m, \quad [u_x(x,0)]_m = [\psi(x)]_m, \tag{19}$$

and

$$u(0,y) = \varphi(y), \quad u_x(x,0) = u_x(x,\omega_2). \tag{20}$$

Here $\gamma_i \in C([0,\omega_1])$ $(i=1,2)$ and $\gamma_1(x) < \gamma_2(x)$ for $x \in [0,\omega_1]$.

Theorem 3. *Let*

$$\det\bigg[\int\limits_{\gamma_1(x)}^{\gamma_2(x)} P_0(x,t)\,dt\bigg]_{mm} \neq 0 \ \textit{ for } \ x \in [0,\omega_1].$$

Then problem (18), (19) *is solvable in the weak sense if and only if the equality*

$$\bigg[\int\limits_{\gamma_1(0)}^{\gamma_2(0)} \big(P_0(0,t)\varphi(t) + P_2(0,t)\varphi'(t) + q(0,t)\big)\,dt\bigg]_m = [\psi(0)]_m \tag{21}$$

holds. Moreover, if equality (21) *holds, then problem* (18), (19) *has a unique weak solution* u *admitting the estimate*

$$\|u\|_{C^{0,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])}\Big),$$

where M is a positive constant independent of φ, ψ and q. Moreover, if $P_j \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ $(j=0,2)$, $q\in C^{1,0}(\Omega;\mathbb{R}^n)$, $\psi\in C^1([0,\omega_1];\mathbb{R}^n)$ and $\gamma_i\in C^1([0,\omega_1])$ $(i=1,2)$, then u is a classical solution admitting the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C^{1,0}([0,\omega_1])}\Big).$$

Corollary 1. *Let*

$$\det\int\limits_0^{\omega_2} P_0(x,t)\,dt \neq 0 \ \ \text{for} \ \ x\in[0,\omega_1].$$

Then problem (18), (20) *is solvable if and only if the equality*

$$\int\limits_0^{\omega_2} \big(P_0(0,t)\varphi(t) + P_2(0,t)\varphi'(t) + q(0,t)\big)\,dt = 0 \tag{22}$$

holds. Moreover, if equality (22) *holds, then problem* (18), (20) *has a unique weak solution u admitting the estimate*

$$\|u\|_{C^{0,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M is a positive constant independent of φ, ψ and q. Moreover, if $P_0\in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ and $q\in C^{1,0}(\Omega;\mathbb{R}^n)$, then u is a classical solution admitting the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big).$$

Let $n=2m$, $u=(v,w)$, and $v,w\in\mathbb{R}^m$. For the systems

$$\begin{aligned} v_{xy} &= w_x + B_{11}(x,y)v_y + B_{12}(x,y)w_y + Q_{11}(x,y)v + Q_{12}(x,y)w + q_1(x,y),\\ w_{xy} &= -v_x + B_{21}(x,y)v_y + B_{22}(x,y)w_y + Q_{21}(x,y)v + Q_{22}(x,y)w + q_2(x,y) \end{aligned} \tag{23}$$

and

$$\begin{aligned} v_{xy} &= w_x + B(x,y)v_y + Q(x,y)v + q_1(x,y),\\ w_{xy} &= -v_x + B(x,y)w_y + Q(x,y)w + q_2(x,y), \end{aligned} \tag{24}$$

consider the initial-periodic conditions

$$v(0,y)=\varphi_1(y), \quad w(0,y)=\phi_2(y), \quad v_x(x,0)=v_x(x,2\pi), \quad w_x(x,0)=w_x(x,2\pi). \tag{25}$$

Theorem 4. *Let*

$$\det\left(\int\limits_0^{2\pi}\begin{pmatrix}\cos t\,I_m & \sin t\,I_m\\ -\sin t\,I_m & \cos t\,I_m\end{pmatrix}\begin{pmatrix}Q_{11}(x,t)-B_{12}(x,t) & Q_{12}(x,t)+B_{11}(x,t)\\ Q_{21}(x,t)-B_{12}(x,t) & Q_{11}(x,t)-B_{12}(x,t)\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}\cos t\,I_m & -\sin t\,I_m\\ \sin t\,I_m & \cos t\,I_m\end{pmatrix}dt\right)\neq 0 \ \ \text{for} \ \ x\in[0,\omega_1].$$

Then problem (23), (25) *is solvable in the weak sense if and only if the equality*

$$\int\limits_0^{2\pi}\begin{pmatrix}\cos t\,I_m & -\sin t\,I_m\\ \sin t\,I_m & \cos t\,I_m\end{pmatrix}$$
$$\times\begin{pmatrix}Q_{11}(0,t)\varphi_1(t)+Q_{12}(0,t)\varphi_2(t)+B_{11}(0,t)\varphi_1'(t)+B_{12}(0,t)\varphi_2'(t)+q_1(0,t)\\ Q_{21}(0,t)\varphi_1(t)+Q_{22}(0,t)\varphi_2(t)+B_{21}(0,t)\varphi_1'(t)+B_{22}(0,t)\varphi_2'(t)+q_2(0,t)\end{pmatrix}dt=\begin{pmatrix}0\\0\end{pmatrix} \tag{26}$$

holds. Moreover, if equality (26) *holds, then problem* (23), (25) *has a unique weak solution* (v,w) *admitting the estimate*

$$\|v\|_{C^{0,1}(\Omega)}+\|w\|_{C^{0,1}(\Omega)}\leq M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M *is a positive constant independent of* φ*,* ψ *and* q*. Moreover, if* $B_{ij}\in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$ *and* $Q_{ij}\in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$ $(i,j=1,2)$*, and* $q_i\in C^{1,0}(\Omega;\mathbb{R}^m)$ $(i=1,2)$*, then* (v,w) *is a classical solution admitting the estimate*

$$\|v\|_{C^{1,1}(\Omega)}+\|w\|_{C^{1,1}(\Omega)}\leq M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}\Big).$$

Corollary 2. *Let*

$$\det\left(\int_0^{2\pi}\begin{pmatrix}Q(x,t) & B(x,t)\\ -B(x,t) & Q(x,t)\end{pmatrix}dt\right)\neq 0\ \ \textit{for}\ \ x\in[0,\omega_1].$$

Then problem (23), (24) *is solvable in the weak sense if and only if the equality*

$$\int_0^{2\pi}\begin{pmatrix}\cos t\, I_m & -\sin t\, I_m\\ \sin t\, I_m & \cos t\, I_m\end{pmatrix}\begin{pmatrix}Q(0,t)\varphi_1(t)+B(0,t)\varphi_1'(t)+q_1(0,t)\\ Q(0,t)\varphi_2(t)+B(0,t)\varphi_2'(t)+q_2(0,t)\end{pmatrix}dt=\begin{pmatrix}0\\0\end{pmatrix}$$

holds. Moreover, if equality (26) *holds, then problem* (23), (24) *has a unique weak solution* u *admitting the estimate*

$$\|v\|_{C^{0,1}(\Omega)}+\|w\|_{C^{0,1}(\Omega)}\leq M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M *is a positive constant independent of* φ*,* ψ *and* q*. Moreover, if* $B\in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$*,* $Q\in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$*, and* $q_i\in C^{1,0}(\Omega;\mathbb{R}^m)$ $(i=1,2)$*, then* u *is a classical solution admitting the estimate*

$$\|v\|_{C^{1,1}(\Omega)}+\|w\|_{C^{1,1}(\Omega)}\leq M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}\Big).$$

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Deep Neural Network for Approximate Solution of One System of Nonlinear Integro-differential Equations

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The analysis of electromagnetic field penetration into materials, its mathematical representation, and subsequent computational solutions represent a crucial aspect of applied mathematics. This phenomenon often involves the production of heat energy, which modifies the medium's permeability and affects diffusion processes. Such effects arise from the strong dependence of the material's conductivity on temperature. The mathematical description of these processes, like many other real-world problems, results in systems of nonlinear partial differential equations and integro-differential equations. In the quasistatic approximation, the system of Maxwell's equations can be expressed in a following form [17]

$$\frac{\partial H}{\partial t} = -\nabla \times (\nu_m \nabla \times H), \quad c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\nabla \times H)^2. \tag{1}$$

The equations (1) describe the evolution of magnetic fields and temperature within the medium, considering Joule heating effects. The coefficients for thermal capacity and electrical conductivity are assumed to depend on temperature. Under specific assumptions, as shown in [5], the system of Maxwell's equations can be reduced to a nonlinear parabolic-type integro-differential equation [5]

$$\frac{\partial H}{\partial t} = -\nabla \times \left[a\left(\int_0^t |\nabla \times H|^2 \, d\tau \right) \nabla \times H \right], \tag{2}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

The integro-differential equation (2) derived in this context is complex, and only specific cases of this model have been studied in depth (see references such as [5–20, 24] and references therein). By assuming a specific structure for the magnetic field, in particular, if $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, the vector equation (2) becomes as the following system of nonlinear integro-differential equations:

$$\begin{aligned} &\frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau \right) \frac{\partial U}{\partial x} \right] = 0, \\ &\frac{\partial V}{\partial t} - \frac{\partial}{\partial x} \left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau \right) \frac{\partial V}{\partial x} \right] = 0. \end{aligned} \tag{3}$$

This note aims to apply Deep Neural Network (DNN) approach implemented in [15] to solve the Dirichlet initial-boundary value problem for system (3). The focus is on approximate solution to the nonlinear equations using neural network architectures, where the diffusion coefficient has the following form $a(S) = (1 + S)^p$, $0 < p \le 1$.

Thus, our goal is to apply DNN for the approximate solution of the the following nonlinear initial-boundary value problem with nonhomogeneous right-hand sides

$$\begin{gathered}
\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial U}{\partial x}\bigg]=f_1(x,t),\\
\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial V}{\partial x}\bigg]=f_2(x,t),\\
U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\ \ t\in[0,T],\\
U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1],
\end{gathered}\tag{4}$$

where f_1, f_2, U_0, and V_0 are the given functions.

Descriptive and measurable features, along with the numerical solutions for problem (4) and one-dimensional analogue of (2) type models have been thoroughly studied in the literature (see, for example, [1, 2, 4–16, 18–20, 24], and the references therein). As mentioned earlier, our goal is to explore an alternative method for solving partial differential equations (PDEs) using Machine Learning techniques. Machine learning, specifically neural networks, is utilized to create surrogate models that predict PDE solutions at any point within the domain. Neural networks, which consist of input, hidden, and output layers, offer flexibility in terms of architecture and the number of neurons per layer (see, for example, [16]). In this approach, the solution to the problem is approximated by a neural network output, and the network parameters are optimized during training. A significant advantage of using DNNs in solving PDEs is their ability to incorporate physical laws into the learning process, reducing the volume of required training data (as discussed in [3, 21–23]).

The methodology allows to define a residual for the nonlinear problem (4), involving the approximate solution $(u(x,t,\rho),\, v(x,t,\rho))$ which is evaluated at specific training points

$$\begin{aligned}
R(x,t,\rho)=&\ \frac{\partial u(x,t,\rho)}{\partial t}+\frac{\partial v(x,t,\rho)}{\partial t}-f_1(x,t)-f_2(x,t)\\
&-\frac{\partial}{\partial x}\bigg\{\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial u(x,t,\rho)}{\partial x}\Big)^2+\Big(\frac{\partial v(x,t,\rho)}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial u(x,t,\rho)}{\partial x}\bigg\}\\
&-\frac{\partial}{\partial x}\bigg\{\bigg(1+\int\limits_0^t\Big[\Big(\frac{\partial u(x,t,\rho)}{\partial x}\Big)^2+\Big(\frac{\partial v(x,t,\rho)}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial v(x,t,\rho)}{\partial x}\bigg\}.
\end{aligned}\tag{5}$$

The neural network is trained by minimizing a cost function that combines residual (5) with boundary and initial condition constraints [3, 15, 21–23].

Test experiments, adopting the setup described in [16], were conducted to validate this approach. The results demonstrate the potential of neural networks to approximate solutions effectively, even for complex nonlinear PDEs. The experiments used TensorFlow for training and incorporated various parameter settings to replicate and extend prior findings.

The source terms f_1 and f_2 were selected such a way that the exact solutions are given as follows:

$$U(x,t)=x(1-x)\sin(2\pi x-t),\quad V(x,t)=x(1-x)\cos(2\pi x-t).$$

In Figures 1 and 2 the results of the exact and approximate solutions are given for U and V respectively. The plots are for the case $p=0.5$ in the problem (4).

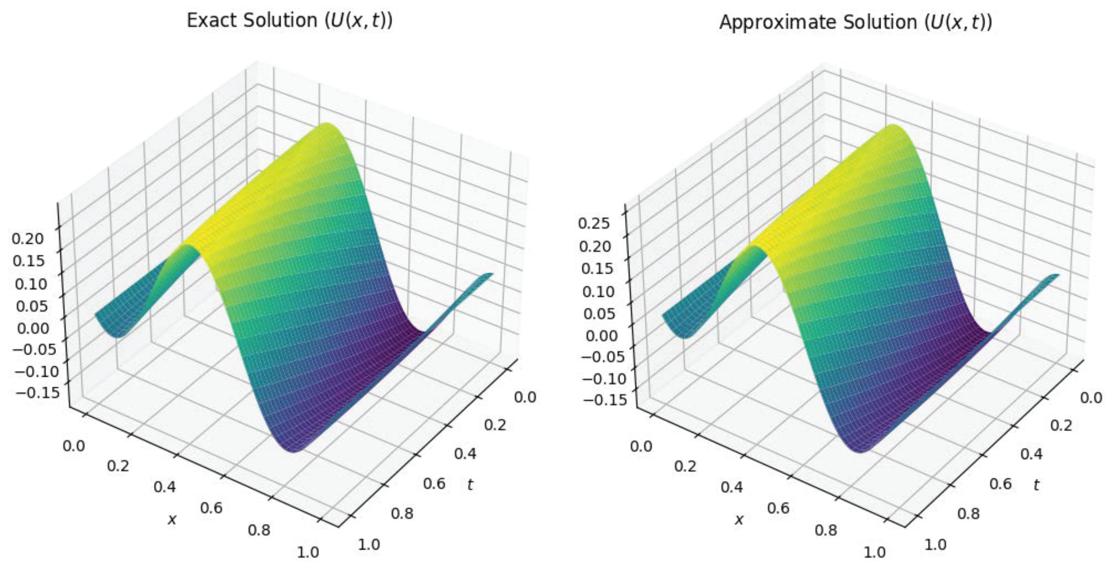

Figure 1. Exact and numerical solutions for $U(x,t)$.

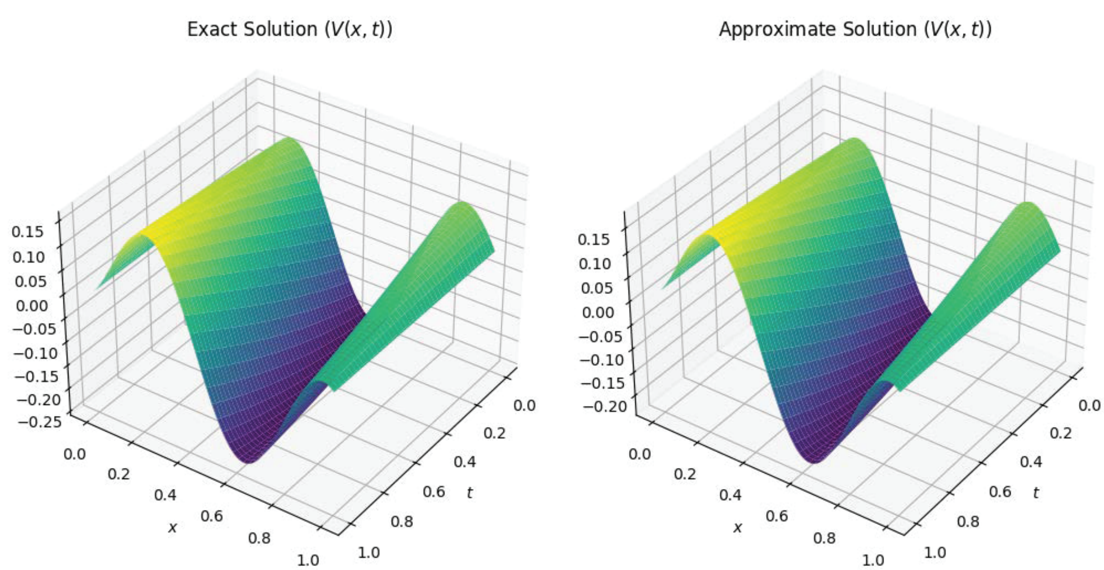

Figure 2. Exact and numerical solutions for $V(x,t)$.

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Median and p-Median

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1 Introduction

In this paper, we introduce the definition of p-median of a measurable function which is the key concept in the definition of p-oscillation and generalized Kurzweil integral. This new integral is based on minimization of sums of p-oscillations instead of ordinary oscillations which leads to a wider class of integrable functions. We introduce its definition in Section 4. However, it is not obvious how to compute p-oscillation of a given function. It leads us to question if it is possible to classify p-medians of a given function for any p. We can answer this question for $p = 1$, $p = 2$ and $p = \infty$.

2 Preliminaries

Let M be an arbitrary subset of $\mathbb{R}$, then $\mu(M)$ is the Lebesgue measure of M and, for $p \in [1, \infty]$, $L^p(M)$ is, as usual, the space of real valued functions measurable on M and such that $\|f\|_p < \infty$, where

$$\|f\|_p = \bigg(\int\limits_M |f(x)|^p \, \mathrm{d}x \bigg)^{\frac{1}{p}} \ \text{ if } \ p \in [1, \infty) \ \text{ and } \ \|f\|_\infty = \operatorname*{ess\,sup}_{x \in M} |f(x)|$$

is the usual norm on $L^p(M)$.

3 Median and p-median

Next definition was used in [5] (c.f. Definition 2.5 therein) and it is an analogue of median of random variable in probability and statistics, cf. e.g. [6, Section 1.4].

Definition 3.1 (Median)**.** Let $f : [a, b] \to \mathbb{R}$ be a measurable function. We say that the number $\lambda \in \mathbb{R}$ is the *median* of the function f on $[a, b]$ if there exists a measurable set $M \subset [a, b]$ such that $\mu(M) = \frac{1}{2}(b - a)$, $f \le \lambda$ on M and $f \ge \lambda$ on $[a, b] \setminus M$.

Definition 3.2 (p-median)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in [1, \infty]$. We say that the number $c(p) \in \mathbb{R}$ is the *p-median* of the function f on I if

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \|f - c(p)\|_p.$$

Remark 3.1. The existence of p-median is obvious. Indeed, since the function $g(c) := \|f - c\|_p$ is non-negative, continuous and its limits at $\pm\infty$ are $+\infty$, it follows that it has a minimum.

As we will prove later, median coincides with p-median for $p = 1$. We will also show that for $p = 2$ the p-median coincides with the integral mean value of the given function, while for $p = \infty$ it is simply the arithmetic mean of essential supremum and infimum of the given function. Furthermore, if $f \in L^\infty(I)$, then the relations $\operatorname{ess\,inf} f \le c(p) \le \operatorname{ess\,sup} f$ hold for all $p \in [1, \infty]$.

First, we will show that median of a measurable function always exist.

Proposition 3.1. *Every measurable function $f : [a, b] \to \mathbb{R}$ has a median.*

Proof. Let a measurable function $f : [a, b] \to \mathbb{R}$ be given and

$$S_1 := \Big\{ \lambda \in \mathbb{R};\ \mu\big(f^{-1}((-\infty, \lambda))\big) \le \frac{b-a}{2} \Big\},$$
$$S_2 := \Big\{ \lambda \in \mathbb{R};\ \mu\big(f^{-1}((\lambda, +\infty))\big) \le \frac{b-a}{2} \Big\}.$$

The monotonicity of measure implies that $h_1(\lambda) := \mu(f^{-1}((-\infty, \lambda)))$ is non-decreasing on $\mathbb{R}$ and $h_2(\lambda) := \mu(f^{-1}((\lambda, +\infty)))$ is non-increasing on $\mathbb{R}$. Moreover,

$$0 \le h_i(\lambda) \le b - a \quad \text{for all} \quad \lambda \in \mathbb{R} \quad \text{and} \quad i \in \{1, 2\}.$$

Denote $A_k := f^{-1}((-\infty, k))$ for $k \in \mathbb{N}$. Then $A_k \subset A_{k+1}$ for each k and in view of the continuity of measure we get

$$\lim_{k\to\infty} \mu(A_k) = \mu\Big(\bigcup_{k=1}^{\infty} A_k \Big) = b - a.$$

Therefore, there is a $k_1 \in \mathbb{N}$ such that $\mu(f^{-1}((-\infty, k_1))) > \frac{b-a}{2}$. Hence, S_1 is bounded from above. Next, we will show that it is non-empty. To this aim, put $B_k := f^{-1}((-\infty, -k))$ for $k \in \mathbb{N}$. We have $B_{k+1} \subset B_k$ for each k and all these sets have finite measures. Thus, using the continuity of measure again, we obtain

$$\lim_{k\to\infty} \mu(B_k) = \mu\Big(\bigcap_{k=1}^{\infty} B_k \Big) = 0.$$

Therefore, there is a $k_2 \in \mathbb{N}$ such that

$$\mu\big(f^{-1}((-\infty, -k_2))\big) < \frac{b-a}{2}\,.$$

In other words, $S_1 \ne \varnothing$. Analogously, we can prove that S_2 is nonempty and bounded from below.

Obviously, $\lambda_1 = \sup S_1 < \infty$ and $-\infty < \lambda_2 = \inf S_2$. Moreover, it is easy to see that $\lambda_2 \le \lambda_1$. Indeed, if the opposite was true, we could find numbers c_1, c_2 such that $\lambda_1 < c_1 < c_2 < \lambda_2$. In such a case we would have

$$\mu\big(f^{-1}((-\infty, c_1))\big) > \frac{b-a}{2} \quad \text{and} \quad \mu\big(f^{-1}((c_2, +\infty))\big) > \frac{b-a}{2}$$

a contradiction, since

$$f^{-1}((-\infty, c_1)) \cap f^{-1}((c_2, \infty)) = \varnothing,$$

while

$$f^{-1}((-\infty, c_1)) \cup f^{-1}((c_2, \infty)) = [a, b].$$

Let $\lambda_2 < \lambda_1$ and let an arbitrary $\xi \in (\lambda_2, \lambda_1)$ be given. Then we can choose $\xi_1 \in S_1$ and $\xi_2 \in S_2$ in such a way that $\lambda_2 < \xi_2 < \xi < \xi_1 < \lambda_1$. By the definitions of the sets S_1, S_2, we have

$$\mu\big(f^{-1}((-\infty, \xi])\big) \le \mu\big(f^{-1}((-\infty, \xi_1))\big) \le \frac{b-a}{2} \quad \text{and} \quad \mu\big(f^{-1}((\xi, \infty))\big) \le \mu\big(f^{-1}((\xi_2, \infty))\big) \le \frac{b-a}{2}\,.$$

Thus,

$$\mu\big(f^{-1}((-\infty,\xi])\big)+\mu\big(f^{-1}((\xi,\infty))\big)\le\mu\big(f^{-1}((-\infty,\xi_1))\big)+\mu\big(f^{-1}((\xi_2,\infty))\big)\le b-a. \tag{3.1}$$

On the other hand, $f^{-1}((-\infty,\xi])\cup f^{-1}((\xi,\infty))=[a,b]$ and this together with (3.1) yields

$$\mu\big(f^{-1}((-\infty,\xi])\big)+\mu\big(f^{-1}((\xi,\infty))\big)=b-a,$$

i.e. any $\xi\in(\lambda_1,\lambda_2)$ is the median of f.

It remains to consider the case $\lambda_1=\lambda_2$. Thus, let $\lambda^*:=\lambda_1=\lambda_2$. Then, since $(-\infty,\lambda^*)\subset S_1$, we have $\mu(f^{-1}((-\infty,\lambda)))\le\frac{1}{2}(b-a)$ for all $\lambda<\lambda^*$ and, thanks to the continuity of measure,

$$\mu\big(f^{-1}((-\infty,\lambda^*))\big)=\lim_{\lambda\to\lambda^*}\mu\big(f^{-1}((-\infty,\lambda))\big)\le\frac{b-a}{2}\,. \tag{3.2}$$

Similarly,

$$\mu(f^{-1}(\lambda^*,\infty))\le\frac{b-a}{2}\,. \tag{3.3}$$

If one of the relations (3.2), (3.3) reduces to the equality, then λ^* will be the median of f. Indeed, if $\mu(f^{-1}(\lambda^*,\infty))=\frac{b-a}{2}$, then for $M=f^{-1}(\lambda^*,\infty))$, we have

$$f(M)=(\lambda^*,\infty),\ \ \mu(M)=\frac{b-a}{2}\ \text{ and }\ f([a,b]\setminus M)\subset(-\infty,\lambda^*].$$

Now, assume that both inequalities (3.2) and (3.3) are strict. Then, as obviously

$$[a,b]=f^{-1}\big(((-\infty,\lambda^*))\big)\cup f^{-1}(\{\lambda^*\})\cup f^{-1}(((\lambda^*,\infty))),$$

the set $f^{-1}(\{\lambda^*\})$ is nonempty and $\mu(f^{-1}(\{\lambda^*\}))>0$. We can define

$$h(t)=\mu\big([a,t]\cap f^{-1}(\{\lambda^*\})\big)\ \text{ for }\ t\in[a,b].$$

As h is continuous on $[a,b]$, $h(a)=0$ and $h(b)=b-a$, we can find a $t_0\in[a,b]$ such that

$$h(t_0)=\mu\big([a,t_0]\cap f^{-1}(\{\lambda^*\})\big)=\frac{b-a}{2}-\mu(f^{-1}(-\infty,\lambda^*))>0.$$

Furthermore,

$$f^{-1}(\{\lambda^*\})=A\cup B,$$

where

$$A:=[a,t_0]\cap f^{-1}\big(\{\lambda^*\}\big)\ \text{ and }\ B:=(t_0,b]\cap f^{-1}\big(\{\lambda^*\}\big)$$

are disjoint. Simultaneously,

$$\mu\big(A\cup f^{-1}(-\infty,\lambda^*)\big)=\frac{b-a}{2}\,,\quad \mu\big(B\cup f^{-1}((\lambda^*,\infty))\big)=\frac{b-a}{2}\,,$$
$$f(x)\le\lambda^*\ \text{ for }\ x\in A\cup f^{-1}((-\infty,\lambda^*))\ \text{ and }\ f(x)\ge\lambda^*\ \text{ for }\ x\in B\cup f^{-1}((\lambda^*,\infty)).$$

It follows easily that λ^* is the median of the function f. □

Example 3.1. Median doesn't have to be uniquely determined, as shown by the following example. The median of the function $f:[0,2]\to\mathbb{R}$ given by the formula

$$f(x)=\begin{cases}0 & \text{if } x\in[0,1),\\ 1 & \text{if } x\in[1,2]\end{cases}$$

can be any number from the interval $[0,1]$. Indeed, let $M=[0,1)$. Then given an arbitrary $\lambda\in[0,1]$, we have $f\le\lambda$ on M and $f\ge\lambda$ on $[0,2]\setminus M$.

On the other hand, it is easy to verify that if $p>1$, then all the p-medians of the function f on $[0,2]$ are equal to $\frac{1}{2}$.

Example 3.2. The median of the function $\sin x$ on $[-\pi, \pi]$ is zero, as well as all its p-medians with $p > 1$ (shown in [3]).

Example 3.3. The median of the function $\sin x$ on the interval $[0, \pi]$ equals $\frac{1}{2}\sqrt{2}$ because $\sin x \geq \frac{\sqrt{2}}{2}$ for all $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ and $\sin x \leq \frac{\sqrt{2}}{2}$ for all $x \in [0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$. On the other hand, its p-median $c(\infty)$ for $p = \infty$ equals $\frac{1}{2}$, while

$$c(2) = \frac{1}{\pi} \int_0^\pi \sin x \, \mathrm{d}x = \frac{2}{\pi} \,.$$

Proposition 3.2. *Median of the* ***continuous*** *function $f : [a, b] \to \mathbb{R}$ is uniquely determined.*

Proof. For a contradiction, let us suppose that f has two medians λ_1, λ_2 such that $\lambda_1 < \lambda_2$. Then, by the definition of the median, there are measurable sets $M_1, M_2 \subset [a, b]$, of measure $\frac{b-a}{2}$ and such that

$$f(x) \leq \lambda_1 \text{ for all|; } x \in M_1 \text{ and } f(x) \geq \lambda_1 \text{ for all|; } x \in [a, b] \setminus M_1$$

and

$$f(x) \leq \lambda_2 \text{ for all|; } x \in M_2 \text{ and } f(x) \geq \lambda_2 \text{ for all|; } x \in [a, b] \setminus M_2.$$

Using these properties, we get

$$\frac{b-a}{2} = \mu(M_2) \geq \mu\big(f^{-1}((-\infty, \lambda_2))\big) \geq \mu(M_1) + \mu\big(f^{-1}((\lambda_1, \lambda_2))\big) = \frac{b-a}{2} + \mu\big(f^{-1}((\lambda_1, \lambda_2))\big).$$

It follows that $\mu(f^{-1}((\lambda_1, \lambda_2))) = 0$. However, the preimage of an open interval (λ_1, λ_2) under a continuous mapping f must be an open set. Therefore, $f^{-1}((\lambda_1, \lambda_2))$ is an open set of measure zero, so it must be empty. Thus, the range H_f of the function f must be a subset of the set $[0, \lambda_1] \cup [\lambda_2, \infty]$. Since the continuous image of the interval $[a, b]$ is again an interval, it must be either $H_f \subset [0, \lambda_1]$ or $H_f \subset [\lambda_2, \infty]$. But, in the former case it is $f(x) \leq \lambda_1$ for all $x \in [a, b]$ which implies that λ_2 can not be the median of f. Similarly, in the latter case we have $f(x) \geq \lambda_2$ for all $x \in [a, b]$ which means that λ_1 can not be the median of f. These conclusions contradicts our assumption, of course. □

Proposition 3.3. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^\infty(I)$. Put*

$$A := \operatorname*{ess\,inf}_{x \in I} f(x) \;\; \textit{and} \;\; B := \operatorname*{ess\,sup}_{x \in I} f(x).$$

Then

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \inf_{c \in [A,B]} \|f - c\|_p \;\; \textit{for all} \;\; p \in [1, \infty].$$

Furthermore,

$$\inf_{c \in \mathbb{R}} \|f - c\|_\infty = \left\| f - \frac{A+B}{2} \right\|_\infty = \frac{B-A}{2} \,.$$

Proof.
(i) First, let us prove the first part of the statement, i.e. that the sought number c will always lie in the interval $[A, B]$. In other words, we want to show that

$$\inf_{z \in [A,B]} \|f - z\|_p \leq \|f - c\|_p \text{ for all } c \in (-\infty, A) \cup (B, \infty).$$

If $c \in (B, \infty)$, then for almost all $x \in [a, b]$ we have

$$|f(x) - c| = c - f(x) > B - f(x) = |f(x) - B|.$$

Consequently, $|f(x)-c|^p > |f(x)-B|^p$ for a.e. $x \in I$ and, thus, $\|f-c\|_p \geq \|f-A\|_p$.

In case $p=\infty$ we have

$$\|f-c\|_\infty \geq \|f-B\|_\infty \ \text{ if } \ c>B \ \text{ and } \ \|f-c\|_\infty \geq \|f-A\|_\infty \ \text{ if } \ c<A.$$

To summarize,

$$\inf_{c\in\mathbb{R}} \|f-c\|_p = \inf_{c\in[A,B]} \|f-c\|_p \ \text{ for all } \ p\in[1,\infty].$$

(ii) Let us prove the remaining part of the statement. If $c \in \mathbb{R}$ is an arbitrary constant, then

$$\operatorname{ess\,inf}(f(x)-c) = A-c \ \text{ and } \ \operatorname{ess\,sup}(f(x)-c) = B-c.$$

Thus

$$\|f-c\|_\infty = \max\big\{|A-c|, |B-c|\big\}.$$

Function $y(c)=\max\{|A-c|,|B-c|\}$ has a minimum for $c=\frac{A+B}{2}$ and, therefore,

$$\inf_{c\in\mathbb{R}} \|f-c\|_\infty = \max\left\{\left|A-\frac{A+B}{2}\right|, \left|B-\frac{A+B}{2}\right|\right\} = \frac{B-A}{2}\,. \qquad \square$$

Remark 3.2. Analogously, if instead of p-norm we consider the supremum norm $\|f\| = \sup\limits_{x\in I}|f(x)|$, we get

$$\inf_{c\in\mathbb{R}} \|f-c\| = \left\|f-\frac{1}{2}\big(\sup f(x)+\inf f(x)\big)\right\|.$$

Proposition 3.4. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^2(I)$. Then*

$$\inf_{c\in\mathbb{R}} \|f-c\|_2 = \left\|f-\frac{1}{\mu(I)}\int_I f(t)\,\mathrm{d}t\right\|_2.$$

In other words, for $p=2$ the p-median of f equals to the integral mean value of f.

Proof. Let $c\in\mathbb{R}$. Since $L^2(I)\subset L^1(I)$ for I bounded, both integrals $\int_I f(x)\,\mathrm{d}x$ and $\int_I f^2(x)\,\mathrm{d}x$ exist and are finite. Therefore,

$$g(c) := \|f-c\|_2^2 = \int_I (f(x)-c)^2\,\mathrm{d}x = \int_I f^2(x)\,\mathrm{d}x - 2c\int_I f(x)\,\mathrm{d}x + c^2\mu(I).$$

This is a quadratic function of c with a positive leading coefficient and thus it must have a minimum. Its derivative is

$$g'(c) = -2\int_I f(x)\,\mathrm{d}x + 2c\,\mu(I).$$

Hence

$$g'(c)=0 \ \text{ if and only if } \ c=\frac{1}{\mu(I)}\int_I f(x)\,\mathrm{d}x.$$

This is its stationary point, and the function g takes a minimum there. Therefore, it is also a minimum of the function $\|f-c\|_2$. $\square$

4 Oscillations and HK^p integral

In this section we introduce the notions of oscillation which is the key concept in the definition of HKS^p integral. Next definition is taken from [5, Definition 2.3].

Definition 4.1 (Oscillations)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $p \in [1, \infty]$. We define the p-oscillation of a measurable function $f : I \to \mathbb{R}$ as

$$\operatorname{osc}_p(f, I) := (\mu(I))^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f - c\|_p.$$

Here and in what follows we set $\frac{1}{p} = 0$ if $p = \infty$.

The following proposition is taken from [5, Proposition 2.6].

Proposition 4.1 (Oscillation and median relation)**.** *Let $\lambda \in \mathbb{R}$ be the median of the function f on the bounded interval $I \subset \mathbb{R}$ and $p \in [1, \infty]$. Then*

$$\operatorname{osc}_p(f, I) \le (\mu(I))^{-\frac{1}{p}} \|f - \lambda\|_p \le 2^{1-\frac{1}{p}} \operatorname{osc}_p(f, I).$$

In particular, for $p = 1$ we get

$$\operatorname{osc}_1(f, I) = (\mu(I))^{-1} \|f - \lambda\|_1. \tag{4.1}$$

It implies that

$$\inf_{c \in \mathbb{R}} \|f - c\|_1 = \|f - \lambda\|_1.$$

In other words, median coincides with p-median for $p = 1$.

Next, we will introduce the new definition of generalized Kurzweil integral based on minimization of sum of p-oscillations instead of ordinary oscillations which leads to a wider class of integrable functions.

Definition 4.2. We say that $\{[a_i, b_i], x_i\}_{i=1}^n$ ($n \in \mathbb{N}$) is a **tagged partition** of the interval $I \subset \mathbb{R}$ if the intervals $[a_i, b_i]$ are non-overlapping, their union is I and $x_i \in [a_i, b_i]$ for every $i \in \{1, \dots, n\}$.

Definition 4.3. Let an arbitrary positive function $\delta : [a, b] \to \mathbb{R}^+$ be given. We say that the tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ is **δ-fine** if

$$[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i)) \text{ for all } i \in \{1, \dots, n\}.$$

Definition 4.4 (Generalized Kurzweil integral)**.** Let $I \subset \mathbb{R}$ be an interval, f, F be functions measurable on I. We say that F is an indefinite **HK$^{\mathbf{p}}$ integral** of a function f if for all $\varepsilon > 0$ there exists $\delta_\varepsilon : I \to \mathbb{R}^+$ such that

$$\sum_{i=1}^n \operatorname{osc}_p \big(F - f(x_i)\, x, [a_i, b_i]\big) < \varepsilon$$

holds for each δ_ε**-fine** tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ of the interval I.

5 Examples

Example 5.1. Next example shows that even if the p-median $c(p)$ is determined uniquely for all $p\in[1,\infty]$, the function $p\mapsto c(p)$ **need not be monotone**, in general. Indeed, for the function

$$f(x)=\begin{cases}\sin^2 x & \text{if } x\in[0,\pi],\\ \sin x & \text{if } x\in(\pi,2\pi]\end{cases}$$

we have $c(1)=c(\infty)=0$. On the other hand, $c(2)$ is negative, as by Proposition 3.4 we have

$$c(2)=\frac{1}{2\pi}\int\limits_0^{2\pi} f(x)\,\mathrm{d}x=\frac{1}{2\pi}\left(\frac{\pi}{2}-2\right)<0.$$

Example 5.2. Let $f(x)=\begin{cases}1 & \text{if } x\in J,\\ 0 & \text{if } x\in I\setminus J,\end{cases}$ where $I\subset\mathbb{R}$ is bounded interval and $J\subset I$ its subinterval.

It was shown in [3, Example 2.2.6] that the p-medians $c(p)$ of this function are uniquely determined and they are explicitly given by the formula

$$c(p)=\left(\left(\frac{\mu(I\setminus J)}{\mu(J)}\right)^{\frac{1}{p-1}}+1\right)^{-1}\ \text{ for } p\in(1,\infty).$$

Notice that the limit of p-medians as $p\to+\infty$ is indeed the arithmetic mean of essential suprema and infima, i.e $\lim\limits_{p\to+\infty} c(p)=\frac{1}{2}=c(\infty)$. Further, notice also that

$$\lim_{p\to1+} c(p)=1=c(1)\ \text{ if } \mu(J)>\mu(I\setminus J)\ \text{ and }\ \lim_{p\to1+} c(p)=0=c(1)\ \text{ if } \mu(J)<\mu(I\setminus J),$$

i.e. the limit of p-medians $c(p)$ as $p\to1+$ is indeed the median of f.

If $\mu(I\setminus J)=\mu(J)$, then $c(p)=\frac{1}{2}$ for all $p\in(1,\infty]$, while the median of f is not unique as it can be any number from the interval $[0,1]$.

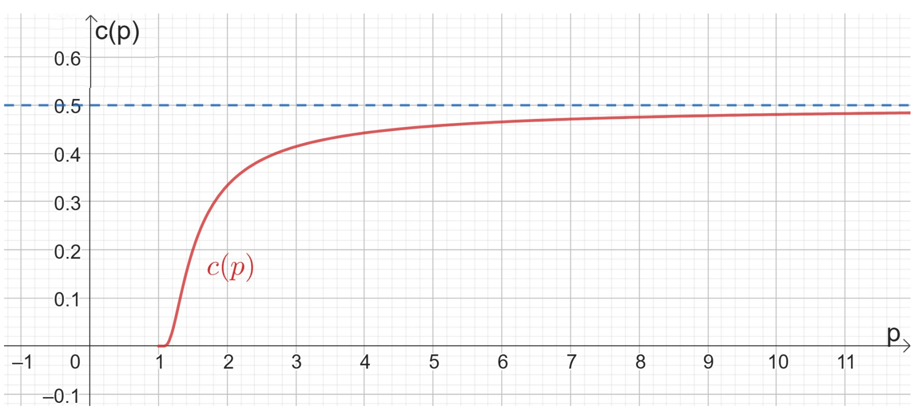

Figure 1. Graph of function $c(p)$ in case $\frac{\mu(I\setminus J)}{\mu(J)}=2$

Example 5.3. Let

$$f(x)=\begin{cases}8 & \text{if } x\in[0,1],\\ 0 & \text{if } x\in(1,6],\\ -4 & \text{if } x\in(6,10].\end{cases}$$

Then the p-median $c(p)$ of f on $[0,10]$ is determined uniquely for any $p \in [1,\infty]$, but no explicit formula for $c(p)$ is available. One can verify that $c(1) = c(3) = 0$, while $c(2) < 0$ and $c(\infty) > 0$. Thus, the function $p \mapsto c(p)$ is not monotone. Notice that the arithmetic mean of suprema and infima is $c(\infty) = 2$, while the integral mean value evaluates $c(2) = -\frac{4}{5}$.

For a given $p > 1$ let us denote again $g(c) := \|f-c\|_p^p$. Obviously, $g(c) = (8-c)^p + 5|c|^p + 4(4+c)^p$ and g is continuous $[-4,8]$. Furthermore, $g'(c) = -p(8-c)^{p-1} + 5p|c|^{p-1}\operatorname{sgn} c + 4p(4+c)^{p-1}$ for $c \neq 0$. One can verify that g' is continuous and increasing on $[-4,0) \cup (0,8]$, while $g'(-4) < 0$, $g'(8) > 0$ and $g'(0-) = g'(0+) = p(4^p - 8^{p-1}))$. In particular, for a given $p \in (1,\infty)$, there is exactly one point $c(p) \in (-4,8)$ such that $g'(c(p)) = 0$. This defines implicitly the function $p \mapsto c(p)$. In addition, g is decreasing on $[-4, c(p)]$ and increasing on $[c(p), 8]$. Finally, notice that $g'(0-) = g'(0+) = 0$ if and only if $8^{p-1} = 4^p$, i.e. if and only if $p = 3$, i.e. $c(3) = 0$.

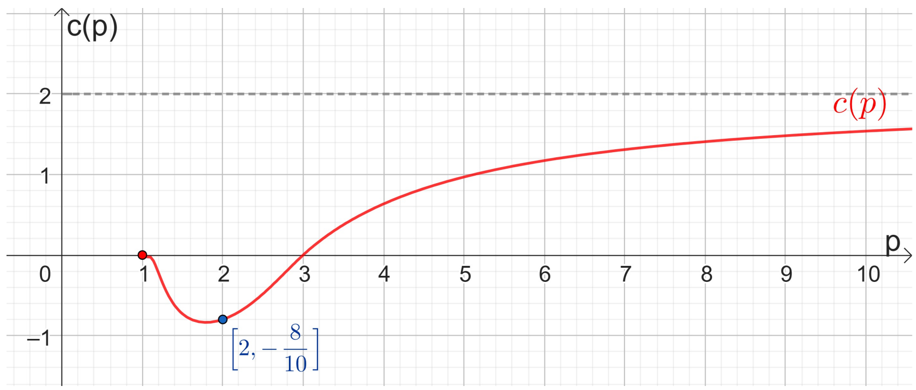

Figure 2. Graph of $c(p)$

6 Open problems

- (Uniqueness of p-median) Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in (1,\infty]$. Is there a unique number $c(p) \in \mathbb{R}$ such that

$$\inf_{c\in\mathbb{R}} \|f-c\|_p = \|f-c(p)\|_p\,?$$

  We have proved the uniqueness of p-medians if $p > 1$ for step function, analogously as in Example 5.3, but still we don't know if there is uniqueness in general for $f \in L^p(I)$. If yes, it would be interesting to investigate properties of function $p \mapsto c(p)$.

- (Limits of p-medians) In Example 5.2 we have seen that for the function f considered there the limit of p-medians $c(p)$ as $p \to \infty$ is $c(\infty)$ and the limit of $c(p)$ as $p \to 1+$ is the median of f. The question is whether this is true in general.

- (Properties of p-medians) Is function $p \mapsto c(p)$ continuous? Is it differentiable? Is it true that $\lim\limits_{p\to 1_+} c'(p) = 0$?

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Application of the Retract Principle to Find Solutions of Discrete Nonlinear Equations

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Let us show using the example of several nonlinear difference equations the possibility of estimating the properties of the solutions using a discrete analogue of the retract principle. To describe this principle, we need to consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \quad k \in \mathbb{N}(k_0), \tag{0.1}$$

where $Y = (Y_0, \ldots, Y_{n-1})^T$ and

$$F(k, Y) = \big(F_1(k, Y), \ldots, F_n(k, Y)\big)^T : \mathbb{N}(k_0) \times \mathbb{R}^n \to \mathbb{R}^n. \tag{0.2}$$

A solution $Y = Y(k)$ of system (0.1) is defined as a function $Y : \mathbb{N}(k_0) \to \mathbb{R}^n$ satisfying (0.1) for each $k \in \mathbb{N}(k_0)$. The initial problem

$$Y(k_0) = Y^0 = (Y^0_0, \ldots, Y^0_{n-1})^T \in \mathbb{R}^n$$

defines a unique solution to (0.1). Obviously, if $F(k, Y)$ is continuous with respect to Y, then the initial problem (0.1), (0.2) defines a unique solution $Y = Y(k_0, Y^0)(k)$, where $Y(k_0, Y^0)$ indicates a dependence of the solution on the initial point (k_0, Y^0), which depends continuously on the value Y^0. Let $b_i, c_i : \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, \ldots, n$ be given functions, satisfying

$$b_i(k) < c_i(k), \quad k \in \mathbb{N}(k_0), \quad i = 1, \ldots, n.$$

Define auxiliary functions $B_i, C_i : \mathbb{N}(k_0) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$ as

$$B_i(k, Y) := -Y_{i-1} + b_i(k), \quad C_i(k, Y) := Y_{i-1} - c_i(k),$$

and auxiliary sets

$$\Omega^i_B := \Big\{(k, Y) : \ k \in \mathbb{N}(k_0), \ B_i(k, Y) = 0, \ B_j(k, Y) \le 0, \ C_p(k, Y) \le 0, \ \forall\, j, p = 1, \ldots, n, \ j \ne i\Big\},$$

$$\Omega^i_C := \Big\{(k, Y) : \ k \in \mathbb{N}(k_0), \ C_i(k, Y) = 0, \ B_j(k, Y) \le 0, C_p(k, Y) \le 0, \ \forall\, j, p = 1, \ldots, n, \ p \ne i\Big\},$$

where $i = 1, \ldots, n$.

Playing a crucial role in the proofs and being suitable for applications, the following lemma is a slight modification of [3, Theorem 1] (see [5, Theorem 2] also).

Definition 0.1. The set Ω is called the *regular polyfacial set* with respect to the discrete system (0.1) if
$$b_i(k+1)-b_i(k)<F_i(k,Y)<c_i(k+1)-b_i(k),$$
for every $i=1,\dots,n$ and every $(k,Y)\in\Omega_B^i$ and if
$$b_i(k+1)-c_i(k)<F_i(k,Y)<c_i(k+1)-c_i(k),$$
for every $i=1,\dots,n$ and every $(k,Y)\in\Omega_C^i$.

To formulate the following theorem, we need to define sets
$$\Omega(k)=\Big\{(k,Y):\ \ Y=(Y_1,\dots,Y_n)\in\mathbb{R}^n,\ \ b_i(k)<Y_i<c_i(k),\ \ i=1,\dots,n\Big\},$$
$$\Omega_i(k)=\Big\{(Y):\ \ Y\in\mathbb{R},\ \ b_i(k)<Y_i<c_i(k),\ \ i=1,\dots,n\Big\}.$$

Theorem 0.1 ([4, Theorem 4])**.** *Let $F:\mathbb{N}(k_0)\times\overline{\Omega}\to\mathbb{R}^n$. Let, moreover, Ω be regular with respect to the discrete system* (0.1)*, and let the function*
$$G_i(w):=w+F_i(k,Y_1,\dots,Y_{i-1},w,Y_{i+1},\dots,Y_n)$$
be monotone on $\overline{\Omega}_i(k)$ for every fixed $k\in\mathbb{N}(k_0)$, each fixed $i\in\{1,\dots,n\}$, and every fixed
$$(Y_1,\dots,Y_{i-1},Y_{i+1},\dots,Y_n)$$
such that $(k,Y_1,\dots,Y_{i-1},w,Y_{i+1},\dots,Y_n)\in\Omega$. Then, every initial problem $Y(k_0)=Y^*$ with $Y^*\in\Omega(k_0)$ defines a solution $Y=Y^*(k)$ of the discrete system (0.1)*, satisfying the relation*
$$Y^*(k)\in\Omega(k)$$
for every $k\in\mathbb{N}(k_0)$.

Now we formulate a result which is proved in [3] by a retract method sometimes called an Anti-Liapunov method due to the assumptions used being often an opposite to those used when Liapunov method is applied (such an approach goes back to Ważewski, who formulated his topological method formulated for ordinary differential equations). The following theorem is a slight modification of [3, Theorem 1] (see [5, Theorem 2] also).

Theorem 0.2. *Assume that the function $F(k,Y)$ satisfies* (0.1) *and is continuous with respect to Y. Let the inequality*
$$F_i(k,Y)<b_i(k+1)-b_i(k)$$
hold for every $i=1,\dots,n$ and every $(k,Y)\in\Omega_B^i$. Let, moreover, the inequality
$$F_i(k,Y)>c_i(k+1)-c_i(k)$$
hold for every $i=1,\dots,n$ and every $(k,Y)\in\Omega_C^i$. Then, there exists a solution $Y=Y(k)$, $k\in\mathbb{N}(k_0)$ of system (0.1)*, satisfying the inequalities*
$$b_i(k)<Y_{i-1}(k)<c_i(k)$$
for every $k\in\mathbb{N}(k_0)$ and $i=1,\dots,n$.

Definition 0.2. A function $u_{upp}:\mathbb{B}\to\mathbb{R}$ is said to be an *approximate solution* to equation (0.1) of an order g, where $g:\mathbb{N}(k_0)\to\mathbb{R}$, if
$$\lim_{k\to\infty}\big[\Delta^3u_{upp}(k)\pm k^\alpha u_{upp}^n(k)\big]g(k)=0.$$

If the main term (i.e. the term being asymptotically leading) in $u_{upp}(k)$ is a power-type function, we say that it is a *power-type* approximate solution.

1 Discrete analogue of Emden–Fowler second-order non-linear equation

Now let us consider the following second-order non-linear equation

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0, \tag{1.1}$$

where $u : \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution, $\Delta u(k)$ is its first-order forward difference, i.e.,

$$\Delta u(k) = u(k+1) - u(k),$$

$\Delta^2(k)$ is its second-order forward difference, i.e.,

$$\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k),$$

and α, m are real numbers. A function $u = u^* : \mathbb{N}(k_0) \to \mathbb{R}$ is called a solution of equation (1.1) if the equality

$$\Delta^2 u^*(k) \pm k^\alpha (u^*(k))^m = 0$$

holds for every $k \in \mathbb{N}(k_0)$.

Equation (1.1) is a discretization of the classical Emden–Fowler second-order differential equation (we refer, e.g., to [2])

$$y'' \pm x^\alpha y^m = 0,$$

where the second-order derivative is replaced by a second-order forward difference and the continuous independent variable is replaced by a discrete one.

Remark 1.1. We need to assume $m \neq 0$, $m \neq 1$, $s + 2 \neq 0$, and $s + 2 - ms \neq 0$, that is, $m \neq 0$, $m \neq 1$, $\alpha \neq -2$, and $\alpha \neq -2m$.

Let us define

$$\begin{aligned} s &= \frac{\alpha+2}{m-1}\,, \\ a &= \big[\mp s(s+1)\big]^{1/(m-1)}, \end{aligned} \tag{1.2}$$

and

$$b = \frac{as(s+1)}{s+2-ms}\,.$$

Remark 1.2. If, in formula (1.2), either the upper variant of sign is in force (i.e. $-$) and $s(s+1) > 0$ or in (1.2) lower variant of sign in force (i.e. $+$) and $s(s+1) < 0$, then the constant m has the form of a ratio m_1/m_2 of relatively prime integers m_1, m_2, and m_2 is odd, the difference $m_1 - m_2$ is odd as well. If this convention holds, formula (1.2) defines two or at least one value. As equation (1.1) splits into two equations, when formulating the results, we assume that a concrete variant is fixed (either with the sign $+$ or with the sign $-$).

Previously in [1, 7, 8] the conditions on the existence of a power-type solution of equation (1.1) were discussed.

Theorem 1.1. *If there exist $\gamma \in (0,1)$, s and $\varepsilon_i > 0$, $i = 1,2,3,4$, such that $P \equiv \frac{\gamma+s+1}{s+1}$ and $Q \equiv \frac{\gamma+s+2}{ms}$ and at least one of the following four conditions is true*

(1) *$ms > 0$, $s > -1$, $\varepsilon_3 < \varepsilon_1 P$, $\varepsilon_4 < \varepsilon_2 P$, $\varepsilon_1 < \varepsilon_3 Q$, $\varepsilon_2 < \varepsilon_4 Q$;*

(2) *$ms < 0$, $s > -1$, $\varepsilon_3 < \varepsilon_1 P$, $\varepsilon_4 < \varepsilon_2 P$, $\varepsilon_2 < -\varepsilon_3 Q$, $\varepsilon_1 < -\varepsilon_4 Q$;*

(3) $ms<0$, $-2\neq s<-1$, $\varepsilon_4<-\varepsilon_1P$, $\varepsilon_3<-\varepsilon_2P$, $\varepsilon_2<-\varepsilon_3Q$, $\varepsilon_1<-\varepsilon_4Q$;

(4) $ms>0$, $-2\neq s<-1$, $\varepsilon_4<-\varepsilon_1P$, $\varepsilon_3<-\varepsilon_2P$, $\varepsilon_1<\varepsilon_3Q$, $\varepsilon_2<\varepsilon_4Q$,

then there exists K such that for all $k_0>K$ there exists a solution $u(k)$ to equation (1.1) *such that for all $k\in\mathbb{N}(k_0)$ the following inequalities*

$$-\frac{\varepsilon_1}{k^\gamma}<\Big(u(k)-\frac{a}{k^s}-\frac{b}{k^{s+1}}\Big)\Big(\frac{b}{k^{s+1}}\Big)^{-1}<\frac{\varepsilon_2}{k^\gamma}\,, \tag{1.3}$$

$$-\frac{\varepsilon_3}{k^\gamma}<\Big(\Delta u(k)-\Delta\Big(\frac{a}{k^s}\Big)-\Delta\Big(\frac{b}{k^{s+1}}\Big)\Big)\Big(\Delta\Big(\frac{b}{k^{s+1}}\Big)\Big)^{-1}<\frac{\varepsilon_4}{k^\gamma}\,, \tag{1.4}$$

$$-\frac{\varepsilon_1}{k^\gamma}+O\Big(\frac{1}{k}\Big)<\Big(\Delta^2u(k)-\Delta^2\Big(\frac{a}{k^s}\Big)-\Delta^2\Big(\frac{b}{k^{s+1}}\Big)\Big)\Big(\Delta^2\Big(\frac{b}{k^{s+1}}\Big)\frac{ms}{s+2}\Big)^{-1}<\frac{\varepsilon_2}{k^\gamma}+O\Big(\frac{1}{k}\Big) \tag{1.5}$$

hold.

Theorem 1.2. *If there exist $s>-1$ and $\varepsilon_i>0$, $i=1,2,3,4$, such that one of the following conditions hold*

(1) $ms>0$, $\varepsilon_3<\varepsilon_1$, $\varepsilon_2>\varepsilon_4$, $\varepsilon_3>\dfrac{ms}{s+2}\,\varepsilon_1$, $\varepsilon_4>\dfrac{ms}{s+2}\,\varepsilon_2$;

(2) $ms<0$, $\varepsilon_3<\varepsilon_1$, $\varepsilon_2>\varepsilon_4$, $\varepsilon_3>-\dfrac{ms}{s+2}\,\varepsilon_2$, $\varepsilon_4>-\dfrac{ms}{s+2}\,\varepsilon_1$,

then for some K for all $k_0>K$ there exist a solution $u(k)$ to equation (1.1) *such that for all $k\in\mathbb{N}(k_0)$ and $\gamma=0$* (1.3)–(1.5) *hold.*

To prove these theorems we had to transform the discrete second-order non-linear equation to the system of two discrete equations, and applying theorems in preliminaries we get the above theorems. For more details to the proof we refer to [1,7].

2 Another second-order non-linear difference equation

Let us consider the problem of the existence of a nontrivial solution to the equation

$$\Delta^2v(k)=-k^s(\Delta v(k))^3 \tag{2.1}$$

such that the limit $\lim_{k\to\infty}v(k)$ exists and is finite. More exactly, under the condition $s>1$, we prove the existence of a solution to equation (2.1) such that the limit

$$\lim_{k\to\infty}v(k)=0.$$

Theorem 2.1. *Let $s>1$. Let ε_i, γ_i, $i=1,2$ be fixed positive numbers such that $\varepsilon_2<\varepsilon_1<1$, $\gamma_2<\gamma_1<1$. Then there exists a solution $v=v(k)$ to equation* (2.1) *such that*

$$-\varepsilon_1|c|k^{-\alpha}<v(k)-ck^{-\alpha}<\gamma_1|c|k^{-\alpha},$$
$$-\varepsilon_2\gamma_2\Delta\big(|c|k^{-\alpha}\big)<\Delta v(k)-(\Delta(ck^{-\alpha}))<\gamma_2\Delta\big(|c|k^{-\alpha}\big),$$

and

$$\Delta^2v(k)=O(1)$$

for all $k\in\mathcal{Z}_{k_0}^\infty$ provided that k_0 is sufficiently large.

Opposite to the equation in the previous chapter where Theorem 0.1 was used, in this case Theorem 0.2 is applied (for details see [6]).

3 Conclusion

In this article we discussed two different non-linear discrete equations. To prove some properties to its solutions, we used the retract principle described in this article. It can be concluded that other nonlinear discrete differential equations can be investigated in a similar way.

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Finding of Periodic Points of Stokes Flow with a Complex Distribution of Motion Velocities of Rectangular Cavity Walls

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1 Introduction

In recent years, there has been a significant development of interest in the implementation of a qualitative process of mixing in flows in two-dimensional rectangular cavities without the participation of physical mixers in the process itself. This becomes possible when the flow of an incompressible viscous liquid is periodically excited in a rectangular cavity with the help of tangential velocities applied to its walls. The results obtained in this direction relate to problems in which the side walls in a rectangular cavity are free from loads, which is physically impossible to implement. The purpose of the research is to build a similar model, which is proposed in [1, 4], considering the case of fixed side walls in a rectangular cavity. Moreover, the goal is to find periodic points of the third order and establish their type.

2 Setting of the problem and the main results

The movement of individual flow particles is considered in a known velocity field and is reduced to solving the advection equations, which are a system of first-order ordinary differential equations with a complex functional dependence in the right-hand parts:

$$\frac{dx_i(t)}{dt} = f(x,y), \quad \frac{dy_i(t)}{dt} = g(x,y), \quad i=\overline{1,n} \tag{2.1}$$

with the initial conditions

$$x_i(t) = x_{i0}, \quad y_i(t) = y_{i0}, \quad i=\overline{1,n}.$$

Two-dimensional slow flow of an incompressible viscous fluid can be represented in terms of a biharmonic problem. If such a motion is so slow that the inertial forces containing the squares of the velocities can be neglected compared to the viscous terms, then the stream function ψ satisfies the biharmonic equation

$$\Delta^2\psi = 0. \tag{2.2}$$

In rectangular coordinates, the Euler components of the velocity vector u and v are defined as

$$u = \frac{\partial\psi}{\partial y}\,, \quad v = -\frac{\partial\psi}{\partial x}\,.$$

Flow in a rectangular cavity $|x| \le a$, $|y| \le b$ is caused by the given tangential velocities $U_{top}(x)$ and $U_{bot}(x)$ on the upper $(y=b)$ and lower $(y=-b)$ walls, respectively, and the side walls $x=a$ are stationary (Figure 1).

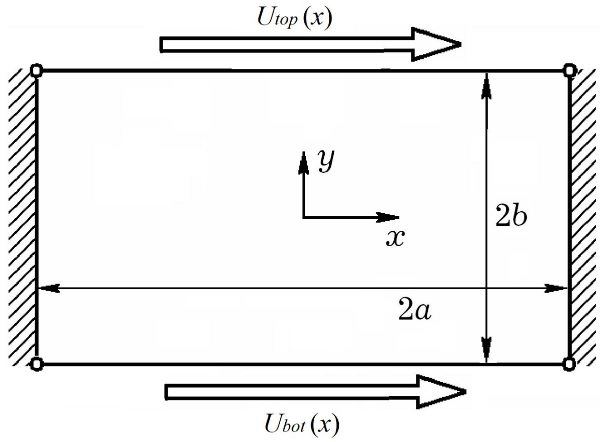

Figure 1. Geometry of a rectangular cavity.

Boundary conditions for equation (2.2) have the form

$$\psi = 0, \quad \frac{\partial \psi}{\partial x} = 0, \quad x = \pm a, \quad |y| \le b, \tag{2.3}$$

$$\psi = 0, \quad \frac{\partial \psi}{\partial y} = \pm U(x), \quad y = \pm b, \quad |x| \le a, \tag{2.4}$$

where

$$U(x) = U_{top}(x) = -U_{bot}(x) = U_1^{(1)} \cos \frac{\pi x}{2a} - U_1^{(2)} \sin \frac{\pi x}{a} \,. \tag{2.5}$$

A detailed description of the construction of the solution to problem (2.2), (2.3), and (2.4) is considered in [2,3]. The resulting solution defines the velocity field, that is, the right-hand sides of the advection equations (2.1).

Important for studying the advection of a passive non-inertial particle is the knowledge of the periodic points of the process of order p, that is, such initial conditions in the advection equation (2.1), when the point accurately returns to its initial position in p periods. A fundamental element of the analysis of the advection process is the classification of periodic points into elliptical and hyperbolic.

We will classify the type of periodic point analytically by determining the eigenvalue λ_1 and λ_2 of the Jacobian matrix of the linearized system (2.1) in the vicinity of the considered point. If λ_1 and λ_2 are complex conjugate, the point is of elliptic type. If λ_1 and $\lambda_2 = \frac{1}{\lambda_1}$ are valid, the time point is of hyperbolic type.

There can also be a situation of $\lambda_1 = \lambda_2 = \pm 1$, which corresponds to the degenerate case where the periodic point is parabolic: in this case, any small change in the velocity field causes the periodic point to become elliptical or hyperbolic.

The Jacobian elements of the matrix M are calculated by solving system (2.1) for four initial conditions $(\overline{x} + \epsilon, \overline{y})$, $(\overline{x} - \epsilon, \overline{y})$, $(\overline{x}, \overline{y} + \epsilon)$, $(\overline{x}, \overline{y} - \epsilon)$, where $(\overline{x}, \overline{y})$ are the rectangular coordinates of a periodic point, and ϵ is an arbitrarily small value,

$$\begin{cases} M_{xx} = \dfrac{x_{(0,pT)}(\overline{x}+\epsilon, \overline{y}) - x_{(0,pT)}(\overline{x}-\epsilon, \overline{y})}{2\epsilon}\,, \quad M_{xy} = \dfrac{x_{(0,pT)}(\overline{x}, \overline{y}+\epsilon) - x_{(0,pT)}(\overline{x}, \overline{y}-\epsilon)}{2\epsilon}\,, \\ M_{yx} = \dfrac{y_{(0,pT)}(\overline{x}+\epsilon, \overline{y}) - y_{(0,pT)}(\overline{x}-\epsilon, \overline{y})}{2\epsilon}\,, \quad M_{xy} = \dfrac{y_{(0,pT)}(\overline{x}, \overline{y}+\epsilon) - y_{(0,pT)}(\overline{x}, \overline{y}-\epsilon)}{2\epsilon}\,, \end{cases} \tag{2.6}$$

where p is the order of the periodic point.

The condition that the determinant of the matrix M must be equal to one is used when checking the accuracy of calculations.

Figure 2 shows periodic points (in red), which have the following coordinates: $A_L = (-2.01, 0)$, $A_C = (0, 0)$, $A_R = (2.01, 0)$. Coordinates of periodic points and parameters $U_1^{(1)}$ and $U_1^{(2)}$ in (2.5) were selected according to the following algorithm:

(1) periodic points A_L and A_C must belong to the same flow line;

(2) points A_L and A_R are equidistant from the central point A_C.

With such values of $U_1^{(1)}$ and $U_1^{(2)}$, the periodic points A_L and A_C pass into each other in a half-period $\tau = \frac{1}{2}T$ ($T = 2$, τ varies from 0 to $\frac{1}{2}T$), exchange positions (the transition occurs clockwise), and the right A_R remains stationary.

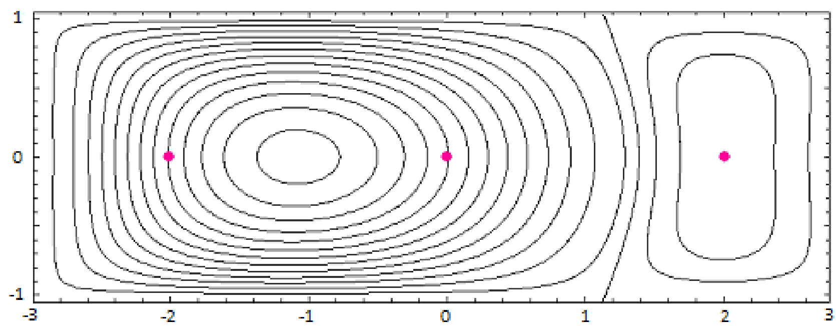

Figure 2. Picture of streamlines in a rectangular cavity $a = 3$ and $b = 1$ with unmovable side walls over the half period $0 < t < \frac{1}{2}T$.

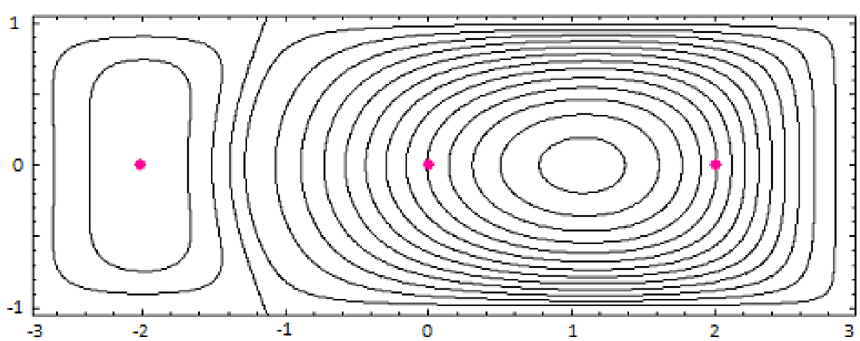

Figure 3. Picture of streamlines in a rectangular cavity $a = 3$ and $b = 1$ with unmovable side walls over the half period $\frac{1}{2}T < t < T$.

In the time period from $\frac{1}{2}T$ to T, the found velocity at the boundaries changes its value to the opposite, begins to act in the opposite direction. In this case, the left periodic point remains stationary, and the central and right point move into each other, changing their positions (the transition occurs counter-clockwise). The corresponding picture of streamlines along with the observed points is shown in Figure 3. In three full periods T, the points will return to their original positions.

Thus, the found points are periodic points of the third order of the elliptic type. The type of these points was determined numerically and analytically according to the methodology proposed in this work. These points play an important role in the theory of mixing liquids and are called “ghost rods”.

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The Averaging Method for Optimal Control Problems of Systems of Integro-Differential Equations

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Abstract

This work is devoted to the optimal control of systems of integro-differential equations with rapidly oscillating coefficients and a small parameter. Using the averaging method, it has been proven that the optimal control of the averaged problem, which is a system of ordinary differential equations, is nearly optimal for the original problem. That is, it minimizes the quality criterion with an accuracy up to ε.

1 Problem statement

We consider the nonlinear optimal control problem of integro-differential system with rapidly oscillating coefficients:

$$\begin{cases} \dot{x} = X\Big(\dfrac{t}{\varepsilon}, x, \displaystyle\int_0^t \varphi(t,s,x(s))\,ds, u(t)\Big), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

and a cost function:

$$J_\varepsilon[u] = \int_0^T L(t, x_\varepsilon(t), u(t))\,dt + \phi(x_\varepsilon(T)) \longrightarrow \inf. \tag{1.2}$$

Here, $\varepsilon > 0$ is a small parameter, $T > 0$ is a constant, x is the phase vector in the domain $D \subset \mathbb{R}^d$, $u(t)$ – m-dimensional control vector from a certain functional set.

Furthermore, $x(t,u)$ is the solution to the Cauchy problem $(1.1), (1.2)$ corresponding to the control $u(t)$. Disregarding the dependence on u, we denote it simply as $x(t)$.

We assume that there exists a function $X_0(x,u)$ such that, for uniformly $x \in \mathbb{R}^d$ and $u \in U$, the following limit exists:

$$\lim_{\varepsilon\to 0} \int_0^t \Big[X\Big(\frac{t}{\varepsilon}, x, \varphi_1(t,x), u\Big) - X_0(x,u)\Big]\,d\tau, \tag{1.3}$$

where

$$\varphi_1(t,x)=\int\limits_0^t \varphi(t,s,x)\,ds,$$

$t\in[0,T]$, $s\in[0,T]$.

Note that condition (1.3) means the integral continuity of the function $X(\frac{\tau}{\varepsilon},x,\varphi(\tau,x),u)$ at the point $\varepsilon=0$ on the $[0,T]$, $x\in D$, $u\in U$.

The optimal control problems (1.1), (1.2) with rapidly oscillating coefficients correspond to a simpler optimal control problem

$$\begin{cases}\dot\xi = X_0(\xi,u(t)),\\ \xi(0)=x_0,\end{cases} \tag{1.4}$$

with a cost function:

$$J_0[u]=\int\limits_0^T L(t,\xi(t),u(t))\,dt+\phi(\xi(T))\longrightarrow \inf. \tag{1.5}$$

For problems (1.1), (1.2), we assume that the following conditions hold.

Condition 1.1. *The admissible controls are m-dimensional vector functions $u(\,\cdot\,)$ such that $u(\,\cdot\,)\in U$ – a compact set in $L^2((0,T))$.*

Condition 1.2. *The function $X(t,x,y,u)$ is defined and continuous with respect to the collection of variables in the domain*

$$Q_0=\Big\{t\in[0,T],\;\; x\in D\subset\mathbb{R}^d,\;\; y\in\mathbb{R}^n,\;\; u\in U\in\mathbb{R}^m\Big\}.$$

(1) *$X(t,x,y,u)$ satisfies the linear growth condition with respect to x, y in Q_0, i.e. there exists a constant $M>0$ such that*

$$|X(t,x,y,u)|\le M\big(1+|x|+|y|\big)$$

for any $(t,x,y,u)\in Q_0$.

(2) *$X(t,x,y,u)$ satisfies the Lipschitz condition with respect to $x\in D\subset\mathbb{R}^d$ and $u\in\mathbb{R}^m$ in Q_0, with constant λ:*

$$\big|X(t,x,y,u)-X(t,x_1,y_1,u_1)\big|\le\lambda\big(|x-x_1|+|y-y_1|+|u-u_1|\big)$$

for any $(t,x,y,u),(t,x_1,y_1,u_1)\in Q_0$.

Condition 1.3. *The function $\varphi(t,s,x)$ is defined and continuous in the domain $Q_1=\{t\in[0,T],\ s\in[0,T],\ x\in D\}$ and satisfies the linear growth and the Lipschitz conditions with respect to x, i.e., $\exists\,L_\varphi$ such that*

$$\begin{aligned}\big|\varphi(t,s,x)-\varphi(t,s,x_1)\big|&\le L_\varphi|x-x_1|,\\ |\varphi(t,s,x)|&\le L_\varphi\big(1+|x|\big).\end{aligned}$$

Condition 1.4. *Uniformly with respect to $x\in D$, $u\in\mathbb{R}^m$, the limit (1.3) exists.*

Condition 1.5. *The function $L(t,x,u)$ is defined and continuous with respect to the collection of arguments in the domain $Q_1=\{t\in[0,T],\ x\in\mathbb{R}^d,\ u\in\mathbb{R}^m\}$, where:*

(1) *$L(t,x,u)$ is uniformly bounded on $[0,T]$ with $u\in\mathbb{R}^m$ and continuous with respect to $x\in\mathbb{R}^d$.*

(2) *$L(t,x,u)$ satisfies the Lipschitz condition with respect to u in Q_1 with constant $\lambda>0$.*

(3) *The function $\phi:\mathbb{R}^d\to\mathbb{R}$ is continuous with respect to x.*

According to Conditions 1.1, 1.2 and Theorem 3.1 from [1], it follows that for any continuous admissible control $u(t)$, the solution of the Cauchy problem $X(t,u)$ exists and is unique on the entire interval $[0,T]$. The problems (1.1), (1.4) make sense for all admissible controls.

2 Main results

The following theorem guarantees the closeness of solutions of the corresponding Cauchy problems (1.1), (1.4) for small ε on a finite time interval.

Theorem 2.1. *Let Conditions 1.1–1.3 hold. Then for any $\eta>0$, there exists $\varepsilon_0=\varepsilon_0(\eta)$ such that $0<\varepsilon\le\varepsilon_0$, for the solutions $x(t,u)$, $\xi(t,u)$ of the Cauchy problems* (1.1) *and* (1.4) *satisfy the following estimate*

$$|x(t,u)-\xi(t,u)|\le\eta,$$

for all $t\in[0,T]$ and all admissible controls $u(t)$.

Proof. We will choose the fixed $\eta>0$. For any $\varepsilon>0$ and any admissible control $u(t)$, we estimate the difference between $x(t,u)$ and $\xi(t,u)$. For simplicity, let's denote $x(t,u)=x(t)$ and $\xi(t,u)=\xi(t)$. We will also omit the dependence of $x(t)$ on ε.

Since U is compact in $L^2((0,T))$, for the given η, there exists a finite grid. Thus, for the chosen control $u(t)$ from the grid such that $\frac{\eta e^{-\lambda}}{4\lambda}$: $u_1(t),\dots,u_n(t)$, where $N=N(\eta)$. Then, for the chosen control $u(t)$, there exists a subsequence $u_j(t)$ from the grid such that

$$\|u(\,\cdot\,)-u_j(\,\cdot\,)\|_{L_2}\le\frac{\eta}{4\lambda}\,e^{-\lambda}.$$

Thus, since $u(t)$ is compact in $L^2((0,T))$, all $u(t)$ satisfy the inequality, where there exists $K>0$ such that

$$\int\limits_0^T|u(t)|\,dt\le K.$$

Then

$$|x(t)|\le|x_0|+MT+M\int\limits_0^T\bigg(|x(s)|+L_\varphi\int\limits_0^s\big(1+|x(\tau)|\big)\,d\tau\bigg)\,ds.$$

Since, by the Bellman–Gronwall inequality, we get

$$|x(t)|\le C,\quad |\xi(t)|\le C, \tag{2.1}$$

where C is a constant. The estimate for $|\xi(t)|$ was obtained in the same way.

Since Assumption 1.2, we get

$$|x(t)-\xi(t)|\le\int\limits_0^t\bigg|X\Big(\frac{s}{\varepsilon},x(s),\int\limits_0^s\varphi(s,\tau,x(\tau))\,d\tau,u_j(s)\Big)-X_0(\xi(s),u_j(s))\bigg|\,ds$$
$$+2\lambda\bigg(\int\limits_0^T|u(s)-u_j(s)|^2\,ds\bigg)^{\frac12}\le I_1+\frac{\eta}{2}\,e^{-\lambda T}.$$

Then we will evaluate I_1 using Conditions 1.2, 1.3, we have

$$I_1 \le \int_0^t \left(\lambda |x(s) - \xi(s)| + \int_0^s |x(t) - \xi(t)| L_\varphi \, d\tau \right) ds$$
$$+ \int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_j(s) \right) - X_0(\xi(s), u_j(s)) \right) ds. \quad (2.2)$$

Since any function from $L^2((0,T))$ can be approximated in the L^2 – norm by a continuous function, and any continuous function on a closed interval can be approximated by a piecewise constant function, for $u_j(t)$ we take a continuous function $u_c(t)$ and a piecewise constant function $u_c(t)$ such that the inequalities hold:

$$\|u_j - u_{c_j}\|_{L^2} < \frac{\eta}{16\lambda} e^{-\lambda T}, \quad (2.3)$$
$$\|u_{c_j}(t) - u_{p_j}(t)\|_{L_2} < \frac{\eta}{16\lambda} e^{-\lambda T} \quad (2.4)$$

for all $t \in [0,T]$.

Using estimates (2.3) and (2.4), we evaluate the last integral from (2.2):

$$\int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_j(s) \right) - X_0(\xi(s), u_j(s)) \right) ds$$
$$\le \int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right) ds + \frac{\eta}{4} e^{-\lambda T}.$$

We split the integral from the last inequality into two integrals, and I_2 and I_3

$$\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right] ds$$
$$= \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s)) \, d\tau, u_p(s) \right) \right] ds$$
$$+ \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right] ds = I_2 + I_3.$$

If necessary, by dividing the segment $[0,T]$ with points $\{t_k\}_0^R$ ($t_0 = 0$, $t_R = T$), it can be assumed that on each interval $[t_k, t_{k+1})$, all components of the vector function $u_p(t)$ take constant values, i.e., $u_p(t_k) = u_p(t_k)$ for $t \in [t_k, t_{k+1})$. Here, the natural $R = R(\eta)$ is fixed for a fixed choice of η.

Now, let us choose a natural n and divide the segment $[0,T]$ into equal n parts using the points $t_i = i \cdot n^{-1}$ ($i = \overline{0, n}$). We assume n is large enough such that each interval $[t_k, t_{k+1})$ contains points t_i. As a result, we obtain n intervals of the form $[t_i, t_{i+1})$. If, for some k and i, $t_i < t_k < t_{i+1}$, the interval $[t_i, t_{i+1})$ is divided into two intervals, $[t_i, t_k)$ and $[t_k, t_{i+1})$. Consequently, the segment $[0,T]$ is divided into no more than $n + R$ intervals, each with a length not exceeding $\frac{1}{n}$. The division points are again denoted as t_i, and the total number of intervals $[t_i, t_{i+1})$ is denoted by $K = K(\eta)$. Clearly, $K \le n + R$, and $u_p(t) = u_p(t_i)$ for $t \in [t_i, t_{i+1})$. Let us denote $\xi_i = \xi(t_i)$, and $u_p(t_i) = u_{pi}$. Then

$$I_2 \le \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \biggl[X\Bigl(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\,d\tau, u_{pi}\Bigr) - X\Bigl(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Bigr) \biggr] ds$$

$$+ \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \biggl[X\Bigl(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Bigr) - X\Bigl(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\,d\tau, u_{pi}\Bigr) \biggr] ds$$

$$\le \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi(\tau) - \xi_i|\,d\tau\,ds + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi_i - \xi(\tau)|\,ds + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi_i - \xi(s)|\,d\tau\,ds$$

$$\le 2 \sum_{i=0}^{K-1} \lambda\, \frac{MT(1+C)}{n^2} \biggl(1 + \int_{t_i}^{t_{i+1}} ds \int_0^s L_\varphi\,d\tau\biggr) \le \lambda MT(1+C)\, \frac{n+R}{n^2} \Bigl(1 + L_\varphi \frac{T}{n}\Bigr).$$

Then, for a chosen $\eta > 0$, there exists a number n such that for all $\varepsilon > 0$, the following holds:

$$I_2 \le \frac{\eta}{8}\, e^{-\lambda T}.$$

For estimating the integral I_3, we split it over the interval $[0, T]$ into a sum of integrals

$$\biggl| \int_0^t \biggl[X\Bigl(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\,d\tau, u_p(s)\Bigr) - X_0(\xi(s), u_p(s)) \biggr] ds \biggr|$$

$$\le \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + \int_{t_i}^{t_{i+1}} ds \int_0^s L_\varphi |\xi(s) - \xi_i|\,d\tau + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + I_4,$$

where

$$I_4 = \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \biggl[X\Bigl(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Bigr) - X_0(\xi_i, u_{pi}) \biggr] ds.$$

In terms of $\varphi_1(t,x)$, we have

$$\int_{t_i}^{t_{i+1}} \Bigl[X\Bigl(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Bigr) - X_0(\xi_i, u_{pi}) \Bigr] ds$$

$$= \int_0^{t_{i+1}} \Bigl[X\Bigl(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Bigr) - X_0(\xi_i, u_{pi}) \Bigr] ds$$

$$+ \int_0^{t_i} \Bigl[X\Bigl(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Bigr) - X_0(\xi_i, u_{pi}) \Bigr] ds. \quad (2.5)$$

To estimate (2.5), it is necessary to use the lemma.

Lemma 2.1. *The convergence in (2.5) is uniform with respect to ξ_i, u_{pi}, and $t_i \in [0,T]$, by subsequence $\varepsilon_n \to 0$.*

Since K is fixed, then, due to the proven lemma, Condition 1.3 holds for small ε_n (depending on K), but independent of ξ_i, u_{pi} and t_i, we have

$$I_6 \le \frac{\eta}{8}\, e^{-\lambda T}.$$

So we have established that for small enough ε_n

$$|x_{\varepsilon_n}(t) - \xi(t)| < \eta, \quad t \in [0,T].$$

We get

$$\sum_{i=0}^{K-1} \int\limits_{t_i}^{t_{i+1}} \left(X\left(\frac{s}{\varepsilon}, \xi_i, \int\limits_0^s \varphi(s,\tau,\xi_i)\, d\tau, u_{pi}\right) - X_0(\xi_i, u_{pi}) \right) ds < \frac{\eta}{16}\, e^{-\lambda T}.$$

So,

$$I_3 \le \frac{\eta}{8}\, e^{-\lambda T}.$$

Hence, the following can be obtained from the proof of I_2,

$$I_1 \le \lambda \left(\int\limits_0^t |x(s) - \xi(s)|\, ds + \int\limits_0^s L_\varphi |x(\tau) - \xi(\tau)|\, d\tau \right) + \frac{\eta}{4}\, e^{-\lambda T} \le \frac{\eta}{2}\, e^{-\lambda T}.$$

The reasoning outlined above can be applied to each function $u_1(t), u_2(t), \dots, u_n(t)$ from the constructed grid. Due to its finiteness, there exists a unique choice i for each function in the system.

Thus, from an arbitrary sequence of solutions $x_{\varepsilon_n}(t)$ of problem (1.1), one can select a subsequence of solutions $x_{\varepsilon_n}(t)$, which converges uniformly for $t \in [0,T]$ to the same limiting function $\xi(t)$. Therefore, the entire family x_ε converges uniformly in $t \in [0,T]$, $u \in U$ as $\varepsilon \to 0$ to $\xi(t)$.

The theorem is proved. □

Theorem 2.2. *Let*

$$J_\varepsilon^* = \inf_{u(\cdot)\in U} J_\varepsilon[u],$$
$$J_0^* = \inf J_0[u].$$

Let Conditions 1.1–1.5 hold. Then problems (1.1), (1.2) *and* (1.4), (1.5) *have solutions* $(x_\varepsilon^*(t), u_\varepsilon^*(t))$, $(\xi^*(t), u^*(t))$, *respectively. Moreover,*

(1)
$$J_\varepsilon^* \to J_0^* \ \text{ as } \ \varepsilon \to 0.$$

(2) *For any $\eta > 0$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$,*
$$|J_\varepsilon^* - J_\varepsilon(u^*)| < \eta,$$
i.e., the optimal control of the averaging problem is nearly optimal for the original problem.

(3) *There exists a sequence $\varepsilon_n \to 0$, $n \to \infty$, such that*
$$x_{\varepsilon_n}^*(t) \to \xi^*(t) \ \text{ uniformly on } [0,T], \tag{2.6}$$
and
$$u_{\varepsilon_n}^*(t) \to u^*(t) \ \text{ in } L^2((0,T)). \tag{2.7}$$

If the averaging problem (1.4), (1.5) *has a unique solution, then the convergence results* (2.6) *and* (2.7) *hold for all* $\varepsilon \to 0$.

Proof.

(1) First, let us prove the continuity of $J_\varepsilon(u)$ with respect to $u \in L^2((0,1))$ for each $\varepsilon > 0$.

Let $u_1(t), u_2(t)$ be arbitrary admissible controls for problem (1.1), (1.2), and let $x(t,u_1), x(t,u_2)$ be the corresponding trajectories.

Using Condition 1.2 and Gronwall's inequality, we have

$$\sup_{t\in[0,1]} |x(t,u_1) - x(t,u_2)| \le \lambda \|u_1 - u_2\|_{L^2} e^{\lambda}. \tag{2.8}$$

Thus,

$$\begin{aligned} |J_\varepsilon(u_1) - J_\varepsilon(u_2)| &\le \lambda \|u_1 - u_2\|_{L^2} \\ &+ \int_0^T \Big[L\big(t, x(t,u_2), u_1(t)\big) - L\big(t, x(t,u_2), u_2(t)\big) \Big]\, dt + \big| \Phi(x(T,u_1)) - \Phi(x(T,u_2)) \big|. \end{aligned} \tag{2.9}$$

Estimate (2.1) is uniform for any admissible control $u(t)$.

Thus, from (2.1), we have that $x(t,u)$ does not go beyond the boundaries of the area B_c-sphere of radius C with center at for $t \in [0,T]$.

Due to (1) from Condition 1.5 and Cantor's theorem, the function $L(t,x,u)$ will be uniformly continuous with respect to $x \in B_c$, uniformly relative to $t \in [0,T]$ and $u \in \mathbb{R}^m$. Therefore, from (2.8) and (2.9), the continuity of $J_\varepsilon(u)$ with respect to the L^2-norm follows.

By similar considerations, we establish the continuity of the functional $J_0(u)$ with respect to u.

Now, considering the compactness of the set of admissible controls, we establish the existence of $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ and $(\xi^*(t), u^*(t))$ – optimal solutions of (1.1), (1.2) and (1.4), (1.5), respectively.

Now, we prove that $J^*_\varepsilon \to J^*_0$ as $\varepsilon \to 0$. Choose an arbitrary $\eta > 0$ and fix it. Then

$$J^*_\varepsilon \le J_\varepsilon(u^*) = J^*_0 + J_\varepsilon(u^*) - J_0(u^*).$$

But

$$|J_\varepsilon(u^*) - J_0(u^*)| \le \int_0^T \Big| L\big(t, x(t,u^*), u^*(t)\big) - L\big(t, \xi(t), u^*(t)\big) \Big|\, dt + \big| \Phi(x(T,u^*)) - \Phi(\xi(T)) \big|. \tag{2.10}$$

From Theorem 2.1 we have

$$\max_{t\in[0,1]} |x(t,u^*) - \xi^*(t)| \to 0, \ \ \varepsilon \to 0. \tag{2.11}$$

Now, considering the uniform continuity of the function $L(t,x,u)$ with respect to $x \in B_c$, uniformly for $t \in (0,T]$ and $u \in \mathbb{R}^m$, it follows from (2.10), (2.11) and Condition 1.5 that there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, we have

$$|J_\varepsilon(u^*) - J_0| < \eta,$$

then

$$J^*_\varepsilon < J^*_0 + \eta. \tag{2.12}$$

From other side, as $\varepsilon < \varepsilon_0$, we get

$$J^*_0 \le J_0(u^*_\varepsilon) = J^*_\varepsilon + \big(J_0(u^*_\varepsilon) - J_\varepsilon(u^*_\varepsilon)\big). \tag{2.13}$$

Therefore

$$J_0^* < J_\varepsilon^* + \eta.$$

From (2.12) and (2.13) it follows that

$$J_\varepsilon^* \to J_0^*, \quad \varepsilon \to 0. \tag{2.14}$$

Then, statement (1) is proved.

The proof of statement (2) follows from the following inequality

$$|J_\varepsilon^* - J_\varepsilon(u^*)| \le |J_\varepsilon^* - J_0^*| + \big|J_0(u^*) - J_\varepsilon(u^*)\big|.$$

Let's move on to the proof of the next statement. Since u is compact in $L^2((0,1))$, it follows that from the family u_ε^*, we can extract a subsequence $u_{\varepsilon_n}^*$ that converges in $L^2((0,1))$.

Let

$$\lim_{\varepsilon_n \to 0} u_{\varepsilon_n}^* = u_0. \tag{2.15}$$

Consider the auxiliary systems. Using the auxiliary systems and Theorem 2.1, through simple considerations, we obtain

$$\sup_{t\in[0,T]} |x_{\varepsilon_n}^*(t) - \xi(t)| \to 0, \quad \varepsilon_n \to 0. \tag{2.16}$$

Accordingly,

$$\begin{aligned} J_{\varepsilon_n}^* &= J_{\varepsilon_n}(u_{\varepsilon_n}^*) = \int_0^T L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big)\, dt + \Phi(x_{\varepsilon_n}^*(T)) \\ &= \int_0^T L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big)\, dt + \phi(X_{\varepsilon_n}^*(T)) + \int_0^T \Big[L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big) - L\big(t, x_{\varepsilon_n}^*(t), u_0(t)\big)\Big]\, dt. \end{aligned} \tag{2.17}$$

From (2) of Condition 1.5 and (2.15), it follows that the last term in (2.17) tends to zero as $\varepsilon_n \to 0$. Let's consider the limit in equation (2.17) as $\varepsilon_n \to 0$, using (2.14) and (2.16), we have

$$J_0^* = \int_0^T L(t, \xi(t), u_0(t))\, dt + \Phi(\xi(T)).$$

Thus, $(\xi(t), u_0(t))$ is the optimal solution of the averaged problem (1.4), (1.5), proving statement (3).

If the problem (1.4), (1.5) has a unique solution, as shown earlier, it follows that any convergent sequence $(x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t))$ converges to the only uniquely defined solution. This completes the proof of statement (4). □

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On Instability of Linear Differential Systems with Smooth Dependence on a Parameter

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Let us consider a one-parameter family of linear differential systems

$$\dot{x} = A_\mu(t)x, \ \ x \in \mathbb{R}^2, \ \ t \geq 0, \tag{1_μ}$$

whose coefficient matrix is of the form

$$A_\mu(t) := \begin{cases} d_k \operatorname{diag}[1,-1], & 2k-2 \leq t < 2k-1, \\ (\mu + b_k)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 2k-1 \leq t < 2k, \ \ k \in \mathbb{N}. \end{cases}$$

Here $\mu k \in \mathbb{R}$ is considered as a parameter; b_k, d_k are arbitrary real numbers.

E. Sorets and T. Spenser have shown in the paper [8] that major characteristic exponent of differential equation

$$\ddot{x} = -\big(K^2\big(\cos t + \cos(\omega t + \theta)\big) + E\big)x, \ \ x \in \mathbb{R}^2, \ \ t \geq 0$$

is positive for all irrational $\omega \in \mathbb{R}$ and for almost all $\theta \in \mathbb{R}$ on the set of energy values $E \geq 0$, such that it's relative Lebesque measure tends to 1 under increasing to infinity K.

L.-S. Young in the article [9], as a part, have established for all sufficiently big values of $d_k \equiv d > 0$ and $b_k = k\omega$, $k \in \mathbb{N}$, where $\omega \in \mathbb{R} \setminus \mathbb{Q}$ satisfies some diophantine condition holding almost everywhere, that the major characteristic exponent of system (1_μ), which coincides for almost all values of $\mu \in \mathbb{R}$, approximately equal to d.

In the papers [2, 3, 6] we considered the case when the inequality $d_k \geq d > 0$, $k \in \mathbb{N}$, holds. Particularly, in [2], we have proved under condition $d_k \equiv d > 4\ln 2$ that major characteristic exponent of system (1_μ), is positive for the set of parameter μ with a positive Lebesque measure.

The theorem of the article [3] implies an absence of uniform on $\mu \in \mathbb{R}$ and $t \geq 0$ upper estimations for a solution norms of system (1_μ). Where as, the method developed in the paper [6] essentially uses Parseval's identity for trygonometric sums. It allows to prove an absence of analogous estimations, which are uniform on μ and subexponential on t. Given there the proof of system (1_μ) major characteristic exponent positiveness unfortunately contains invalid statements. The theorem of article [4], that implies the same conclusion, is wrong as well.

In this report we offer the way sufficient to complete the correct proof of specified result.

For all $n \in \mathbb{N}$, an arbitrary $\alpha \in \mathbb{R}$ and set $\chi = \{x_1, \dots, x_n\}$, $x_i \in \mathbb{R}$, $i = \overline{1,n}$, let us denote

$$f_i(x) = f_i(x, x_i) := \ln|x - x_i|, \ \ x \neq x_i,$$

and

$$f(x) = f(x, \alpha, \chi) := \alpha + n^{-1}\sum_{i=1}^{n} f_i(x).$$

Lemma ([7]). *For all* $a, k, l, \widehat{l} \in \mathbb{R}$ *such that* $l \geq 1$, $\widehat{l} > 0$, $k > 3 + 2\widehat{l}^{-1}$, *and for every set* $\chi = \{x_1, \dots, x_n\}$ *and number* $\alpha \in \mathbb{R}$, *that satisfy the conditions* $f(a) > -l$, $\sup\{f(x) : |x - a| \leq 1/2\} < l$, *for Lebesque measure of the set*

$$M = M(\alpha, \chi, a, k, l, \widehat{l}) := \Big\{x \in K : \sup_{y \in K} f(y) > f(x) + \widehat{l}\Big\},$$

where $\widetilde{d} := e^{-lk}$, $K := [a - \widetilde{d}/k, a + \widetilde{d}/k]$, *the estimation holds* $\operatorname{mes} M \leq 48k^{-2}\widetilde{d}/\widehat{l}$.

Let us denote by $X_{A_\mu}(t, s)$, $t, s \geq 0$, Cauchy matrix of system (1_μ).

Theorem. *The major characteristic exponent of system* (1_μ), *considered as a function of parameter* μ, *is positive on the set of positive Lebesgue measure in the case when the condition* $d_k \geq d > 0$, $k \in \mathbb{N}$, *holds.*

Proof. Under

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

we denote the rotation matrix on the angle $\varphi \in \mathbb{R}$ counterclockwise.

According to estimations (40) from paper [6], the inequality holds

$$\int_0^{2\pi} X_{A_\mu}(2k, 0\, d\mu \geq 2\pi \prod_{j=1}^{k} \operatorname{ch} d_j \geq 2\pi(1 + 2^{-1}d^2)^k.$$

Hence, and because of the equality $X_{A_\mu}(2k, 2k-1) = U(\mu + b_k)$, we have the relation

$$\int_0^{2\pi} X_{A_\mu}(2k - 1, 0)\, d\mu \geq 2\pi(1 + 2^{-1}d^2)^{k-1}. \tag{2}$$

Remark. In cited article F_k should been defined by the formula $F_k = \kappa_k E + \kappa_{k-1} \operatorname{sh} d_k I$. Followed by estimations (40) an equality in (41) is in general incorrect. Really, for every continuous function $f(\cdot) : \mathbb{R} \to (0, +\infty)$ and numbers $p > q$ the next formula holds [1, p. 167]

$$\exp\left\{\frac{1}{p - q}\int_q^p \ln f(t)\, dt\right\} \leq \int_q^p \frac{1}{p - q} \ln f(t)\, dt. \tag{3}$$

Whereas estimation (41) from paper [6] demands the opposite to (3) inequality. So all subsequent statements of this article are not justified. Hence the conclusion of Theorem 2 in [6] cannot be thought as sufficiently proved.

Here we give another way that allows to avoid the indicated failures.

From estimation (2) it follows the existence of $\gamma_k \in [0, 2\pi]$ such that the inequality holds

$$\|X_{A_{\gamma_k}}(2k, 0)\| \geq (1 + 2^{-1}d^2)^k. \tag{4}$$

Denote by $x_{ij}(t, \mu)$, $i, j = \overline{1, 2}$, the elements of matrix $X_{A_\mu}(t, 0)$.

In the papre [7] after the formula (36) we have proved that $x_{ij}(2n-1, \mu)$, $i, j = \overline{1, 2}$, is a uniform polynome $P_{n,i,j}(\sin\mu, \cos\mu)$ degree $n - 1$ on $\sin\mu$ and $\cos\mu$.

For every real $\mu \neq \pi(2^{-1} + m)$, $m \in \mathbb{Z}$, the equality holds

$$P_{n,i,j}(\sin\mu, \cos\mu) = \cos^n \mu P_{n,i,j}(\operatorname{tg}\mu, 1).$$

In the opposite case when $\mu \neq \pi m$, $m \in \mathbb{Z}$, we have the formula

$$P_{n,i,j}(\sin\mu, \cos\mu) = \sin^n \mu P_{n,i,j}(1, \operatorname{ctg}\mu).$$

Denote

$$\delta_n = \delta_n(\mu) := \begin{cases} 0, & \text{if } |\cos\mu| \geq \dfrac{1}{\sqrt{2}}\,, \\ 1, & \text{if } |\cos\mu| < \dfrac{1}{\sqrt{2}}\,. \end{cases}$$

The next relation is correct

$$P_{n,i,j}(\sin\mu, \cos\mu) = \cos^n\big(\mu + 2^{-1}\pi\delta_n(\mu)\big)\,P_{n,i,j}\big(\operatorname{tg}^{1-\delta_n}\mu, \operatorname{ctg}^{\delta_n}\mu\big). \tag{5}$$

The equality

$$\widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\mu)}\mu) = \sum_{i=1}^{2}\sum_{j=1}^{2} P^2_{n,i,j}\big(\operatorname{tg}^{1-\delta_n}\mu, \operatorname{tg}^{-\delta_n}\mu\big)$$

defines a polynome $\widehat{P}_n(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$.

Next formulas hold

$$\begin{aligned} \|X_{A_\mu}(2n-1,0)\|^2 = \max_{y\in\mathbb{R}^2} \frac{\|X_{A_\mu}(2n-1,0)y\|^2}{\|y\|^2} = \max_{\zeta\in\mathbb{R}} \left\| (x^2_{ij}(2n-1,\mu))^2_{i,j=1} \begin{pmatrix}\cos\zeta\\ \sin\zeta\end{pmatrix} \right\|^2 \\ = \max_{\zeta\in\mathbb{R}} \sum_{i=1}^{2} \Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2. \end{aligned} \tag{6}$$

They imply the inequalities

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} x^2_{ij}(2n-1,\mu) &\leq \sum_{i=1}^{2} \max_{j\in\{1,2\}} x^2_{ij}(2n-1,\mu) \\ &= \sum_{i=1}^{2} \max_{\zeta\in\{0,2^{-1}\pi\}} \Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2 \\ &\leq \sum_{i=1}^{2} \max_{\zeta\in\mathbb{R}} \Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2 \overset{(5)}{=} \|X_{A_\mu}(2n-1,0)\|^2 \\ &\overset{(6)}{\leq} \max_{\zeta\in\mathbb{R}} \sum_{i=1}^{2} \big(x_{i1}(2n-1,\mu)\cos\zeta\big)^2 + \big(x_{i2}(2n-1,\mu)\sin\zeta\big)^2 \leq \sum_{i=1}^{2}\sum_{j=1}^{2} x^2_{ij}(2n-1,\mu). \end{aligned} \tag{7}$$

Hence, for some $\varkappa \in [1,2]$ we have the equalities

$$\begin{aligned} \widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\mu)}\mu) &\overset{(5)}{=} \sum_{i=1}^{2}\sum_{j=1}^{2} \cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big) P^2_{n,i,j}(\sin\mu, \cos\mu) \\ &= \cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big) \sum_{i=1}^{2}\sum_{j=1}^{2} x^2_{ij}(2n-1,\mu) \\ &\overset{(7)}{=} \varkappa \cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big) \|X_{A_\mu}(2n-1,0)\|^2. \end{aligned} \tag{8}$$

For all $\mu \in \mathbb{R}$, such that $\delta_n(\mu) = 0$, the next estimation is correct

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| = |\cos\mu| \geq \frac{1}{\sqrt{2}}\,.$$

In the opposite case the formulas hold

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| = \big|\cos(\mu + 2^{-1}\pi)\big| = |\sin\mu| = \sqrt{1-\cos^2\mu} \geq \frac{1}{\sqrt{2}}\,.$$

The both cases united imply the inequality

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| \geq \frac{1}{\sqrt{2}}\,, \quad \mu \in \mathbb{R}. \tag{9}$$

According to relation (10) from the paper [5], we have the estimation

$$\|X_{A_\mu}(t,0)\| \leq e^{th}, \quad \text{where} \;\; h := \sup_{k\in\mathbb{N}} d_k. \tag{10}$$

From formulas (8)–(10) the next estimations follow

$$\widehat{P}_n(\mathrm{tg}^{1-2\delta_n(\mu)}\,\mu) \overset{(8),\,(9)}{\leq} 2^{n/2}\|X_{A_\mu}(2n-1,0)\|^2 \overset{(10)}{\leq} 2^{n/2}e^{h(2n-1)}. \tag{11}$$

The relations (4) and (8) imply the inequalities

$$\widehat{P}_n(\mathrm{tg}^{1-2\delta_n(\gamma_n)}\,\gamma_n) \overset{(8)}{\geq} \|X_{A_{\gamma_n}}(2n-1,0)\| \overset{(4)}{\geq} (1+2^{-1}d^2)^{n-1}. \tag{12}$$

Due to main algebra theorem, there exist $\alpha \in \mathbb{R}$ and $\beta_j \in \mathbb{C}$, $j = \overline{1, 2n-2}$, such that

$$\widehat{P}_n(\nu) = \alpha \prod_{j=1}^{2n-2}(\nu - \beta_j). \tag{13}$$

Let us put in lemmas conditions (here $[\,\cdot\,]$ denotes a whole part of the number)

$$l := 1+h, \quad \widehat{l} := \frac{\widehat{d}}{4}, \quad k := \max\big\{2^{11}\widehat{d}^{-1}, 4 + 2[\widehat{l}^{-1}]\big\}, \quad \widetilde{d} := e^{-lk}, \quad f(\,\cdot\,) := \frac{1}{2n-2}\ln \widehat{P}_n(\,\cdot\,).$$

Denote $\widetilde{\gamma}_n = \mathrm{tg}^{1-2\delta_n(\gamma_n)}\,\gamma_n$.

For all $\nu \in [\widetilde{\gamma}_n - \widetilde{d}/k, \widetilde{\gamma}_n + \widetilde{d}/k]$ there exists $\mu = \mu(\nu) \in [\gamma_n - \widetilde{d}/k, \gamma_n + \widetilde{d}/k]$ such that $\nu = \mathrm{tg}^{1-2\delta_n(\mu)}\,\mu$.

Hence, as a consequence of formula (11), for such ν the estimation holds

$$f(\nu) \overset{(11)}{\leq} \frac{1}{2n-2}\ln(2^{n/2}e^{h(2n-1)}) = \frac{n\ln 2 + h(2n-1)}{2n-2} \leq 1+h. \tag{14}$$

Denote $\widehat{d} := \frac{1}{2}\ln(1+2^{-1}d^2)$.

Inequalities (12) imply the relation

$$f(\widetilde{\gamma}_n) \overset{(12)}{\geq} \frac{1}{2n-2}\ln(1+2^{-1}d^2)^{n-1} \geq \frac{1}{2}\,. \tag{15}$$

Then, considering (13) and (14), due to lemma we have the inequality

$$\overline{\mathrm{mes}}\Big\{\mu \in \Big[\gamma_n - \frac{\widetilde{d}}{k}, \gamma_n + \frac{\widetilde{d}}{k}\Big] : \;\; \frac{1}{2n-2}\ln \widehat{P}_n(\mathrm{tg}^{\delta_n}\gamma_n) > \frac{1}{2n-2}\ln \widehat{P}_n(\mathrm{tg}^{\delta_n}\mu) + \frac{\widehat{d}}{4}\Big\} \leq 48k^{-2}\widetilde{d}\,\frac{4}{\widehat{d}}. \tag{16}$$

For all $\mu=\mu(\nu)\in[\gamma_n-\widetilde{d}/k,\gamma_n+\widetilde{d}/k]$ the next formulas are correct

$$\begin{aligned}\big|\cos\big(\mu+2^{-1}\pi\delta_n(\mu)\big)\big|-\big|\cos\big(\gamma_n+2^{-1}\pi\delta_n(\mu)\big)\big|\\ \geq-\Big|\cos\big(\mu+2^{-1}\pi\delta_n(\mu)\big)-\cos\big(\gamma_n+2^{-1}\pi\delta_n(\mu)\big)\Big|\geq-\frac{\widetilde{d}}{k}\,. \quad (17)\end{aligned}$$

Thus, denote $\varepsilon:=\widetilde{d}/k$, for all $\mu\in[\gamma_n-\varepsilon,\gamma_n+\varepsilon]$ with exception of the set W_n which Lebesgue measure $\operatorname{mes}W_n\leq\dfrac{\varepsilon}{4}$ by the cause of (16) we have the estimations

$$\begin{aligned}\frac{1}{2n-1}\,\ln\|X_{A_\mu}(2n-1,0)\|\overset{(8),\,(16)}{\geq}\frac{1}{2n-1}\,\ln\widehat{P}_n(\operatorname{tg}^{\delta_n}\gamma_n)\\ +\frac{1}{2n-1}\,\ln\big|\cos^n(\mu-2^{-1}\pi\delta_n(\mu))\big|-\big|\cos(\gamma_n+2^{-1}\pi\delta_n(\mu))\big|-\frac{\widehat{d}}{4}\\ \overset{(8),\,(17)}{\geq}\frac{1}{2n-1}\,\ln\|X_{A_{\gamma_n}}(2n-1,0)\|-\frac{\widetilde{d}}{k}-\frac{\widehat{d}}{4}\overset{(15)}{\geq}\frac{\widehat{d}}{5}\,. \quad (18)\end{aligned}$$

The set of limit points of sequence $\{\gamma_k\}_{k=1}^{\infty}$ is not empty.

Let us denote by γ_∞ some of them.

For an arbitrary $n\in\mathbb{N}$, there exists $k(n)\geq n$ such that $|\gamma_{k(n)}-\gamma_\infty|<\frac{\varepsilon}{2}$.

Denote also

$$W_\infty:=\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}W_{k(n)}=\operatorname*{Lim}_{m\to+\infty}\bigcap_{n\geq m}W_{k(n)}.$$

The next relations hold

$$\operatorname{mes}W_\infty=\lim_{m\to+\infty}\operatorname{mes}\bigcap_{n\geq m}W_{k(n)}\leq\lim_{m\to+\infty}\sup_{n\geq m}\operatorname{mes}W_{k(n)}\leq\frac{\varepsilon}{4}\,. \quad (19)$$

We have the inclusions

$$\begin{aligned}\widetilde{M}:=\big[\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon\big]\setminus W_\infty\\ =\bigcap_{m\in\mathbb{N}}\bigcup_{n\geq m}\Big([\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon]\setminus W_{k(n)}\Big)\\ \subset\bigcap_{m\in\mathbb{N}}\bigcup_{n\geq m}\Big(\big[\gamma_{k(n)}-\varepsilon,\gamma_{k(n)}+\varepsilon\big]\setminus W_{k(n)}\Big). \quad (20)\end{aligned}$$

Thus for all $\mu\in\widetilde{M}$, as a consequence of formula (18), the next estimations are correct

$$\lambda_{\max}(A_\mu)\geq\varlimsup_{n\to+\infty}\frac{1}{2k(n)-1}\,\ln\big\|X_{A_\mu}(2k(n)-1,0)\big\|\overset{(18),\,(20)}{\geq}\frac{\widehat{d}}{5}>0.$$

As well, relations (19) imply the inequality

$$\operatorname{mes}\widetilde{M}\leq\operatorname{mes}\big[\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon\big]-\operatorname{mes}W_\infty\geq\frac{\varepsilon}{4}\,.$$

The theorem is proved. □

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On Bounded and Periodic Solutions of Planar Systems of ODE

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We consider the system

$$u' = f_1(u,v), \quad v' = f_2(u,v) \tag{S}$$

subject to the conditions

$$u(0) = u(\omega), \quad v(0) = v(\omega), \tag{P}$$

and

$$u(0) = c, \quad \sup\big\{|u(t)| + |v(t)| : \ t \geq 0\big\} < +\infty. \tag{B}$$

Here, f_1 and f_2 are Carathéodory functions on $[0,\omega] \times \mathbb{R}^2$, and ω-periodic with respect to the independent variable.

Definition 1. Solutions (u_1,v_1) and (u_2,v_2) of $\text{(S)},\text{(P)}$ are said to be consecutive if $u_1(t) \leq u_2(t)$ for $t \in [0,\omega]$, $u_1 \not\equiv u_2$, and for every solution (u,v) of $\text{(S)},\text{(P)}$ satisfying $u_1(t) \leq u(t) \leq u_2(t)$ for $t \in [0,\omega]$, either $u_1 \equiv u$ or $u_2 \equiv u$ holds.

Property O. We will say that $\text{(S)},\text{(P)}$ possesses Property O if there exists $\varepsilon > 0$ such that every solution (u,v) of $\text{(S)},\text{(P)}$ satisfies

$$\min\big\{|v(t)| : \ t \in [0,\omega]\big\} \leq \varepsilon.$$

Remark 1. Consider the problem

$$u' = \lambda \cos^2(3u)\psi(v), \quad v' = \cos^2 t \sin u - \frac{1}{4}\,, \quad u(0) = u(\omega), \quad v(0) = v(\omega). \tag{$*$}$$

It is clear that for every c, function $(u,v) := (\frac{\pi}{6}\,, c + \frac{1}{8}\,\sin(2t))$ is a solution of $(*)$, and consequently, $(*)$ does not have Property O.

Hypothesis B. We will say that $f_1 : [0,\omega] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Hypothesis B if

$$f_1(t,x,\cdot\,) : \mathbb{R} \to \mathbb{R} \ \text{ is non-decreasing for a.e. } \ t \in [0,\omega], \ \ x \in \mathbb{R}, \tag{1}$$

and

$$f_1(t,x,y)\operatorname{sgn} y \geq 0 \ \text{ for } \ t \in [0,\omega], \ \ x,y \in \mathbb{R}.$$

Proposition 1. *Let Hypothesis B hold, and*

$$\operatorname{meas}\big\{t \in [0,\omega] : \ f_1(t,x,y) \neq 0\big\} > 0 \ \textit{ for } \ x \in \mathbb{R}, \ \ y \in \mathbb{R} \setminus \{0\}.$$

Then problem $\text{(S)},\text{(P)}$ *has Property O.*

Property V. We will say that $\text{(S)},\text{(P)}$ possesses Property V if for every pair (u_1,v_1) and (u_2,v_2) of solutions of $\text{(S)},\text{(P)}$, satisfying $u_1 \equiv u_2$, the identity $v_1 \equiv v_2$ is fulfilled.

Example presented in Remark 1 shows that $(*)$ does not have Property V.

Proposition 2. *Let* $f_1(t,x,y) := f_0(t,x)\psi(y)$*, where* $f_0(t,x) \geq h_0(t) \geq 0$ *for* $t \in [0,\omega]$*,* $x \in \mathbb{R}$*,* $h_0 \not\equiv 0$*, and* ψ *is an increasing continuous function with* $\psi(0) = 0$*. Then* (S), (P) *possesses Property V.*

Hypothesis L_x. We will say that f_1 satisfies Hypothesis L_x if (1) holds and for every $r > 0$ and $\varepsilon > 0$, there exist $p_{r\varepsilon} \in L([0,w])$ such that

$$\big|f_1(t,x,y) - f_1(t,x',y)\big| \leq p_{r\varepsilon}(t)|x - x'| \ \text{ for } \ t \in [0,w], \ \ |x - x'| \leq \varepsilon, \ \ |y| \leq r.$$

Definition 2. Function $(\alpha,\beta) : [0,w] \to \mathbb{R}^2$ is said to be a lower function of (S), (P) if $\beta = \beta_0 + \beta_1$, $\alpha, \beta_0 \in AC([0,w])$, β_1 is non-decreasing, $\beta_1'(t) = 0$ for a.e. $t \in [0,\omega]$, $\alpha(0) = \alpha(\omega)$, $\beta(0+) \geq \beta(\omega-)$, and

$$\alpha'(t) = f_1(t,\alpha(t),\beta(t)), \quad \beta'(t) \geq f_2(t,\alpha(t)) \ \text{ for a.e. } \ t \in [0,\omega].$$

Analogously, $(\gamma,\delta) : [0,w] \to \mathbb{R}^2$ is said to be an upper function of (S), (P) if $\delta = \delta_0 + \delta_1$, $\gamma, \delta_0 \in AC([0,w])$, δ_1 is non-increasing, $\delta_1'(t) = 0$ for a.e. $t \in [0,\omega]$, $\gamma(0) = \gamma(\omega)$, $\delta(0+) \leq \delta(\omega-)$, and

$$\gamma'(t) = f_1(t,\gamma(t),\delta(t)), \quad \delta'(t) \leq f_2(t,\gamma(t)) \ \text{ for a.e. } \ t \in [0,\omega].$$

Definition 3. Solution (u,v) of (S), (P) is said to be upper weakly stable (lower weakly stable) if for every $\varepsilon > 0$, there exist a lower function (α,β) (resp. an upper function (γ,δ)) of (S), (P) such that

$$u(t) \leq \alpha(t) \leq u(t) + \varepsilon \ \text{ for } \ t \in [0,\omega], \ \ \alpha \not\equiv u$$
$$\Big(\text{resp. } \ u(t) - \varepsilon \leq \gamma(t) \leq u(t) \ \text{ for } \ t \in [0,\omega], \ \ \gamma \not\equiv u\Big).$$

Remark 2. Let $f_1(t,x,0) \equiv 0$, $f_2(t,0) \equiv 0$, $\varepsilon_0 > 0$ and $f_2(t,\cdot)$ is non-increasing on $[-\varepsilon_0,\varepsilon_0]$. It is not difficult to verify that solution $(u,v) := (0,0)$ is both u.w.s and l.w.s.

The next proposition (partially) justifies introduced terminology.

Proposition 3. *Let Hypothesis* L_x *be fulfilled and* (S), (P) *possess Property V. Let, moreover,* (u,v) *be a Lyapunov stable solution of* (S), (P)*. Then* (u,v) *is both u.w.s and l.w.s.*

Definition 4. Let $\alpha, \gamma \in C([0,\omega])$, $\alpha(t) \leq \gamma(t)$ for $t \in [0,\omega]$, $a \in [0,\omega[$, $\alpha(a) < \gamma(a)$ and $c \in]\alpha(0),\gamma(0)[$. We say that (S), (P) possesses property $Z_{\alpha\gamma}(a,c)$ if for every solution (u,v) of (S), (P) satisfying $\alpha(t) \leq u(t) \leq \gamma(t)$ for $t \in [0,\omega]$, the inequality $u(a) \neq c$ holds.

Remark 3. It is clear that if (u_1,v_1) and (u_2,v_2) are consecutive solutions of (S), (P), then there exist $a \in [0,\omega[$ and $c \in]u_1(a),u_2(a)[$ such that (S), (P) possesses Property $Z_{u_1u_2}(a,c)$.

Now we are able to formulate results.

Consecutive solutions

Theorem 1. *Suppose that* (S), (P) *possesses Property O and* (u_1,v_1) *and* (u_2,v_2) *are solutions of* (S), (P) *satisfying* $u_1(t) \leq u_2(t)$ *for* $t \in [0,\omega]$*. Let, moreover,* $a \in [0,\omega[$*,* $u_1(a) < u_2(a)$*,* $c \in]u_1(a),u_2(a)[$ *and* (S), (P) *possesses Property* $Z_{u_1u_2}(a,c)$*. Then there exist consecutive solutions* (u_*,v_*) *and* (u^*,v^*) *of* (S), (P) *such that*

$$u_1(t) \leq u_*(t) \leq u^*(t) \leq u_2(t) \ \text{ for } \ t \in [0,\omega], \ \ u_*(a) < c < u^*(a).$$

Proposition 4. *Let Hypothesis B hold,* (u_*,v_*) *and* (u^*,v^*) *are consecutive solutions of* (S), (P) *and* $u_*(t) < u^*(t)$ *for* $t \in [0,\omega]$*. Then, if* (u_*,v_*) *is u.w.s, then* (u^*,v^*) *is not l.w.s and vice versa, if* (u^*,v^*) *is l.w.s, then* (u_*,v_*) *is not u.w.s.*

Unstable solution

Theorem 2. *Let Hypothesis B and Hypothesis L_x hold and* (S), (P) *possess Property O. Let, moreover,* (α,β) *and* (γ,δ) *be lower and upper functions of* (S), (P), $\alpha(t)\le\gamma(t)$ *for* $t\in[0,\omega]$, $a\in[0,\omega[\,$, $c\in\,]\alpha(a),\gamma(a)[$ *and* (S), (P) *possess property* $Z_{\alpha\gamma}(a,c)$. *Then, there exist unstable solution* (u,v) *of* (S), (P) *such that*

$$\alpha(t)\le u(t)\le\gamma(t) \ \text{ for } \ t\in[0,w].$$

Corollary. *Let Hypothesis B and Hypothesis L_x hold, problem* (S), (P) *possess Property O, and*

$$f_1(t,x+\omega_1,y)=f_1(t,x,y), \quad f_2(t,x+\omega_1)=f_2(t,x) \ \text{ for } \ t\in[0,\omega], \ \ x,y\in\mathbb{R},$$

where $\omega_1>0$. *Let, moreover,* (S), (P) *be solvable and possess no more than countable many solutions. Then* (S), (P) *has countably many unstable solutions.*

Bounded solutions

Theorem 3. *Let Hypothesis B and Hypothesis L_x hold,* $r_0>0$, $h_0\in L([0,\omega])$ *be nontrivial non-negative, and*

$$|f_1(t,x,\sigma r_0)|\ge h_0(t) \ \text{ for } \ t\in[0,\omega], \ \ x\in\mathbb{R}, \ \ \sigma\in\{-1,1\}.$$

Let, moreover, (u_*,v_*) *and* (u^*,v^*) *be consecutive solutions of* (S), (P), $u_*(t)<u^*(t)$ *for* $t\in[0,\omega]$, *and* (u_*,v_*) *is u.w.s* ((u^*,v^*) *is l.w.s*). *The, for every* $c\in\,]u_*(0),u^*(0)[\,$, *problem* (S), (B) *has a solution* (u,v) *such that*

$$\begin{gathered} u_*(t)\le u(t)\le u^*(t), \quad u(t)\le u(t+\omega) \ \text{ for } \ t\ge 0 \\ \Big(u_*(t)\le u(t)\le u^*(t), \quad u(t)\ge u(t+\omega) \ \text{ for } \ t\ge 0\Big) \end{gathered} \tag{2}$$

and

$$\begin{gathered} \lim_{n\to+\infty}\max\big\{|u^*(t)-u(t)|: \ t\in[n\omega,(n+1)\omega]\big\}=0 \\ \Big(\lim_{n\to+\infty}\max\big\{|u_*(t)-u(t)|: \ t\in[n\omega,(n+1)\omega]\big\}=0\Big). \end{gathered}$$

If, moreover, the Cauchy problem for (S) *is uniquely solvable, then all inequalities in* (2) *hold in the strong sense.*

As an example, we consider the system

$$u'=f_0(t,u)\psi(v), \quad v'=p_0(t,u)\sin u+q(t). \tag{S$'$}$$

Here, we suppose that

$$\begin{gathered} p_0(t,x)\le p(t) \ \text{ for } \ t\in[0,\omega], \ \ x\in\mathbb{R}, \\ 0\le h_0(t)\le f_0(t,x)\le h(t) \ \text{ for } \ t\in[0,\omega] \ \ x\in\mathbb{R}, \ \ h_0\not\equiv 0, \end{gathered}$$

and $\psi\in C(\mathbb{R})$,

$$\psi(y)\operatorname{sgn} y\ge 0, \quad |\psi(y)|\le 1 \ \text{ for } \ y\in\mathbb{R}.$$

Solvability of (S′), (P)

Theorem 4. *Let* $\|h\|_L < 2\pi$ *and*

$$\big\|[p]_+\big\|_L + \left|\int\limits_0^\omega g(s)\,\mathrm{d}s\right| \le \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{4}\,.$$

Then, for every $k \in \mathbb{Z}$, *there exists a solution* (u_k, v_k) *of* (S′), (P) *such that*

$$\operatorname{Range}(u_k - 2k\pi) \subseteq \Big[\frac{\pi}{2} - \frac{1}{4}\,\|h\|_L, \frac{3\pi}{2} + \frac{1}{4}\,\|h\|_L\Big]$$

and

$$\Big[\frac{\pi}{2} + \frac{1}{4}\,\|h\|_L, \frac{3\pi}{2} - \frac{1}{4}\,\|h\|_L\Big] \cap \operatorname{Range}(u_k - 2k\pi) \neq \varnothing.$$

In the next theorem, another localization of solutions is stated.

Theorem 5. *Let* $\|h\|_L < \pi$ *and*

$$\big\|[p]_+\big\|_L + \left|\int\limits_0^\omega g(s)\,\mathrm{d}s\right| < \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2}\,. \tag{3}$$

Then, for every $k \in \mathbb{Z}$, *there exists solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *of* (S′),(P) *such that*

$$\operatorname{Range}(u_{1k} - 2k\pi) \subset \Big]-\frac{\pi}{2}\,, \frac{\pi}{2}\Big[, \quad \operatorname{Range}(u_{2k} - 2k\pi) \subset \Big]\frac{\pi}{2}\,, \frac{3\pi}{2}\Big[\,.$$

It is not difficult to verify the validity of

Proposition 5. *Let* $\|h\|_L < \pi$, $i \in \{0, 1\}$ *and*

$$(-1)^{i+1}\int\limits_0^\omega q(s)\,\mathrm{d}s > \big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2}\,.$$

Then, every solution (u, v) *of* (S′), (P) *satisfies*

$$\Big\{(-1)^i\,\frac{\pi}{2} + 2\pi n:\ n \in \mathbb{Z}\Big\} \cap \operatorname{Range} u = \varnothing.$$

Conservative solutions of (S′), (P)

Suppose in addition that

$$\psi \quad \text{is increasing on } \mathbb{R}. \tag{4}$$

Then, by virtue of Proposition 1 and 2, (S′), (P) possesses Property O and Property V. Taking into account Theorem 1, 4, 5 and Proposition 5, we get

Theorem 6. *Let* (4) *hold,* $\|h\|_L < \pi$ *and*

$$\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2} < \left|\int\limits_0^\omega q(s)\,\mathrm{d}s\right| \le \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{4} - \big\|[p]_+\big\|_L\,.$$

Then, for every $k \in \mathbb{Z}$, *there exist a pair of consecutive solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *of* (S′), (P) *such that:*

(1) *If* $\int\limits_0^\omega q(s)\,\mathrm{d}s \geq 0$, *then*

$$\mathrm{Range}(u_{1k}-2k\pi) \subseteq \Big[\frac{\pi}{2}-\frac{1}{4}\,\|h\|_L\,,\frac{3\pi}{2}\Big[,\quad \mathrm{Range}(u_{2k}-2k\pi) \subset \Big]\frac{3\pi}{2}\,,\frac{7\pi}{2}\Big[;$$

(2) *If* $\int\limits_0^\omega q(s)\,\mathrm{d}s \leq 0$, *then*

$$\mathrm{Range}(u_{1k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{5\pi}{2}\Big[,\quad \mathrm{Range}(u_{2k}-2k\pi) \subseteq \Big[\frac{5\pi}{2}\,,\frac{7\pi}{2}+\frac{1}{4}\,\|h\|_L\Big[.$$

Theorem 7. *Let* (4) *hold,* $\|h\|_L < \pi$ *and* (3) *be fulfilled. Then, for every* $k \in \mathbb{Z}$, *there exist two pairs of consecutive solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *and* (u_{3k}, v_{3k}) *and* (u_{4k}, v_{4k}) *of* $(\mathrm{S}'), (\mathrm{P})$ *such that* $u_{2k}(t) \leq u_{3k}(t)$ *for* $t \in [0,\omega]$,

$$\mathrm{Range}(u_{1k}-2k\pi) \subseteq \Big]-\frac{\pi}{2}\,,\frac{\pi}{2}\Big[,\quad \mathrm{Range}(u_{2k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[$$

and

$$\mathrm{Range}(u_{3k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[,\quad \mathrm{Range}(u_{4k}-2k\pi) \subseteq \Big]\frac{3\pi}{2}\,,\frac{5\pi}{2}\Big[.$$

If, moreover, $p(t) \leq 0$ *for* $t \in [0,\omega]$, *then* (u_{1k}, v_{1k}) *is u.w.s and* (u_{4k}, v_{4k}) *is l.w.s.*

Unstable solutions of $(\mathrm{S}'), (\mathrm{P})$

First note that Hypothesis L_x now reads as follows: for every $\varepsilon > 0$, there exists $p_\varepsilon \in L([0,\omega])$ such that

$$\big|f_0(t,x)-f_0(t,x')\big| \leq p_\varepsilon(t)|x-x'| \ \text{ for } \ t \in [0,\omega], \ \ |x-x'| \leq \varepsilon. \tag{5}$$

Theorem 8. *Let* (5) *be fulfilled, and the conditions of Theorem 6 (resp. Theorem 7) hold. Then from every pair of consecutive solutions of* $(\mathrm{S}'), (\mathrm{P})$, *at least one of them is unstable. In particular,* $(\mathrm{S}'), (\mathrm{P})$ *possesses at least countably many unstable solutions.*

Theorem 9. *Let* (4) *and* (5) *hold,* $p(t) \leq 0$ *for* $t \in [0,\omega]$, $\|h\|_L < \pi$, *and*

$$\bigg|\int\limits_0^\omega g(s)\,\mathrm{d}s\bigg| < \|p\|_L \cos\frac{\|h\|_L}{2}\,. \tag{6}$$

Then, for every $k \in \mathbb{Z}$, *the problem* $(\mathrm{S}'), (\mathrm{P})$ *has an unstable solution* (u_k, v_k) *such that*

$$\mathrm{Range}(u_k-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[.$$

Bounded solution of (S') and its asymptotics

First mention that under the assumptions of Theorem 4, one can show that for every $c \in \mathbb{R}$, the problem $(\mathrm{S}'), (\mathrm{B})$ is solvable. However, we are interested in the existence of non-periodic solutions of $(\mathrm{S}'), (\mathrm{B})$.

Theorem 10. *Let* (5) *hold, and the conditions of Theorem 6 be fulfilled. Let, moreover,* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *be solutions of* $(\mathrm{S}'), (\mathrm{P})$ *appearing in the conclusion of Theorem 6. Then, for every* $k \in \mathbb{Z}$*, there exists a non-periodic solution* (u_k, v_k) *of* $(\mathrm{S}'), (\mathrm{B})$ *such that*

$$u_{1k}(t) \le u_k(t) \le u_{2k}(t) \ \text{ for } \ t \ge 0.$$

Theorem 11. *Let* (4) *and* (5) *hold,* $\|h\|_L < \pi$*, and* (6) *be fulfilled (clearly, conditions of Theorem 7 hold). Let, moreover,* $k \in \mathbb{Z}$ *and* (u_{ik}, v_{ik})*,* $i = 1, 2, 3, 4$*, be solutions of* $(\mathrm{S}'), (\mathrm{P})$*; their existence is stated in Theorem 7.*

Then, for every $c \in]u_{1k}(0), u_{2k}(0)[$ *, the problem* $(\mathrm{S}'), (\mathrm{B})$ *has a solution* (u_k, v_k) *such that*

$$u_{1k}(t) \le u_k(t) \le u_{2k}(t), \quad u_k(t) \le u_k(t+\omega) \ \text{ for } \ t \ge 0, \tag{7}$$

and

$$\lim_{n \to +\infty} \max \big\{ |u_k(t) - u_{2k}(t)| : \ t \in [n\omega, (n+1)\omega] \big\} = 0,$$

while, for every $c \in]u_{3k}(0), u_{4k}(0)[$ *, the problem* $(\mathrm{S}'), (\mathrm{B})$ *possesses a solution* (u_k, v_k) *such that*

$$u_{3k}(t) \le u_k(t) \le u_{4k}(t), \quad u_k(t) \ge u_k(t+\omega) \ \text{ for } \ t \ge 0, \tag{8}$$

and

$$\lim_{n \to +\infty} \max \big\{ |u_k(t) - u_{3k}(t)| : \ t \in [n\omega, (n+1)\omega] \big\} = 0,$$

If, moreover, ψ *is a Lipschitz function, then all inequalities in* (7) *and* (8) *hold in the strict sense.*

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Approximate Averaged Bounded Feedback Control with Two Switching Points for a Parabolic Process

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Let $\Omega \subset \mathbb{R}^n$ be a bounded region with a smooth boundary and for $\varepsilon \in (0,1)$ controlled process in a cylinder $\overline{Q}_T = [0,T] \times \overline{\Omega}$ be described by a boundary value problem

$$\begin{cases} y_t^\varepsilon(x,t) = A^\varepsilon(y^\varepsilon(x,t)) + g^\varepsilon(x)v(t), & (t,x) \in Q_T, \\ y^\varepsilon(x,t) = 0, & x \in \partial\Omega, t \in [0,T], \\ y^\varepsilon(x,0) = y_0^\varepsilon(x), & x \in \overline{\Omega}, \end{cases} \tag{1}$$

where $A^\varepsilon := \operatorname{div}(a^\varepsilon \overrightarrow{\nabla})$, $a^\varepsilon(x) = ((a_{ij}^\varepsilon(x)))_{i,j=1}^n$ – measured symmetric matrix satisfying the conditions of uniform ellipticity and boundedness, $g^\varepsilon, y_0^\varepsilon \in L_2(\Omega)$, ε – small parameter.

Control is limited by

$$v(\,\cdot\,) \in U = \Big\{ v \in L_2(0,T) : \ \ |v(t)| \leq \xi \ \text{ almost everywhere on } [0;T] \Big\}. \tag{2}$$

The problem of optimal control is to minimize the semi-definite quality criterion

$$J^\varepsilon(v) = \bigg(\int\limits_\Omega q(x) y^\varepsilon(x,T)\,dx \bigg)^2 + \gamma \int\limits_0^T v^2(t)\,dt, \tag{3}$$

at solutions (1), where $q \in L_2(\Omega)$, $\gamma > 0$.

We will assume the following convergences of the coefficients of the optimal control problem (1)–(3):

$$\begin{gathered} g^\varepsilon \to g^0, \ \ y_0^\varepsilon \to y_0 \ \text{ weakly in } L_2(\Omega) \ \text{ as } \ \varepsilon \to 0, \\ a^\varepsilon \to a^0 \ \text{ in sense of } G\text{-convergence of matrices as } \ \varepsilon \to 0. \end{gathered} \tag{4}$$

For definitions and properties of G-convergence of matrices see [1]. Further using the limit functions from (4) we can assume that the optimal control problem (1)–(3) is defined also for $\varepsilon = 0$ (so-called averaged problem). It is known that for each fixed control $v \in U$ problem (1) has a unique solution in the class $C([0,T]; L_2(\Omega))$ [3]. In addition, for an arbitrary $\varepsilon \in [0,1)$ the optimal control problem (1)–(3) also has a unique solution $v^\varepsilon(\,\cdot\,) \in U$ [4].

Applying the Fourier method and decomposing the data of the initial optimal control problem by the eigenbasis $\{X_i^\varepsilon(x)\}_{i=1}^\infty$ of the operator

$$A^\varepsilon : A^\varepsilon X_i^\varepsilon + (\lambda_i^\varepsilon)^2 X_i^\varepsilon = 0, \ \ i = \overline{1,\infty},$$

the problem (1)–(3) is split and reduced to an equivalent optimal control problems for the first order ordinary differential equation [6]. In addition, we will assume that the spectrum of the averaged operator A^0 is simple, i.e.,

$$0 < (\lambda_1^0)^2 < (\lambda_2^0)^2 < \cdots . \tag{5}$$

Let g_i^ε, q_i^ε be the Fourier coefficients of the corresponding functions according to the system $\{X_i^\varepsilon(x)\}_{i=1}^\infty$. Let us denote

$$f^\varepsilon(t)=\sum_{i=1}^\infty e^{(\lambda_i^\varepsilon)^2(t-T)}q_i^\varepsilon g_i^\varepsilon,\quad a_0^\varepsilon=\sum_{i=1}^\infty e^{-(\lambda_i^\varepsilon)^2T}q_i^\varepsilon y_i^{0\varepsilon},$$
$$\Re^\varepsilon(x,t)=\sum_{i=1}^\infty q_i^\varepsilon e^{(\lambda_i^\varepsilon)^2(t-T)}X_i^\varepsilon(x).$$

The resulting optimal control problem for an ordinary differential equation has already been studied in [2] under the condition that $f^\varepsilon(t)$ is a positive and strictly monotonically increasing function. We consider another condition and assume that for each $\varepsilon\in[0,1)$ the function $f^\varepsilon(t)$ remains strictly monotonically increasing and changes sign, i.e., there exists such $t_0^\varepsilon\in(0,T)$ that

$$f^\varepsilon(0)<0,\;\; f^\varepsilon(t_0)=0,\;\; f^\varepsilon(T)>0,\;\; f^\varepsilon(t)\text{ strictly monotonically increases on }[0,T]. \tag{6}$$

Refusion of the requirement of function $f^\varepsilon(t)$ positiveness allows us to observe a more complex behavior of the optimal control. In particular, it becomes possible for the optimal control to have two switching points when reaching both the lower $v=-\xi$ and upper $v=-\xi$ restrictions. This qualitatively new case we will pay our further attention to.

Applying the maximum principle, under the condition that the system of inequalities

$$\frac{\mp a_0^\varepsilon f^\varepsilon(0)}{\gamma+\int_0^T(f^\varepsilon(s))^2\,ds}<-\xi,\quad \frac{\mp a_0^\varepsilon f^\varepsilon(T)}{\gamma+\int_0^T(f^\varepsilon(s))^2\,ds}>\xi,\quad \xi\int_0^T|f^\varepsilon(s)|\,ds<\mp a_0^\varepsilon \tag{7}$$

is fulfilled for all $\varepsilon\in[0,1)$, we obtain the optimal control in the feedback form for optimal control problem (1)–(3)

$$u^\varepsilon[t,y^\varepsilon(t)]=\begin{cases}\mp\xi, & t\in[0,t_-^\varepsilon],\\ -\dfrac{(\Re^\varepsilon(t),y^\varepsilon(t))\pm\xi\int_{t_+^\varepsilon}^T f^\varepsilon(s)\,ds}{\gamma+\int_{t_-^\varepsilon}^{t_+^\varepsilon}(f^\varepsilon(s))^2\,ds}\,f^\varepsilon(t), & t\in[t_-^\varepsilon,t_+^\varepsilon],\\ \pm\xi, & t\in[t_+^\varepsilon,T],\end{cases} \tag{8}$$

where $y^\varepsilon(x,t)$ – solution of the boundary value problem (1) with control (8), and the switching points t_-^ε and t_+^ε are determined from the system

$$\begin{cases}\dfrac{a_0^\varepsilon\mp\xi\int_0^{t_-^\varepsilon}f^\varepsilon(s)\,ds\pm\xi\int_{t_+^\varepsilon}^T f^\varepsilon(s)\,ds}{\gamma+\int_{t_-^\varepsilon}^{t_+^\varepsilon}(f^\varepsilon(s))^2\,ds}\,f^\varepsilon(t_-^\varepsilon)=\pm\xi\,,\\[2ex] \dfrac{a_0^\varepsilon\mp\xi\int_0^{t_-^\varepsilon}f^\varepsilon(s)\,ds\pm\xi\int_{t_+^\varepsilon}^T f^\varepsilon(s)\,ds}{\gamma+\int_{t_-^\varepsilon}^{t_+^\varepsilon}(f^\varepsilon(s))^2\,ds}\,f^\varepsilon(t_+^\varepsilon)=\mp\xi\,.\end{cases} \tag{9}$$

It can be seen from the formulas (8), (9) that optimal control contains coefficients which are expressed by series and, in addition, it irregularly depends on a small parameter ε. Thus, for practical application it is natural to restrict infinite series by finite sums and to get rid of dependence on a small parameter using average procedure.

For an arbitrary $\varepsilon \in [0;1)$ and $N \in \mathbb{N}$, we define

$$
\begin{aligned}
a_{0N}^{\varepsilon} &= \sum_{i=1}^{N} e^{-(\lambda_i^{\varepsilon})^2 T} q_i^{\varepsilon} y_i^{0\varepsilon}, \\
f_N^{\varepsilon}(t) &= \sum_{i=1}^{N} e^{(\lambda_i^{\varepsilon})^2 (t-T)} q_i^{\varepsilon} g_i^{\varepsilon}, \\
\Re_N^{\varepsilon}(x,t) &= \sum_{i=1}^{N} q_i^{\varepsilon} e^{(\lambda_i^{\varepsilon})^2 (t-T)} X_i^{\varepsilon}(x).
\end{aligned}
$$

Let $(8)_N$ and $(9)_N$ be the formulas (8) and (9) respectively in which all occurrences of a_0^{ε}, $f^{\varepsilon}(t)$ and $\Re^{\varepsilon}(x,t)$ are replaced with a_{0N}^{ε}, $f_N^{\varepsilon}(t)$ and $\Re_N^{\varepsilon}(x,t)$, respectively, and $t_{-N}^{\varepsilon}, t_{+N}^{\varepsilon}$ are the corresponding switching points. It can be proved that for each $N \in \mathbb{N}$ the points t_{-N}^{ε}, t_{+N}^{ε} are determined uniquely, and for an arbitrary $\varepsilon \in [0;1)$ the convergences $t_{-N}^{\varepsilon} \to t_{-}^{\varepsilon}$, $t_{+N}^{\varepsilon} \to t_{+}^{\varepsilon}$ as $N \to \infty$ are fulfilled.

We will construct the law of approximate averaged synthesis for problem (1)–(3), and make sure of the closeness of the values of the quality criterion (3) on the optimal control and constructed approximate control. Further, the main object of our research is the approximate averaged control

$$
v_N[t, z_N^{\varepsilon}(t)] = \begin{cases}
\mp\xi, & t \in [0, t_{-N}^0], \\
-\dfrac{(\Re_N^0(t), z_N^{\varepsilon}(t)) \pm \xi \int\limits_{t_{+N}^0}^{T} f_N^0(s)\,ds}{\gamma + \int\limits_{t_{-N}^0}^{t_{+N}^0} (f_N^0(s))^2\,ds}\, f_N^0(t), & t \in [t_{-N}^0, t_{+N}^0], \\
\pm\xi, & t \in [t_{+N}^0, T],
\end{cases} \tag{10}
$$

where $z_N^{\varepsilon}(x,t)$ – solution of problem (1) with control (10), t_{-N}^0, t_{+N}^0 – switching points that are defined from system $(9)_N$ at $\varepsilon = 0$.

Using the ideas from [2], it is possible to prove the correctness of the proposed approximate averaged synthesis (10), that is, the control in the feedback form (10) provides close to the optimal value of the objective functional (10). Namely, we have

Theorem. *Let $g^{\varepsilon}, y_0^{\varepsilon}$, $q \in L_2(\Omega)$ and the assumptions* (4)–(7) *be fulfilled. Then for an arbitrary small $\eta > 0$ there exist $N_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $N \geq N_0$ and $\varepsilon \in (0, \varepsilon_0)$ for the controls* (8), (10) *we have the inequalities*

$$
\begin{gathered}
\big\| u^{\varepsilon}[\,\cdot\,, y^{\varepsilon}] - v_n[\,\cdot\,, z_n^{\varepsilon}] \big\|_{L_2(0,T)} < \eta, \\
\big\| y^{\varepsilon}(\,\cdot\,, t) - z_n^{\varepsilon}(\,\cdot\,, t) \big\| < \eta \ \ \textit{for all} \ \ t \in [0, T], \\
\big| J^{\varepsilon}\big(u^{\varepsilon}[t, y^{\varepsilon}]\big) - J^{\varepsilon}\big(u_n[t, z_n^{\varepsilon}]\big) \big| < \eta.
\end{gathered}
$$

So, besides the law of the optimal synthesis (8), (9) also approximate averaged feedback control (10), which provides control system behavior that is close to optimal one and thus has a series of advantages from the practical application point of view, is proposed and proved.

In [5] the efficiency of approximate averaged control construction procedure consisted of cutting to finite sums of series in optimal control and replacing of Fourier coefficients nonregular depended on small parameter by corresponding average values is illustrated by concreate example of controlled system for parabolic process. We compare the properties of the averaged control and the sequence of optimal controls calculated at the different values of the small parameter. For comparison, the switching points of optimal control and averaged control, deviation between optimal control and averaged control, the difference in the values of the quality criterion on optimal control and averaged control are used. In considered example the precision of quality criteria value on approximate control has ε-order and for sufficiently small ε the precision is one order better then ε.

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Some Sufficient Conditions for Almost Reducibility of Millionshchikov Systems

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Consider a linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with continuous and bounded coefficient matrix A.

System (1) is said to be almost reducible [2] (or approximately similar, see [6]) if for any $\delta > 0$ there exists a Lyapunov transformation reducing system (1) to the form

$$\dot{x} = Bx + Q_\delta(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0,$$

where B is a constant matrix and Q_δ satisfy the condition $\|Q_\delta(t)\| \leq \delta$. Note that the matrix B is the same for all $\delta > 0$.

The property of almost reducibility plays a crucial role in many issues related to Erugin's problem on Lyapunov regularity of linear systems with almost periodic coefficients. This problem was posed by N. P. Erugin at a mathematical seminar at the Institute of Physics and Mathematics of Byelorussian Academy of Sciences in 1956. The original formulation of Erugin's problem involved proving the hypothesis of Lyapunov regularity of all systems with almost periodic coefficients, see [4, pp. 121, 137] and also [5].

Erugin's problem has been solved by V. M. Millionshchikov who has proved the following two statements.

(i) *Let $\mathcal{H}(A)$ be the hull of A, i.e. the uniform closure of all shifts $A_\tau(t) := A(t+\tau)$. If A is almost periodic, then almost all systems with coefficient matrices from $\mathcal{H}(A)$ are Lyapunov regular* [17].

(ii) *There exists some Lyapunov irregular system* (1) *with almost periodic coefficients* [19].

It should be noted that the proof in [19] is not completely constructive and use the following result from [18].

(iii) *If there exists a non-almost reducible system with coefficient matrix from $\mathcal{H}(A)$, then there exists an irregular system with coefficient matrix from $\mathcal{H}(A)$* [18].

By virtue of (iii), to prove statement (ii) it is sufficient to construct some non-almost reducible system with almost periodic coefficients. To this end V. M. Millionshchikov introduced a special class of limit periodic linear systems and constructed the required system within that class. Now such systems are usually called Millionshchikov systems. A comprehensive study of such systems was made by A. V. Lipntskii in [8–15]. In particular, an explicit example of Lyapunov-irregular Millionshchikov system is given in [8] (see also [21]). However, no effective tools are known for recognising almost reducibility for these systems.

A number of almost reducibility criteria are known for general systems and systems with almost periodic coefficients, see e.g. [3, 7, 20]. However, most of these criteria are based on properties of some solution sets for such systems. In [16] we propose a sufficient condition for almost reducibility of Millionshchikov systems based on properties of periodic approximations to the system under consideration. Our goal here is to give some corollaries of this result.

Let coefficient matrix A has the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t + \tau_k), \tag{2}$$

where A_k, $k = 0, \ldots, +\infty$, are periodic matrices with the periods T_k and τ_k are arbitrary real numbers. If each matrix A_k is everywhere continuous and series (2) converges uniformly on the entire time axis $\mathbb{R}$, then the matrix A is limit-periodic [1, p. 32] and, therefore, almost periodic.

In what follows we suppose that $T_0 = 2$, $T_k \in \mathbb{N}$, and $T_{k+1}/T_k = m_i \in \mathbb{N}$ for all $k = 0, \ldots, +\infty$. We also suppose that $m_k > 1$, $k = 0, \ldots, +\infty$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take some continuous function $\omega : [0, 1] \to \mathbb{R}$ such that $\omega(0) = \omega(1) = 0$ and $\int\limits_0^1 \omega(t)\, dt = 1$. Take also a sequence $\varphi : \mathbb{N} \to [0, \pi/2[$. As usually, the values of the sequence φ we denote by φ_k, $k \in \mathbb{N}$.

Now let us define the matrices A_k by the following equalities:

$$A_0(t) = \begin{cases} \omega(t)D, & \text{for } t \in [0, 1[, \\ 0, & \text{for } t \in [1, 2[\end{cases} \tag{3}$$

for $k = 0$ and

$$A_k(t) = \begin{cases} -\varphi_k \omega(t) J, & \text{for } t \in [0, 1[, \\ 0, & \text{for } t \in [1, T_i[\end{cases} \tag{4}$$

for all $k = 1, \ldots, +\infty$.

It can be easily shown that if

$$\sum_{k=1}^{\infty} \varphi_k < +\infty,$$

then system (1) with the coefficient matrix A defined by (3) and (4) is limit periodic.

Definition 1. We say that system (1) with the coefficient matrix A defined by (3), (4), and (2) with $\tau_k = 0$ is a gathered Millionshchikov system.

Definition 2. System (1) with the coefficient matrix A defined by (3), (4), and (2) with $\tau_k \in 2\mathbb{Z}$, is said to be a Millionshchikov system. We say that this system corresponds to a gathered Millionshchikov system with the same matrices A_k, $k = 0, \ldots, +\infty$.

Note that any gathered Millionshchikov system has the coefficient matrix of the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t).$$

Let

$$S_m(t) = \sum_{k=0}^{m} A_k(t), \quad m = 1, \ldots, +\infty,$$

where A_k are defined by (3) and (4). It can be easily seen that each matrix S_m is T_m-periodic. Now for arbitrary $m \in \mathbb{N}$ consider a periodic linear system

$$\dot{z} = S_m(t)z, \quad z \in \mathbb{R}^2, \quad t \in \mathbb{R}. \tag{5}$$

Denote the Cauchy matrix of system (5) by Z_m. Then the monodromy matrix of system (5) can be written as $Z_m(T_m, 0)$ and the eigenvalues of $Z_m(T_m, 0)$ are the Floquet multipliers of system (5). If these numbers are real, then we can find some real eigenvectors of $Z_m(T_m, 0)$ and the angle β_m between them.

Definition 3 ([16])**.** We say that gathered Millionshchikov system (1) is a real-type system if all Floquet multipliers of each corresponding system (5) with $m \in \mathbb{N}$ are real.

Theorem 1 ([16])**.** *Suppose that system* (1) *is a real-type gathered Millionshchikov system. If the angle β_m is separated from zero for all $m \in \mathbb{N}$, then system* (1) *is almost reducible.*

Definition 4. We say that a gathered Millionshchikov system (1) minorises another gathered Millionshchikov system if the angles φ_k of the first system are not greater than the corresponding angles of the second system.

Theorem 2. *If system* (1) *satisfies conditions of Theorem 1, then all minorizing it gathered Millionshchikov systems are almost reducible.*

Theorem 3. *If system* (1) *satisfies conditions of Theorem 1, then all corresponding to it Millionshchikov systems are almost reducible.*

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On Control Problems for Systems with Fractional Derivative and Aftereffect

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1 Introduction

We consider a controlled system of linear functional differential equations with a fractional derivative and aftereffect. The well-known definition of the fractional Caputo derivative of the order $\alpha \in (0,\,1)$ is used. The system under study includes, in addition to the Caputo derivative, a linear Volterra operator of a general form defined on the trajectories and a linear Volterra operator defined on the controls. For the system, an initial state is fixed. The aim of controlling is given by a prescribed value of a linear target vector-functional. The question of the solvability to the control problem is studied for two cases: the case without constraints regarding control and the case of linear constraints with respect to control. The approach used is based on the theory of abstract functional differential equations (AFDE) developed by the heads of the Perm Seminar, professors N. V. Azbelev and L. F. Rakhmatullina, and systematically presented in [2].

2 Preliminaries

Let $L_\infty = L_\infty^n[0,T]$ be the space of measurable and bounded in essence functions $z : [0,T] \to \mathbb{R}^n$ with the norm $\|z\|_{L_\infty} = \operatorname{vraisup}\,(|z(t)|,\ t \in [0,T])$ (here and below $|\cdot|$ stands for a norm in $\mathbb{R}^n$). $AC_\infty = AC_\infty^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ with the derivative $\dot{x} \in L_\infty$ and the norm $\|x\|_{AC_\infty} = |x(0)| + \|\dot{x}\|_{L_\infty}$, $L_2 = L_2^r[0,T]$ is the space of square summable functions $u : [0,T] \to \mathbb{R}^r$ with the inner product $\langle u,\, v\rangle = \int\limits_0^T u'(t)\cdot v(t)\,dt$ (the symbol $(\,\cdot\,)'$ stands for transposition).

Consider the linear fractional functional differential system

$$\mathcal{D}^\alpha x = \mathcal{T}x + f, \tag{2.1}$$

where $\mathcal{D}^\alpha$ is the Caputo derivative of the order $\alpha \in (0,1)$ (see, for instance, [4]),

$$(\mathcal{D}^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\dot{x}(s)}{(t-s)^\alpha}\,ds$$

($\Gamma(\,\cdot\,)$ is the Euler gamma-function), $\mathcal{T} : AC_\infty^n[0,T] \to L_\infty^n[0,T]$ is linear bounded Volterra operator with the property: there exists $p > 0$ such that the inequality

$$|(\mathcal{T}x)(t)| \le p \max_{s\in[0,t]} |x(s)|, \ \ t \in [0,T] \tag{2.2}$$

holds for any $x \in AC_\infty^n[0,T]$.

All our constructions below are based on the representation of solutions to (2.1) with the initial condition $x(0) = 0$.

Let us denote

$$(Kz)(t) = (\mathcal{T} J^{\alpha} z)(t),$$

where the fractional integration operator $J^{\alpha} : L^n_{\infty}[0,T] \to AC^n_{\infty}[0,T]$ is defined by the equality

$$(J^{\alpha} z)(t) = \int\limits_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, z(s)\, ds$$

(see, for instance, [4]).

Remark. Equations (2.1) and $\dot{x} = \mathcal{T} x + f$ are two different representatives of the AFDE [2] $\delta x = \mathcal{T} x + f$. Therewith the theory of the latter uses the representation $x = V \frac{d}{dt} x + x(0)$, where $(Vz)(t) = \int\limits_0^t z(s)\, ds$, while the theory of (2.1) is based on the representation $x = J^{\alpha} \mathcal{D}^{\alpha} x + x(0)$. The space AC_{∞} is isomorphic to the direct product $L_{\infty} \times \mathbb{R}^n$ with two possible isomorphisms: $x = Vz + \beta$ and $x = J^{\alpha} z + \beta$, correspondingly.

Throughout the following, we assume that $K : L^n_{\infty}[0,T] \to L^n_{\infty}[0,T]$ is a regular integral Volterra operator:

$$(Kz)(t) = \int\limits_0^t K(t,s)\, z(s)\, ds.$$

This condition is fulfilled for wide classes of operators $\mathcal{T}$ including the operators of inner superposition with delay [1] and Volterra integral ones. In such cases, the representation of the kernel $K(t,s)$ can be obtained in an explicit form.

Under above condition (2.2), the operator K has the resolvent operator $R : (I-K)^{-1} = I + R$, where I is the identity operator, see [9], and R is an integral Volterra operator too:

$$(Rf)(t) = \int\limits_0^t R(t,s) f(s)\, ds$$

with the resolvent kernel $R(t,s)$ [11, Theorem 2.2, p. 119].

As is shown in [9], the Cauchy problem for (2.1) with the initial condition $x(0) = 0$ is uniquely solvable, and its solution has the representation

$$x(t) = (Cf)(t) = \int\limits_0^t C(t,s) f(s)\, ds, \tag{2.3}$$

where $C(t,s)$ is the Cauchy matrix that is defined by the equality

$$C(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, E + \int\limits_s^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\, R(\tau,s)\, d\tau \tag{2.4}$$

(here and below E is the identity $(n \times n)$-matrix). Note that, for $\alpha = 1$, (2.4) takes the form [5]

$$C(t,s) = E + \int\limits_s^t R(\tau,s)\, d\tau.$$

Operator $C : L_{\infty} \to AC_{\infty}$ is called the Cauchy operator.

3 The problem formulation

The development of the theory of fractional dynamics led to the works on the dynamics of fractional systems with control. A detailed review of the fundamental works on the control theory of systems with fractional derivatives is given in [3].

Consider the fractional functional differential system under control

$$\mathcal{D}^\alpha x = \mathcal{T}x + Fu + f, \tag{3.1}$$

where $F : L_2^r[0,T] \to L_\infty^n[0,T]$ is a linear Volterra operator that is responsible for the implementation of control actions $u \in L_2^r[0,T]$.

Without loss of generality we assume that the initial state of the system is zero:

$$x(0) = 0. \tag{3.2}$$

The goal of control we set by the equality

$$\ell x \equiv \sum_{i=1}^{m} A_i x(t_i) + \int\limits_0^T B(\tau)x(\tau)\,d\tau = \beta \in \mathbb{R}^N, \tag{3.3}$$

where t_i, $i = 1,\dots,m$ are fixed points from $[0,T]$, A_i are constant $(N \times n)$-matrices, $B(\,\cdot\,)$ is $(N \times n)$-matrix with summable elements, β is prescribed constant vector. We study the solvability of the control problem (3.1)–(3.3) for two cases: the case without constraints regarding control and the case of linear constraints with respect to control. Some results are presented in the next section.

4 Main results

Let's denote

$$\Theta_i(s) = A_i\big\{F^*\big[\chi_i(\,\cdot\,)C(t_i,\,\cdot\,)\big]\big\}(s),\ \ i = 1,\dots,m; \quad \Phi(s) = \int\limits_s^T B(\tau)C(\tau,s)\,d\tau; \tag{4.1}$$

$$\Theta_{m+1}(s) = \big\{F^*\Phi(\,\cdot\,)\big\}(s); \quad M(s) = \sum_{i=1}^{m+1}\Theta_i(s); \quad W = \int\limits_0^T M(s)M'(s)\,ds. \tag{4.2}$$

Here $\chi_i(\,\cdot\,)$ is the characteristic function of the segment $[0,t_i]$, F^* is the adjoint operator to F, M is the so called moment matrix.

Theorem 4.1 ([10])**.** *Control Problem* (3.1)–(3.3) *is solvable for any $f \in L_\infty^n[0,T]$ and $\beta \in \mathbb{R}^N$ if and only if the $(N \times N)$-matrix W defined by the equalities* (4.1), (4.2) *is invertible. The control $u_0(t) = M'(t)\,d$ with $d = W^{-1}[\beta - \ell Cf]$, solves the problem.*

In the case of polyhedral constraints with respect to the control u:

$$\Lambda \cdot u(t) \le \gamma, \ \ \gamma \in \mathbb{R}^{N_1}, \ \ t \in [0,T], \tag{4.3}$$

with constant $(N_1 \times r)$-matrix (it is assumed that the set of solutions to the system $\Lambda \cdot v \le \gamma$ is nonempty), there arises the problem of the description to the set of all target values β such that the problem (3.1)–(3.3), (4.3) is solvable. Such a set is called the attainability set to the problem.

It should be noted that, in most works, the attainability is understood in relation to the terminal target vector-functional $\ell x = x(T)$. In contrary to this, we consider the essentially more general case of the target vector-functional, and the term ℓ-attainability seems to be more proper. Here we consider the question on the inner (lower by inclusion) estimates to the ℓ-attainability set. In the case of the systems with the derivative of the integer order, this question is studied in [6–8]. All constructions used to obtain the inner estimates are based on the representation (2.3) of solutions to the system (3.1).

Using the moment matrix M, the equality (3.3) that defines the aim of control is reduced to the integral form with respect to control u:

$$\int_0^T M(t) \cdot u(t)\, dt = \beta.$$

Thus the control problem (3.1)–(3.3), (4.3) is reduced to the system

$$\int_0^T M(t) \cdot u(t)\, dt = \beta \in \mathbb{R}^N, \quad \Lambda \cdot u(t) \le \gamma, \quad t \in [0, T].$$

The inner estimate of the ℓ-attainability set is based on the following constructions. Let us split the segment $[0, T]$ onto partial ones by the points $\vartheta_1, \dots, \vartheta_{\mathcal{K}-1} : 0 = \vartheta_0 < \vartheta_1 < \dots < \vartheta_{\mathcal{K}-1} < T = \vartheta_{\mathcal{K}}$, and denote by $\chi_i(t)$ the characteristic function of the interval $(\vartheta_{i-1}, \vartheta_i]$. We restrict the class of controls by piecewise constant ones of the form

$$u(t) = \sum_{i=1}^{\mathcal{K}} d_i \chi_i(t), \tag{4.4}$$

where $d_i \in \mathbb{R}^m$ are constant vectors. Next we define constant $(N \times r)$-matrices M_i by the equalities

$$M_i = \int_{\vartheta_{i-1}}^{\vartheta_i} M(t)\, dt, \quad i = 1, \dots, \mathcal{K}.$$

Let us fix a collection of vectors $\lambda_1, \dots, \lambda_j, \dots, \lambda_{\mathcal{N}} \in R^N$, and, for every j, set the linear programming problem

$$\sum_{i=1}^{\mathcal{K}} \lambda_j' \cdot M_i d_i \to \max, \quad \Lambda \cdot d_i \le \gamma, \quad i = 1, \dots, \mathcal{K}. \tag{4.5}$$

Let $\lambda_{j_1}, \dots, \lambda_{j_{\mathcal{N}_1}}$ be a subset of the collection $\{\lambda_j\}$, $j = 1, \dots, \mathcal{N}$ such that, for any its element the problem (4.5) has a solution $D^{j_k} = (d_1^{j_k}, \dots, d_{\mathcal{K}}^{j_k})$, $k = j_1, \dots, j_{\mathcal{N}_1}$. Every such solution, after substitution of it into (4.4), defines a program control $u^{j_k}(t)$ that gives an attainable value of the target vector-functional ℓ:

$$\ell x = \int_0^T M(t) \cdot u^{j_k}(t)\, dt = \rho^{j_k}.$$

The collection of such values (points in $\mathbb{R}^N$) allows one to obtain an inner estimate to the ℓ-attainability set.

Theorem 4.2. *Let $\mathcal{P} \subset \mathbb{R}^N$ be the set of all linear convex combinations of the points ρ^{j_k}, $k = j_1, \dots, j_{\mathcal{N}_1}$. Then any value $\beta \in \mathcal{P}$ is an ℓ-attainable value in the problem* (3.1)–(3.3), (4.3).

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Two Point Boundary Value Problems for the Fourth Order Ordinary Differential Equations

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1 Introduction

We study, on the interval $I := [a, b]$, the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t) \tag{1.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{1.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{1.3$_1$}$$

$$u^{(j)}(a) = 0 \;\; (j = 0, 1, 2), \quad u(b) = 0, \tag{1.3$_2$}$$

where $p, h \in L(I;\, R)$, $f \in K(I \times R; R)$.

By a solution of problem $(1.2), (1.3_i)$ $(i \in \{1, 2\})$ we understand a function $u \in \widetilde{C}^3(I; R)$ which satisfies equation (1.2) a.e. on I, and conditions (1.3_i).

We use the following notations here.

$\widetilde{C}^{(3)}(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their third derivatives;

$L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\, ds$;

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions.

For arbitrary $x, y \in L(I;\, R)$, the notation

$$x(t) \preccurlyeq y(t) \;\; (x(t) \succcurlyeq y(t)) \;\; \text{for} \;\; t \in I$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$; We also use the notations $[x]_\pm = (|x| \pm x)/2$.

The aim of our work is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ depends only on the nonnegative (nonpositive) part of the coefficient p if this nonnegative (nonpositive) part is small enough. On the basis of these results for the nonlinear problems, we have proved sufficient conditions of solvability, which in some sense improve previously known results.

The results of the given work are based on our previous results from the papers [1] and [2].

Below we present some definitions and results from these papers.

Definition 1.1. Equation
$$u^{(4)}(t) = p(t)u(t) \ \text{ for } \ t \in I \tag{1.4}$$
is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less than four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 1.2. We will say that $p \in D_+(I)$ if $p \in L(I; R_0^+)$, and problem (1.4), (1.3_1) has a solution u such that
$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,. \tag{1.5}$$

Definition 1.3. We will say that $p \in D_-(I)$ if $p \in L(I; R_0^-)$, and problem (1.4), (1.3_2) has a solution u such that inequality (1.5) holds.

Theorem 1.1 ([1])**.**

(a) *Let the equation*
$$u^{(4)}(t) = [p(t)]_+ u(t)$$
be disconjugate on I. Then problem (1.1), (1.3_1) *is uniquely solvable for arbitrary* $[p]_-$ *and* q*.*

(b) *Let the equation*
$$u^{(4)}(t) = -[p(t)]_- u(t)$$
be disconjugate on I. Then problem (1.1), (1.3_2) *is uniquely solvable for arbitrary* $[p]_+$ *and* q*.*

Theorem 1.2 ([2])**.** *Let $p \in L(I;\, R_0^+)$. Then for the discojugacy of equation* (1.4) *on I it is necessary and sufficient the existence of $p^* \in D_+(I)$, such that*
$$p(t) \preccurlyeq p^*(t) \ \textit{ for } \ t \in I.$$

Theorem 1.3 ([2])**.** *Let $p \in L(I;\, R_0^-)$. Then for the dsconjugacy of equation* (1.4) *on I it is necessary and sufficient the existence of $p_* \in D_-(I)$ such that*
$$p_*(t) \preccurlyeq p(t) \ \textit{ for } \ t \in I.$$

2 Linear problems

The proofs of the following results of the unique solvability of problems (1.1), (1.3_1) and (1.1), (1.3_2) are based on Theorems 1.1–1.3.

Theorem 2.1 ([3])**.** *Let $i \in \{1,2\}$ and the function $p_0 \in L(I;R)$ be such that the equation*
$$u^{(4)}(t) = [p_0(t)]_+ u(t) \ \textit{ if } \ i = 1,$$
$$u^{(4)}(t) = -[p_0(t)]_- u(t) \ \textit{ if } \ i = 2,$$
is disconjugate on I. Then if the inequality
$$(-1)^{i-1}[p(t) - p_0(t)] \le 0 \ \textit{ for } \ t \in I$$
holds, problem (1.1), (1.3_i) *is uniquely solvable.*

From the last theorem with $p_0 = [p]_+$ or $p_0 = -[p]_-$ it immediately follows

Corollary 2.1. *Let there exist $p^* \in D_+(I)$ $(p_* \in D_-(I))$ such that the inequality*

$$[p(t)]_+ \preccurlyeq p^*(t) \ \ \big(-[p(t)]_- \succcurlyeq p_*(t) \big) \ \text{ for } \ t \in I$$

holds. Then problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ is uniquely solvable.

Corollary 2.2. *Let inequality*

$$p(t) \le \frac{\lambda_1^4}{(b-a)^4} \approx \frac{500}{(b-a)^4} \ \ \Big([p(t)]_- \le \frac{\lambda_2^4}{(b-a)^4} \approx \frac{949}{(b-a)^4} \Big)$$

holds, where λ_1 (λ_2) is the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \ \ u^{(j)}(0) = 0, \ \ u^{(j)}(1) = 0 \ \ (j = 0, 1) \ \ \ \big(u^{(j)}(0) = 0 \ \ (j = 0, 1, 2), \ \ u(1) = 0 \big).$$

Then problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ is uniquely solvable.

3 Nonlinear problem

On the basis of our results for the linear problems, for the nonlinear problems we have proved the following solvability theorem.

Theorem 3.1 ([3])**.** *Let $i \in \{1, 2\}$ and there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a.e. on I the inequality*

$$-g(t)|x| \le (-1)^{i-1} f(t, x) \operatorname{sgn} x \le \delta(t, |x|) \ \text{ for } \ |x| > r \tag{3.1}$$

holds, where the function $\delta \in K(I \times R_0^+; R_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \delta(s, \rho)\, ds = 0.$$

Then if the equation

$$u^{(4)}(t) = [p(t)]_+ u(t) \ \ \text{ if } \ i = 1, \quad u^{(4)}(t) = -[p(t)]_- u(t) \ \ \text{ if } \ i = 2$$

is disconjugate, problem $(1.2), (1.3_i)$ has at least one solution.

From the last theorem and Corollary 2.2 it easily follows

Corollary 3.1. *Let there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a.e. on I inequalities (3.1) and*

$$[p(t)]_+ \le \frac{500}{(b-a)^4} \ \ \Big([p(t)]_- \le \frac{949}{(b-a)^4} \Big),$$

hold. Then problem $(1.2), (1.3_1)$ $((1.2), (1.3_2))$ has at least one solution.

The following theorems of the uniqueness of the solution for our nonlinear problem follows from Theorems 1.2, 1.3 and 2.1.

Theorem 3.2 ([3])**.** *Let there exist $p^* \in D_+(I)$ such that a. e. on I the inequality*

$$\big[f(t, x_1) - f(t, x_2)\big] \operatorname{sgn}(x_1 - x_2) < \big[p^*(t) - p(t)\big] |x_1 - x_2|$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem $(1.2), (1.3_1)$ has at most one solution.

Theorem 3.3 ([3])**.** *Let there exist $p_* \in D_-(I)$ such that a.e. on I the inequality*

$$\big[f(t, x_1) - f(t, x_2)\big] \operatorname{sgn}(x_1 - x_2) > \big[p_*(t) - p(t)\big] |x_1 - x_2|$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem $(1.2), (1.3_2)$ has at most one solution.

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ODE-Systems with Boundary Inhomogeneous Conditions Containing Higher-Order Derivatives

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The study of systems of ODE is one of the part of investigations in modern analysis and its applications. Unlike Cauchy problems, the solutions to inhomogeneous boundary-value problems for differential systems may not exist and/or may not be unique. Therefore, the question about the solvability character of such problems is fundamental for the theory of differential equations.

The topic is most fully studied for linear ODE. Thus, Kiguradze [2] investigated the solutions of first order differential systems with general inhomogeneous boundary conditions of the form

$$y'(t)+A(t)y(t)=f(t),\ \ t\in(a,b),\quad By=c. \tag{1}$$

Here, the matrix-valued function $A(\,\cdot\,)$ is Lebesgue integrable over the finite interval (a,b); the vector-valued function $f(\,\cdot\,)$ belongs to $L((a,b);\mathbb{R}^m)$; the vector c pertains to $\mathbb{R}^m$, and B is an arbitrary linear continuous operator from the Banach space $C([a,b];\mathbb{R}^m)$ to $\mathbb{R}^m$ with $m\in\mathbb{N}$.

The boundary condition in (1) covers the main types of classical boundary conditions; namely: Cauchy problems, two-point and multipoint problems, integral and mixed problems. The Fredholm property with zero index was established for problems of the form (1). Moreover, the conditions for the problems to be well posed were obtained, and the limit theorem for their solutions was proved.

These results were further developed in a series of articles by Mikhailets and his colleagues. Specifically, they allow the differential system to have an arbitrary order $r\in\mathbb{N}$ and the boundary operator B to be any linear continuous operator from the space $C^{r-1}([a,b];\mathbb{C}^m)$ to $\mathbb{C}^{rm}$. They obtained conditions for the boundary-value problems to be well posed and proved limit theorems for solutions to these problems.

We arbitrarily choose a finite interval $(a,b)\subset\mathbb{R}$ and the following parameters:

$$n\in\mathbb{N}\cup\{0\},\ \ \{m,r,l\}\subset\mathbb{N},\ \text{ and }\ 1\le p\le\infty.$$

As usual,

$$W_p^{n+r}([a,b];\mathbb{C}):=\Big\{y\in C^{n+r-1}([a,b];\mathbb{C}):\ \ y^{(n+r-1)}\in AC[a,b],\ \ y^{(n+r)}\in L_p[a,b]\Big\}$$

is a complex Sobolev space; set $W_p^0:=L_p$. This space is Banach with respect to the norm

$$\|y\|_{n+r,p}=\sum_{k=0}^{n+r}\|y^{(k)}\|_p,$$

with $\|\cdot\|_p$ standing for the norm in the Lebesgue space $L_p([a,b];\mathbb{C})$. We need the Sobolev spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \ \text{ and } \ (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They respectively consist of vector-valued functions and matrix-valued functions whose elements belong to W_p^{n+r}. The norms in these spaces are defined to be the sums of the relevant norms in W_p^{n+r} of all elements of a vector-valued or matrix-valued function.

We preserve the same notation $\|\cdot\|_{n+r,p}$ for these norms. It will be clear from the context to which space (scalar or vector-valued or matrix-valued functions) relates the designation of the norm. The same concerns all other Banach spaces used in the sequel. Certainly, the above Sobolev spaces coincide in the $m=1$ case. If $p<\infty$, they are separable and have a Schauder basis.

Consider the linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \ \ t\in(a,b), \tag{2}$$

$$By = c. \tag{3}$$

Here, the matrix-valued functions $A_{r-j}(\cdot)\in (W_p^n)^{m\times m}$, vector-valued function $f(\cdot)\in (W_p^n)^m$, vector $c\in\mathbb{C}^l$, linear continuous operator

$$B : (W_p^{n+r})^m \to \mathbb{C}^l \tag{4}$$

are arbitrarily chosen; $y(\cdot)\in (W_p^{n+r})^m$ is unknown.

The boundary condition (3) consists of l scalar condition for system of m differential equations of r-th order, we representing vectors and vector-valued functions as columns.

A solution to the boundary-value problem (2), (3) is understood as a vector-valued function $y(\cdot)\in (W_p^{n+r})^m$ that satisfies both equation (2) (everywhere if $n\geq 1$, and almost everywhere if $n=0$) on (a,b) and equality (3).

If the parameter n increases, so does the class of linear operators (4). When $n=0$, this class contains all operators that set the general boundary conditions described above. Hence, the condition (3) with operator (4) is generic condition for this equation. It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order $\geq rm$. Thus, boundary conditions can contain derivatives whose order is greater than the order of the equation. If $l<rm$, then the boundary conditions are underdetermined. If $l>rm$, then the boundary conditions are overdetermined.

In case $1\leq p<\infty$, the linear continuous operator $B:(W_p^{n+r})^m\to\mathbb{C}^l$ admits the unique analytic representation

$$By = \sum_{i=0}^{n+r-1} \alpha_i\, y^{(i)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\,\mathrm{d}t, \ \ y(\cdot)\in (W_p^{n+r})^m,$$

for certain number matrices $\alpha_s\in\mathbb{C}^{rl\times m}$ and matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{rl\times m})$; as usual, $1/p+1/p'=1$. If $p=\infty$, this formula also defines a bounded operator $B:(W_\infty^{n+r})^m\to\mathbb{C}^{rl}$. However, there exist other operators of this class generated by integrals over finitely additive measures. Hence, unlike $p<\infty$, the case of $p=\infty$ contains additional analytical difficulties.

We rewrite the inhomogeneous boundary-value problem (2), (3) in the form of a linear operator equation

$$(L,B)y = (f,c).$$

Here, (L,B) is a bounded linear operator on the pair of Banach spaces

$$(L,B):(W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l, \tag{5}$$

which follows from the definition of the Sobolev spaces involved and from the fact that W_p^n is a Banach algebra.

Let E_1 and E_2 be Banach spaces. A linear bounded operator $T: E_1 \to E_2$ is called a Fredholm one if its kernel and co-kernel are finite-dimensional. If T is a Fredholm operator, then its range $T(E_1)$ is closed in E_2, and its index is finite

$$\operatorname{ind} T := \dim\ker T - \dim \frac{E_2}{T(E_1)} \in \mathbb{Z}.$$

Theorem 1. *The bounded linear operator* (5) *is a Fredholm one with index $rm-l$.*

The proof of Theorem 1 uses the well-known theorem on the stability of the index of a linear operator with respect to compact additive perturbations.

Theorem 1 naturally raises the question of finding d-characteristics of the operator (L,B), i.e. $\dim\ker(L,B)$ and $\dim\operatorname{coker}(L,B)$. This is a quite difficult task because the Fredholm numbers may vary even with arbitrarily small one-dimensional additive perturbations.

To formulate the following result, let us introduce some notation and definitions.

For each number $i \in \{1,\dots,r\}$, we consider the family of matrix Cauchy problems:

$$Y_i^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_i^{(r-j)}(t) = O_m, \;\; t \in (a,b),$$
$$Y_i^{(j-1)}(a) = \delta_{i,j} I_m, \;\; j \in \{1,\dots,r\},$$

where $Y_i(\,\cdot\,)$ is an unknown $m \times m$ matrix-valued function.

Let $[BY_i]$ denote the number $l \times m$ matrix whose j-th column is the result of the action of B on the j-th column of the matrix-valued function Y_i.

Definition 1. A block rectangular number matrix

$$M(L,B) := \big([BY_1],\dots,[BY_r]\big) \in \mathbb{C}^{l\times rm} \tag{6}$$

is called the characteristic matrix to problem (2), (3).

Note that this matrix consists of r rectangular block columns $[BY_k] \in \mathbb{C}^{m\times l}$.

Theorem 2. *The dimensions of the kernel and co-kernel of the operator* (5) *are equal to the dimensions of the kernel and co-kernel of the characteristic matrix* (6)*, resp; i.e.,*

$$\dim\ker(L,B) = \dim\ker(M(L,B)),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}(M(L,B)).$$

Theorem 2 implies the following necessary and sufficient conditions for the invertibility of (5).

Corollary. *The operator* (5) *is invertible if and only if $l = rm$ and the square matrix $M(L,B)$ is nonsingular.*

If all the coefficients of the differential expression L are constant, then the characteristic matrix can be explicitly found. In this case, the characteristic matrix is an analytic function of a certain square number matrix and coincides hence with some polynomial of this matrix.

Example 1 (One-point problem)**.** Consider a linear one-point boundary-value problem

$$(Ly)(t) := y'(t) + Ay(t) = f(t), \ \ t \in (a,b), \tag{7}$$

$$By := \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) = c. \tag{8}$$

Here, A is a constant $(m\times m)$-matrix, $f(\,\cdot\,) \in (W_p^n)^m$, $\alpha_k \in \mathbb{C}^{l\times m}$, $c \in \mathbb{C}^l$, $y(\,\cdot\,) \in (W_p^{n+1})^m$,

$$B : (W_p^{n+1})^m \to \mathbb{C}^l, \quad (L,B) : (W_p^{n+1})^m \to (W_p^n)^m \times \mathbb{C}^l.$$

Let $Y(\,\cdot\,) = (y_{i,j})_{i,j=1}^m \in (W_p^{n+1})^{m\times m}$ be the unique solution of the linear homogeneous matrix equation with the initial Cauchy condition

$$Y'(t) + AY(t) = O_m, \ \ t \in (a,b), \quad Y(a) = I_m.$$

Hence,

$$Y(t) = \exp(-A(t-a)), \quad Y(a) = I_m;$$
$$Y^{(k)}(t) = (-A)^k \exp(-A(t-a)), \quad Y^{(k)}(a) = (-A)^k, \ \ k \in \mathbb{N}.$$

Recall that

$$M(L,B) = \left(B \begin{pmatrix} y_{1,1} \\ \vdots \\ y_{m,1} \end{pmatrix}, \dots, B \begin{pmatrix} y_{1,m} \\ \vdots \\ y_{m,m} \end{pmatrix} \right) \in \mathbb{C}^{l\times m}.$$

Substituting these values into the equality (8), we have

$$M(L,B) = \sum_{k=0}^{n-1} \alpha_k (-A)^k.$$

It follows from Theorem 1 that $\operatorname{ind}(L,B) = \operatorname{ind}(M(L,B)) = m - l$. Therefore, owing to Theorem 2, we obtain

$$\dim\ker(L,B) = \dim\ker\Big(\sum_{k=0}^{n-1} \alpha_k(-A)^k\Big) = m - \operatorname{rank}\Big(\sum_{k=0}^{n-1} \alpha_k(-A)^k\Big),$$
$$\dim\operatorname{coker}(L,B) = -m + l + \dim\ker\Big(\sum_{k=0}^{n-1} \alpha_k(-A)^k\Big) = l - \operatorname{rank}\Big(\sum_{k=0}^{n-1} \alpha_k(-A)^k\Big).$$

It follows from these formulas that d-characteristics of the problem do not depend on the length of the interval (a,b).

Example 2 (Multipoint problem)**.** Let us consider a multipoint boundary-value problem for the differential system (7) with $A(t) \equiv O_m$. The boundary conditions contain derivatives of integer and/or fractional orders (in the sense of Caputo) at certain points $t_k \in [a,b]$, $k = \{0,\dots,N\}$. These conditions become

$$Ly(t) := y'(t) = f(t), \ \ t \in (a,b),$$
$$By := \sum_{k=0}^{N} \sum_{j=0}^{s} \alpha_{k,j} ({}^{C}\mathrm{D}_{a+}^{\beta_{k,j}} y)(t_k) = c.$$

Here, all $\alpha_{k,j} \in \mathbb{C}^{l\times m}$, whereas the nonnegative numbers $\beta_{k,j}$ satisfy

$$\beta_{k,0} = 0 \ \text{ whenever } \ k \in \{1, 2, \dots, N\}.$$

Theorem 1 asserts that index of the operator (L, B) equals $m - l$. Let us find its Fredholm numbers. Since $Y(\cdot) = I_m$, we have

$$M(L,B) = \sum_{k=0}^{N}\sum_{j=0}^{s} \alpha_{k,j}(\,{}^{C}\mathrm{D}_{a+}^{\beta_{k,j}} I_m) = \sum_{k=0}^{N} \alpha_{k,0},$$

because the derivatives $(\,{}^{C}\mathrm{D}_{a+}^{\beta_{k,j}} I_m) = 0$ whenever $\beta_{k,j} > 0$. Hence, by Theorem 2, we conclude that

$$\dim\ker(L,B) = \dim\ker\Big(\sum_{k=0}^{N}\alpha_{k,0}\Big) = m - \operatorname{rank}\Big(\sum_{k=0}^{N}\alpha_{k,0}\Big),$$

$$\dim\operatorname{coker}(L,B) = -m + l + \dim\operatorname{coker}\Big(\sum_{k=0}^{N}\alpha_{k,0}\Big) = l - \operatorname{rank}\Big(\sum_{k=0}^{N}\alpha_{k,0}\Big).$$

These formulas show that d-characteristics of the problem do not depend on the length of the interval (a, b) and on the choice of the points $\{t_k\}_{k=0}^{N} \subset [a, b]$ and matrices $\alpha_{k,j}$ with $j \geq 1$.

Conclusions

We prove that the generic problem (2), (3) is a Fredholm one and find its Fredholm numbers, i.e. the dimensions of its kernel and cokernel. Along the way, we find the index of the problem. Note that, unlike the index, the Fredholm numbers are unstable with respect to one-dimensional additive perturbations with an arbitrarily small norm. To find these numbers, we introduce a rectangular number characteristic matrix $M(L, B)$ of the problem and prove that the Fredholm numbers of this matrix coincide with the Fredholm numbers of the problem. We give examples in which the characteristic matrix can be explicitly found [1, 3].

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On Admissible Perturbations of 3D Quadratic ODE System with an Infinite Number of Limit Cycles

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1 Introduction

The qualitative study of parameter depending systems of autonomous ordinary differential equations requires the study of limit sets and their bifurcations. With respect to applications, equilibria, limit cycles, homoclinic orbits and invariant manifolds play a crucial role. In case of planar autonomous systems a lot of methods for qualitative investigations has been established (see [1, 8]), nevertheless there are unsolved basic questions [9]. According to the results of H. Dulac, Yu. Ilyasenko and J. Ecale a planar polynomial autonomous system has only a finite number of limit cycles (individual finiteness) [7, 12]. The question for the maximum number of limit cycles of polynomial systems in dependence of the the degree of the polynomials and their bifurcations (Hilberts sixteenth problem) is still open. It has been proved that the cyclicity of a focus and of a period annulus (continuum of periodic orbits) of quadratic systems is three [27].

It is well known that already in the case of of quadratic polynomial three-dimensional autonomous systems new limit sets exist and new bifurcation scenarios occur [2, 10, 11, 14, 29]. The motivation for our work is to due two papers [4, 5] of V. Bulgakov devoted to the bifurcation of limit cycles in polynomial three-dimensional systems. In the first paper the focus is on Hopf bifurcation using the approach of Y. Bibikov [3] which essentially coincides with the center manifold approach [28]. In the second paper Bulgakov and Grin [5] proved that the system

$$\begin{aligned}
\dot{x} &= a_0x - a_1y + a_2xy + a_3y^2 + a_4xz + a_5yz,\\
\dot{y} &= a_1x + a_0y - a_2x^2 - a_3xy + a_4yz - a_5xz,\\
\dot{z} &= 2(a_0z + a_4z^2);\ \ (x,y,z)\in\mathbb{R}^3,
\end{aligned}\tag{1.1}$$

where $a_i\in\mathbb{R}$ $(i=\overline{0,5})$ are system parameters, has infinitely many (continuum) limit cycles, which represent intersection curves of the family of invariant surfaces $z=(x^2+y^2)/k$, $k\in\mathbb{R}\setminus\{0\}$ and the plane $z=-a_0/a_4$. But the results presented in the mentioned paper [5] are local. For system (1.1) under consideration, the non-local existence of an infinite number of limit cycles is proved in [25]. The focus was also on Hopf bifurcation of the reduced system on the invariant manifolds, but since these manifolds are no center manifolds this type of bifurcation did not explain the existence of limit cycles in the three-dimensional system. The underlying mechanism to generate a continuum of limit cycles is related to the existence of a period annulus.

In planar systems it is usual to define a limit cycles as an isolated periodic solution [1, 8]. To be able to speak about a continuum of limit cycles we have to use another definition of a limit cycle. In the monograph of C. Chicone [6] we find the following definition: A limit cycle Γ is a periodic orbit that is either the ω-limit set or the α-limit set of some point in the phase space with the periodic orbit removed. This monograph further emphasizes that the above definitions are not equivalent to each other in general, but they are equivalent in the case of real analytical systems. We will use Chicones definition in what follows.

The goal of our work is to perturb system (1.1) and find such a perturbed non-autonomous system in which the continuum of limit cycles retained its existence.

2 Preliminaries

In paper [15] V. I. Mironenko introduced the concept of a reflecting function to study the qualitative behavior of solutions of ODE systems. This function is now known as the *Mironenko reflecting function* (MRF) and has been successfully used to solve many problems in qualitative theory of ODE [16–18, 20].

ODE systems with the same MRF have the same translation operator (see [13]) on any interval $(-\beta, \beta)$, and 2ω-periodic ODE systems with the same MRF have the same mapping on the period $[-\omega, \omega]$ (Poincare mapping). Therefore, some qualitative properties (such as the existence of periodic solutions and their stability) of solutions of ODE systems that have the same MRF are common.

So it is advisable to look for perturbations that do not change the MRF (the so-called *admissible perturbations*) of known (well-studied) systems. If we manage to find admissible perturbations, then we thereby know which perturbations not change the qualitative properties of the solutions inherent in the solutions of the original unperturbed system.

For example, in papers [21–24, 26], admissible perturbations of various systems, such as the Lorenz-84 system, Langford system, generalized Langford system and Hindmarsh-Rose system, were obtained, and the qualitative properties of solutions of perturbed systems were also studied.

To search for admissible perturbations, we can use theorem from [19], which we formulate here in the form of the following lemma.

Lemma 2.1. *Let the vector functions $\Delta_i(t,x)$ $(i = \overline{1,m}$, where $m \in \mathbb{N}$ or $m = \infty)$ be solutions of the equation*

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X - \frac{\partial X}{\partial x} \Delta = 0 \tag{2.1}$$

and $\alpha_i(t)$ be any scalar continuous odd functions. Then the MRF of any perturbed system of the form

$$\dot{x} = X(t,x) + \sum_{i=1}^{m} \alpha_i(t) \Delta_i(t,x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n$$

is equal to the MRF of the system

$$\dot{x} = X(t,x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n. \tag{2.2}$$

3 Main results

For system (1.1), we look for admissible perturbations of the form $\Delta \cdot \alpha(t)$, where $\alpha(t)$ is an arbitrary continuous scalar odd function and

$$\Delta = \left(\sum_{i+j+k=0}^{l} q_{ijk}\, x^i y^j z^k, \sum_{i+j+k=0}^{l} r_{ijk}\, x^i y^j z^k, \sum_{i+j+k=0}^{l} s_{ijk}\, x^i y^j z^k \right)^{\mathrm{T}},$$

where $q_{ijk}, r_{ijk}, s_{ijk} \in \mathbb{R}$, $i, j, k, l \in \mathbb{N} \cup \{0\}$. For the polynomial Δ under consideration, relation (2.1) takes the form

$$\frac{\partial \Delta(x,y,z)}{\partial (x,y,z)} X(x,y,z) \equiv \frac{\partial X(x,y,z)}{\partial (x,y,z)} \Delta(x,y,z). \tag{3.1}$$

Substituting Δ into relation (3.1) and using the method of indefinite coefficients we obtain a system of equations for q_{ijk}, r_{ijk}, s_{ijk}. As a result, we obtained the following theorem.

Theorem 3.1. *Let $\alpha_i(t)$ $(i=\overline{1,3})$ be arbitrary scalar continuous odd functions. Then for $a_2 = a_3 = 0$, the MRF of system* (1.1) *coincides with the MRF of the system*

$$\begin{aligned}\dot{x} &= (a_0x - a_1y + a_4xz + a_5yz)(1+\alpha_1(t)) + x\alpha_2(t) + y\alpha_3(t),\\ \dot{y} &= (a_1x + a_0y + a_4yz - a_5xz)(1+\alpha_1(t)) + y\alpha_2(t) - x\alpha_3(t),\\ \dot{z} &= 2z(a_0 + a_4z)(1+\alpha_1(t)).\end{aligned} \tag{3.2}$$

Proof. For $a_2 = a_3 = 0$, the right-hand side of system (1.1) is

$$X = \Big(a_0x - a_1y + a_4xz + a_5yz, a_0y + a_1x + a_4yz - a_5xz, 2(a_0 + a_4z)z\Big)^{\mathrm{T}}$$

and its Jacobi matrix is

$$\frac{\partial X}{\partial(x,y,z)} = \begin{pmatrix} a_0 + a_4z & -a_1 + a_5z & a_4x + a_5y\\ a_1 - a_5z & a_0 + a_4z & a_4y - a_5x\\ 0 & 0 & 2(a_0 + 2a_4z)\end{pmatrix}.$$

Let us write out the vector factors for $\alpha_i(t)$ from the right-hand side of system (3.2):

$$\Delta_1 = \begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz\\ a_0y + a_1x + a_4yz - a_5xz\\ 2z(a_0 + a_4z)\end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} x\\ y\\ 0\end{pmatrix}, \quad \Delta_3 = \begin{pmatrix} y\\ -x\\ 0\end{pmatrix}.$$

By successively checking identity (3.1) for each vector-multiplier Δ_i we will make sure that it is true. Let us show this, for example, for Δ_1. The Jacobi matrix is

$$\frac{\partial \Delta_1}{\partial(x,y,z)} = \begin{pmatrix} a_0 + a_4z & -a_1 + a_5z & a_4x + a_5y\\ a_1 - a_5z & a_0 + a_4z & a_4y - a_5x\\ 0 & 0 & 2(a_0 + 2a_4z)\end{pmatrix}.$$

Hence we obtain

$$\begin{aligned}\frac{\partial \Delta_1}{\partial(x,y,z)}\,X &= \begin{pmatrix} a_0 + a_4z & -a_1 + a_5z & a_4x + a_5y\\ a_1 - a_5z & a_0 + a_4z & a_4y - a_5x\\ 0 & 0 & 2(a_0 + 2a_4z)\end{pmatrix}\begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz\\ a_0y + a_1x + a_4yz - a_5xz\\ 2(a_0 + a_4z)z\end{pmatrix}\\ &\equiv \begin{pmatrix} a_0 + a_4z & -a_1 + a_5z & a_4x + a_5y\\ a_1 - a_5z & a_0 + a_4z & a_4y - a_5x\\ 0 & 0 & 2(a_0 + 2a_4z)\end{pmatrix}\begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz\\ a_0y + a_1x + a_4yz - a_5xz\\ 2z(a_0 + a_4z)\end{pmatrix} = \frac{\partial X}{\partial(x,y,z)}\,\Delta_1.\end{aligned}$$

Then the assertion of the theorem follows from Lemma 2.1. □

If, as usual, we consider non-negative time, then the requirement that the functions $\alpha_i(t)$ be odd is not essential, since they can be continued in an odd way continuously to the negative semi-axis of time (assuming that $\alpha_i(0) = 0$).

In some cases, it is possible to find solutions of system (1.1) corresponding to limit cycles.

Lemma 3.1. *Suppose $a_2 = a_3 = 0$ and $a_4 \neq 0$. Then $\forall\, k \in \mathbb{R} \setminus \{0\}$ such that $a_0k/a_4 < 0$, system* (1.1) *has a solution*

$$x(t) = \sqrt{\frac{-a_0k}{a_4}}\,\cos\Big(\Big(\frac{a_0a_5}{a_4} + a_1\Big)t\Big),$$
$$y(t) = \sqrt{\frac{-a_0k}{a_4}}\,\sin\Big(\Big(\frac{a_0a_5}{a_4} + a_1\Big)t\Big), \tag{3.3}$$
$$z(t) = -\frac{a_0}{a_4}$$

corresponding to the cycle $x^2 + y^2 = -a_0k/a_4$, $z = -a_0/a_4$. Moreover, for $a_1 \neq -a_0a_5/a_4$ this solution is $\frac{2\pi|a_4|}{|a_0a_5+a_1a_4|}$-periodic.

The assertions of the lemma are proved by direct substitution of (3.3) into system (1.1).

The following theorem tells us about the cases when system (3.2) has infinitely many periodic solutions, and what is the character of the stability of these solutions.

Theorem 3.2. *Let $\alpha_i(t)$ $(i = \overline{1,3})$ be scalar twice continuously differentiable odd functions, $a_4 \neq 0$, $a_1 \neq -a_0a_5/a_4$ and the right-hand side of system* (3.2) *be $\frac{2\pi|a_4|}{|a_0a_5+a_1a_4|}$-periodic with respect to time t. Then $\forall\, k \in \mathbb{R}\setminus\{0\}$ such that $a_0k/a_4 < 0$, a solution of system* (3.2)*, satisfying the initial conditions*

$$x\Big(\frac{-\pi|a_4|}{|a_0a_5 + a_1a_4|}\Big) = \sqrt{\frac{-a_0k}{a_4}}\,, \quad y\Big(\frac{-\pi|a_4|}{|a_0a_5 + a_1a_4|}\Big) = 0, \quad z\Big(\frac{-\pi|a_4|}{|a_0a_5 + a_1a_4|}\Big) = -\frac{a_0}{a_4}\,, \tag{3.4}$$

is $\frac{2\pi|a_4|}{|a_0a_5+a_1a_4|}$-periodic. Moreover, the character of stability of this solution and solution (3.3) *of system* (1.1) *with the same initial conditions* (3.4) *coincides.*

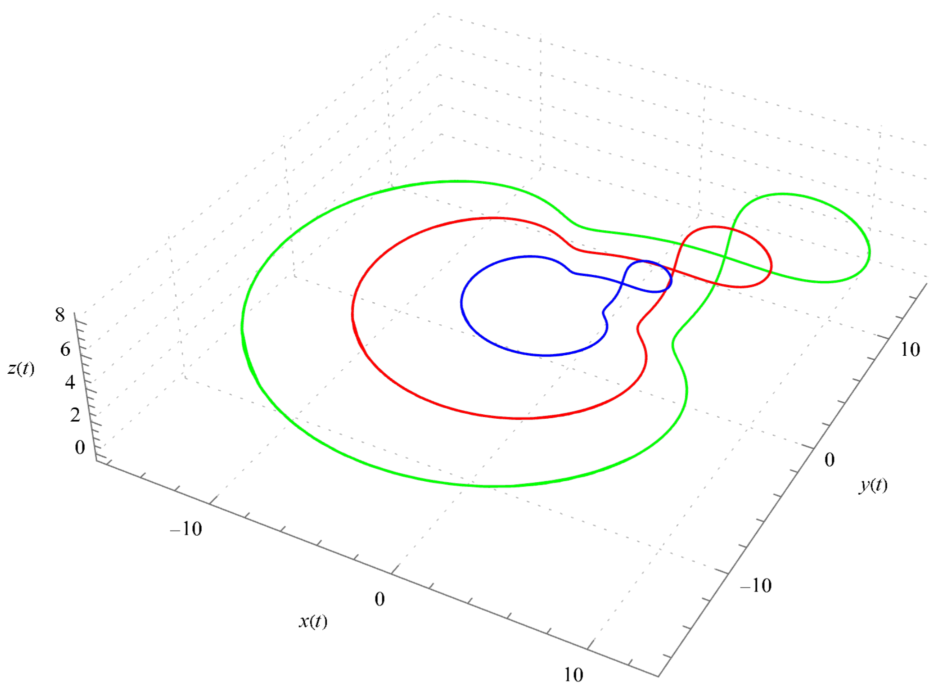

Figure 1. Phase portrait of periodic solutions of system (3.2) for $a_0 = 4$, $a_1 = 5$, $a_4 = -1$, $a_5 = 1$, $\alpha_i(t) = \sin(i \cdot t)$ $(i = \overline{1,3})$ and satisfying the initial conditions $x(-\pi) = 2\sqrt{k}$, $y(-\pi) = 0$, $z(-\pi) = 4$ (blue for $k = 1$, red for $k = 4$, and green for $k = 9$).

The proof of the theorem follows from the coincidence of the mappings over the period for systems (1.1) and (3.2).

Example. Let $a_0 = 4$, $a_1 = 5$, $a_2 = a_3 = 0$, $a_4 = -1$, $a_5 = 1$. Then, by Lemma 3.1, $\forall k \in (0, +\infty)$ system (1.1) has 2π-periodic solution (3.3). If $\alpha_i(t) = \sin(i \cdot t)$ $(i = \overline{1,3})$, then the right-hand side of system (3.2) is 2π-periodic. Therefore, by Theorem 3.2, $\forall\, k \in (0, +\infty)$ system (3.2) has 2π-periodic solution which satisfies the initial conditions $x(-\pi) = 2\sqrt{k}$, $y(-\pi) = 0$, $z(-\pi) = 4$ (see Figure 1).

4 Conclusion

Admissible perturbations were found for system (1.1) in the case when $a_2 = a_3 = 0$. The resulting perturbed non-autonomous systems have the same Mironenko reflecting function as the original unperturbed system. Solutions of different systems of ODEs with the same Mironenko reflecting function have many of the same qualitative properties. In particular, we proved that admissibly perturbed systems have infinitely many periodic solutions and that the character of their stability coincides with the character of stability of the corresponding solutions of unperturbed systems.

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Positive Solutions of Superlinear Mean Curvature Problems in Planar Domains: Topological Versus Variational Approach

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It is a rather common belief that, for analysing problems having a variational structure, variational methods are more powerful and give better results than topological ones. In this note we exhibit a class of variational problems for which the topological approach reveals instead much more effective. Namely, we look for positive regular solutions of the prescribed mean curvature problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded planar domain, the function f grows superlinearly, and λ is a real parameter.

In a recent paper Figueiredo and Rădulescu proved the existence of positive solutions for (1) assuming that f is a superlinear function having critical exponential growth at infinity with respect to the Moser–Trudinger inequality in $\mathbb{R}^2$, also providing in their article detailed history, motivations, and references concerning this topic. Precisely, in [6] they proved the following result.

Theorem 1 ([6, Theorem 1.1])**.** *Assume that*

(h_1) *Ω is a bounded domain in $\mathbb{R}^2$ having a smooth boundary $\partial\Omega$,*

(h_2) *$f : \mathbb{R} \to \mathbb{R}$ is a continuous function,*

(h_3) *there exists $\alpha_0 > 0$ such that*

$$\lim_{s\to+\infty} \frac{f(s)}{\exp(\alpha s^2)} = 0 \ \text{ for } \alpha > \alpha_0 \ \text{ and } \ \lim_{s\to+\infty} \frac{f(s)}{\exp(\alpha s^2)} = +\infty \ \text{ for } \alpha < \alpha_0,$$

(h_4) $\lim\limits_{s\to 0} \frac{f(s)}{s} = 0$,

(h_5) *the function $s \mapsto \frac{f(s)}{s}$ is increasing in $(0, +\infty)$,*

(h_6) *there exists $p > \frac{32}{7}\sqrt{2} \approx 6.465$ and $\lambda > 0$ such that, for all $s > 0$,*

$$f(s) \geq s^{p-1},$$

(h_7) *for all $s > 0$,*

$$f(s)s \geq p \int_0^s f(t)\, dt,$$

where p is the same exponent as in (h_6).

Then, there exists a constant $\lambda^* > 0$ such that problem (1) *has at least one positive weak solution $u \in C^1(\overline{\Omega})$ provided that $\lambda > \lambda^*$.*

Example. A paradigmatic model for f is provided by the function

$$f(s) = (s^+)^{p-1} \exp(\alpha_0 s^2), \tag{2}$$

where $p > 2$ and $\alpha_0 > 0$ are given exponents and $s^+ = \max\{s, 0\}$.

The proof of Theorem 1 produced in [6] strongly exploits the variational structure of problem (1) and cleverly combines a truncation argument [8], along with the use of the Nehari manifold method [4], Moser iteration techniques [10], and Stampacchia estimates [9].

The aim of this note is to show that Theorem 1 can be significantly improved, as well as extended in several directions, through a few minor modifications of a result recently established by Omari and Sovrano in [11], namely Theorem 2.2 therein. Like there, we consider here a more general problem than (1), specifically

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(x, u, \nabla u; \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where λ plays the role of a parameter and

(k_1) *Ω is a bounded domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$ of class C^2,*

(k_2) *$f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ is a continuous function.*

The following notion of solution for problem (3) is adopted in the sequel.

Definition. By a solution of (3) we mean a function $u \in W^{2,q}(\Omega)$ for all finite $q \geq 1$, which satisfies the equation almost everywhere in Ω and the boundary condition everywhere on $\partial\Omega$. A solution u is said strictly positive if $u(x) > 0$ in Ω and $\partial_\nu u(x) < 0$ on $\partial\Omega$, $\nu = \nu(x)$ being the unit outer normal to Ω at $x \in \partial\Omega$.

We also introduce the set

$$\mathscr{S} = \Big\{(u, \lambda) \in C^1(\overline{\Omega}) \times (0, +\infty) : \ \ u \text{ is a strictly positive solution of (3) for some } \lambda > 0\Big\}$$

and we endow $\mathscr{S}$ with the product topology of $C^1(\overline{\Omega}) \times \mathbb{R}$.

Since the right-hand side of the equation in (3) also depends on the gradient of the solution, the variational structure of this problem may be lost, thus ruling out the use of critical point theory in any existence proof. As a consequence, Theorem 2.2 in [11] is proven via topological methods and perturbative techniques. Specifically, assuming a structure condition on f expressed by (k_3) below, the quasilinear problem (3) is first interpreted, when λ is large, as a small perturbation of a limiting semilinear problem for which the existence of a priori bounds for the possible positive solutions is known from [7] or [1, 2, 5]. Then, the existence of a connected branch $\mathscr{C}$ of positive solutions $(u, \lambda) \in \mathscr{S}$ of (3), bifurcating from 0 as $\lambda \to +\infty$, is eventually established by relying on a fixed point index calculation inspired to [1] and by using a general Leray-Schauder continuation theorem on metric ANRs stated in [3].

Hence, the following two results can be obtained. The first one, Theorem 2 below, improves and generalises the result in [6]. Indeed, with respect to Theorem 1, Theorem 2 allows to remove, as far as problem (1) is concerned, assumptions (h_3), (h_5), (h_7), which therefore reveal to be of a merely technical nature related to the method of proof, as well as to extend the range of the admissible exponents p from the interval $(\frac{32}{7}\sqrt{2}, +\infty)$, considered in Theorem 1, to the natural one $(2, +\infty)$. Accordingly, eliminating (h_3) permits f to exhibit a totally arbitrary behaviour at infinity. In particular, when considering the model given by (2), no additional restrictions on $p \in (2, +\infty)$ are needed.

Theorem 2. *Assume* (k_1), (k_2),

(k_3) $f(x, s, \xi; \lambda) = \lambda g(x, s, \xi) + h(x, s, \xi)$, *where*

$(k_{3,1})$ $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function for which there exist a finite exponent* $p > 2$ *and a function* $w \in C^0(\overline{\Omega})$ *such that*

$$\lim_{(s,\xi)\to(0,0)} \frac{g(x, s, \xi)}{|s|^{p-2}s} = w(x) \ \text{ uniformly in } \overline{\Omega},$$

$(k_{3,2})$ $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is a continuous function such that*

$$\lim_{(s,\xi)\to(0,0)} \frac{h(x, s, \xi)}{s} = 0 \ \text{ uniformly in } \overline{\Omega},$$

(k_4) $w(x) > 0$ *for all* $x \in \overline{\Omega}$.

Then, there exist a constant $\lambda^* \geq 0$ *and a connected component* $\mathscr{C}$ *of* $\mathscr{S}$ *such that* $\operatorname{proj}_{\mathbb{R}}\mathscr{C} = (\lambda^*, +\infty)$ *and*

$$\lim_{\lambda\to+\infty} \max\left\{\|u\|_{C^1} : \ (u, \lambda) \in \mathscr{C}\right\} = 0.$$

Proof. The proof of Theorem 2 essentially exploits the same argument we developed for establishing Theorem 2.2 in [11], under the choice $\mu = 0$ in assumption (H_3) therein. While Steps 3.1, 3.3, and 3.4 in [11] remain unchanged, a few differences occur in Step 3.2 in order to get the conclusions of Lemmas 3.5 and 3.7 in [11] for the semilinear problem

$$\begin{cases} -\Delta v = \sigma v + w(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where w and p come from $(k_{3.1})$ and $\sigma \in \mathbb{R}$ is a given constant. Indeed, if (k_4) holds, the non-existence result of Lemma 3.5 in [11] can be obtained just by testing (4) against a positive principal eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, whereas the a priori estimates of Lemma 3.7 in [11] now follow directly from Theorem 1.1 in [7] and the linear elliptic regularity theory. □

The second result is a variant of Theorem 3 where the function w, considered in $(k_{3.1})$ and (k_4), is allowed to change sign, provided that its nodal domains satisfy certain conditions, introduced in [1, 5] and exploited in [11] in a context similar to the present one. Namely, we assume that

(k_5) $w \in C^2(\overline{\Omega})$,

(k_6) $\Omega^+ = \{x \in \Omega : \ w(x) > 0\} \neq \varnothing$, $\Omega^- = \{x \in \Omega : \ w(x) < 0\} \neq \varnothing$, *and* $\Omega^0 = \{x \in \Omega : \ w(x) = 0\}$ *is such that* $\partial\Omega^0 \subset \Omega$; *the boundaries* $\partial(\operatorname{int}\Omega^0)$, $\partial\Omega^+$, *and* $\partial\Omega^-$ *are of class* C^2; Ω^0 *has a finite number of connected components, that we denote by* D_i^+, D_j^-, *and* $D_k^{\pm}$.

Hence, we can represent Ω^0 in the form

$$\Omega^0 = \bigcup_i D_i^+ \cup \bigcup_j D_j^- \cup \bigcup_k D_k^{\pm},$$

where the components D_i^+, D_j^-, and $D_k^{\pm}$ are supposed to satisfy:

(k_7) *for each* i, $\partial D_i^+ \subset \overline{\Omega}^+$ *and there exist* $\gamma_{1,i} > 0$, *a neighbourhood* U_i^+ *of* ∂D_i^+, *and* $\omega_i^+ : \overline{U}_i^+ \to]0, +\infty[$ *such that*

$$w(x) = \omega_i^+(x) \operatorname{dist}(x, \partial D_i^+)^{\gamma_{1,i}} \text{ for all } x \in \Omega^+ \cap U_i^+,$$

(k_8) *for each* j, $\partial D_j^- \subset \overline{\Omega}^-$ *and there exist* $\gamma_{2,j} > 0$, *a neighbourhood* U_j^- *of* ∂D_j^-, *and* $\omega_j^- : \overline{U}_j^- \to]-\infty, 0[$ *such that*

$$w(x) = \omega_j^-(x) \operatorname{dist}(x, \partial D_j^-)^{\gamma_{2,j}} \text{ for all } x \in \Omega^- \cap U_j^-,$$

(k_9) *for each* k, *the following alternative holds*

$(k_{9,1})$ *if* $\operatorname{int}(D_k^\pm) = \varnothing$, *then*

- $\partial D_k^\pm = \Gamma_k$ *are of class* C^2,
- *there exist* $\gamma_{3,k} > 0$, *a neighbourhood* U_k^+ *of* Γ_k, *and* $\omega_k^+ : \overline{U}_k^+ \to]0, +\infty[$ *such that*
$$w(x) = \omega_k^+(x) \operatorname{dist}(x, \Gamma_k)^{\gamma_{3,k}} \text{ for all } x \in \Omega^+ \cap U_k^+, \tag{5}$$
- *there exist* $\gamma_{4,k} > 0$, *a neighbourhood* U_k^- *of* Γ_k, *and* $\alpha_k^- : \overline{U}_k^- \to]-\infty, 0[$ *such that*
$$w(x) = \omega_k^-(x) \operatorname{dist}(x, \Gamma_k)^{\gamma_{4,k}} \text{ for all } x \in \Omega^- \cap U_k^-, \tag{6}$$

$(k_{9,2})$ *if* $\operatorname{int}(D_k^\pm) \neq \varnothing$, *then*

- $\partial D_k^\pm = \Gamma_k^+ \cup \Gamma_k^-$, *with* $\Gamma_k^+ \cap \Gamma_k^- = \varnothing$, $\Gamma_k^+ \subset \overline{\Omega}^+$, $\Gamma_k^- \subset \overline{\Omega}^-$ *of class* C^2,
- *there exist* $\gamma_{3,k} > 0$, *a neighbourhood* U_k^+ *of* Γ_k^+, *and* $\omega_k^+ : \overline{U}_k^+ \to]0, +\infty[$ *satisfying condition* (5),
- *there exist* $\gamma_{4,k} > 0$, *a neighbourhood* U_k^- *of* Γ_k^-, *and* $\omega_k^- : \overline{U}_k^- \to]-\infty, 0[$ *satisfying condition* (6).

Let us define

$$D^+ = \bigcup_i D_i^+, \quad D^- = \bigcup_j D_j^-, \quad D^\pm = \bigcup_k D_k^\pm.$$

The set D^+ (respectively, D^-) is constituted by the connected components D_i^+ (respectively, D_j^-) of Ω^0, that are surrounded by regions of positivity (respectively, negativity) of w. Instead, $D^\pm$ is constituted by the connected components D_j^- of Ω^0, that are in between a region of positivity and one of negativity of w. $D^\pm$ can be either a "thin" nodal set, like when assuming condition (7) below, or a "thick" nodal set, that is, a set of positive measure. Figure 1, taken from [11], illustrates an admissible nodal configuration for the function w.

Remark. Suppose that the function $w \in C^2(\overline{\Omega})$ satisfies the following condition introduced in [2]:

$$\Omega^+ \neq \varnothing, \quad \Omega^- \neq \varnothing, \quad \Omega^0 = \overline{\Omega}^+ \cap \overline{\Omega}^- \subset \Omega, \text{ and } \nabla w(x) \neq 0 \text{ for all } x \in \Omega^0. \tag{7}$$

In this case, D^+, D^-, and $\operatorname{int}(D^\pm)$ are all empty sets and assumption $(H_{9.1})$ holds. Indeed, let Γ_k be a connected component of Ω^0. Then, condition (5) is satisfied taking $\gamma_{1,k} = 1$ and $\omega_k^+ : U_k^+ \to]0, +\infty[$ defined by

$$\omega_k^+(x) = \begin{cases} -|\nabla w(x)| & \text{if } x \in U_k^+ \setminus \Gamma_k, \\ \dfrac{w(x)}{\operatorname{dist}(x, \Gamma_k)} & \text{if } x \notin U_k^+ \setminus \Gamma_k, \end{cases}$$

where U_k^+ is a suitable tubular neighbourhood of Γ_k. Condition (6) can be verified similarly.

Theorem 3. *Assume* (k_1)–(k_3) *and* (k_5)–(k_9). *Then, the same conclusions of Theorem 2 hold.*

Proof. Theorem 3 is a direct consequence of Theorem 2.2 in [11], when the choice $\mu = 0$ in assumption (H_3) therein is made. □

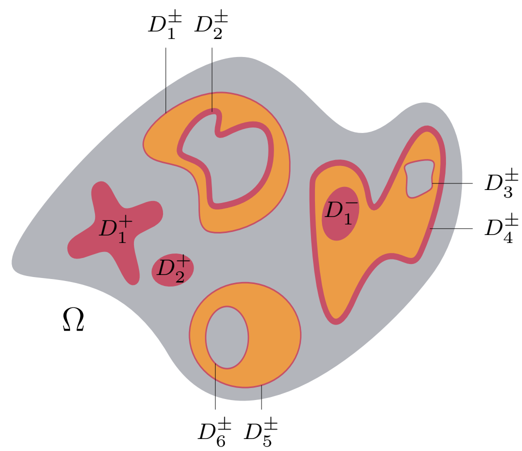

Figure 1. Example of an admissible nodal configuration for the weight function w. The sets Ω^+, Ω^0, and Ω^- are respectively the union of the grey, the red, and the yellow regions. Here, $D^+ = \bigcup_{i=1}^{2} D_i^+$, $D^\pm = \bigcup_{k=1}^{6} D_k^\pm$, and $D^- = D_1^-$.

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Initial Value Problem on an Infinite Interval for First Order Advanced Differential Equations

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In the present report, we give conditions guaranteeing, respectively, the existence and uniqueness of a solution to the Cauchy initial value problem

$$u'(t)=f(t,u(\tau(t))), \tag{1}$$

$$u(0)=c_0, \tag{2}$$

defined on the interval $\mathbb{R}_+=[0,+\infty[$.

Everywhere below it is assumed that c_0 is a positive number, $\tau:\mathbb{R}_+\to\mathbb{R}_+$ is a measurable and bounded on every finite interval contained in $\mathbb{R}_+$ function, satisfying the inequality

$$\tau(t)\geq t \ \text{ for } \ t\in\mathbb{R}_+, \tag{3}$$

while $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is a function from the Carathéodory space.

We use the following notation and definitions.

$L_{loc}(\mathbb{R}_+)$ is the space of real functions, defined on $\mathbb{R}_+$, which are Lebesgue integrable on every finite interval contained in $\mathbb{R}_+$;

$$f^*(t,y)=\max\big\{|f(t,x)|:\ |x|\leq y\big\} \ \text{ for } \ t\in\mathbb{R}_+, \ \ y>0;$$

$$f_*(t,y)=\min\big\{|f(t,x)|:\ y\leq x\leq c_0\big\} \ \text{ for } \ t\in\mathbb{R}_+, \ \ 0<y\leq c_0.$$

We say that a function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ belongs to the Carathéodory space if $f(t,\cdot\,):\mathbb{R}\to\mathbb{R}$ is continuous for almost all $t\in\mathbb{R}_+$,

$$f(\,\cdot\,,x)\in L_{loc}(\mathbb{R}_+) \ \text{ for } \ x\in\mathbb{R},$$

and

$$f^*(\,\cdot\,,y)\in L_{loc}(\mathbb{R}_+) \ \text{ for } \ y\in\mathbb{R}_+.$$

A solution to problem $(1),(2)$ is sought in the space of functions $u:\mathbb{R}_+\to\mathbb{R}$ which are absolutely continuous on every finite interval contained in $\mathbb{R}_+$.

The solution u to problem $(1),(2)$ is said to be vanishing at infinity if

$$\lim_{t\to+\infty}u(t)=0.$$

If $\tau(t)\equiv t$ and on the set $\mathbb{R}_+\times\mathbb{R}$ the inequality

$$|f(t,x)|\leq g(t)|x|+h(t)$$

is fulfilled, where $g,\, h \in L_{loc}(\mathbb{R}_+)$, then, according to the Wintner theorem (see [5]), problem (1), (2) has at least one solution in $\mathbb{R}_+$ and each maximally extended to the right solution to this problem is defined on $\mathbb{R}_+$.

In the general case, when inequality (3) holds and $\tau(t) \not\equiv t$, Wintner's condition does not guarantee the solvability of problem (1), (2).

Moreover, the following proposition is valid.

Proposition 1. *Let the function f admit the estimate*

$$f(t,x) \le -g(t)|x| - h(t) \;\; \text{for} \;\; t \in \mathbb{R}_+, \;\; x \in \mathbb{R},$$

where $g,\, h \in L_{loc}(\mathbb{R}_+)$ are nonnegative functions. If, moreover, the function τ is nondecreasing and the inequalities

$$\limsup_{t\to+\infty} \int_t^{\tau(t)} g(s)\,ds > 1, \tag{4}$$

$$\int_0^{+\infty} h(s)\,ds > c_0 \tag{5}$$

hold, then problem (1), (2) *has no solution.*

Proposition 2. *Let the function f admit the estimate*

$$f(t,x) \ge g(t)|x| \;\; \text{for} \;\; t \in \mathbb{R}_+, \;\; x \in \mathbb{R},$$

where $g \in L_{loc}(\mathbb{R}_+)$ is a nonnegative function. If, moreover, the function τ is nondecreasing and inequality (4) *holds, then problem* (1), (2) *has no solution.*

As examples, we consider the differential equations

$$u'(t) = -g(t)|u(\tau(t))| - h(t), \tag{6}$$

$$u'(t) = g(t)|u(\tau(t))| + h(t), \tag{7}$$

where $g,\, h \in L_{loc}(\mathbb{R}_+)$ are nonnegative functions.

Propositions 1 and 2 yield the following corollary.

Corollary 1. *If inequalities* (4) *and* (5) *hold* (*inequality* (4) *holds*), *then problem* (6), (2) (*problem* (7), (2)) *has no solution.*

It is easy to see that if for some $r > 0$ the function $f^*(\,\cdot\,, r)$ is integrable on $\mathbb{R}_+$ and satisfies the inequality

$$c_0 + \int_0^{+\infty} f^*(t,r)\,dt \le r,$$

then problem (1), (2) has at least one solution.

The above Propositions 1 and 2, containing the sufficient conditions for the unsolvability of problem (1), (2), concern the case, where

$$\int_0^{+\infty} f^*(t,y)\,dt = +\infty \;\; \text{for} \;\; y > 0.$$

In this case the questions on the solvability and unique solvability of the above mentioned problem still remain unstudied (see, for example, [1–4,6] and the references therein). The results we obtained fill this gap to some extent.

The following theorem is valid.

Theorem 1. *If*

$$f(t,0)=0,\quad f(t,x)\leq 0 \ \textit{for} \ t>0,\quad x>0, \tag{8}$$

then problem (1), (2) *has at least one nonnegative solution. And if along with* (8) *the condition*

$$\int\limits_{0}^{+\infty} f_*(t,y)\,dt=+\infty \ \textit{for} \ 0<y\leq c_0 \tag{9}$$

holds, then that solution is vanishing at infinity.

Sketch of the Proof of Theorem 1. Since the function τ is bounded on every finite interval, there exists a sequence of positive numbers $(a_k)_{k=1}^{+\infty}$ such that for every natural k in the interval $[0,a_k]$ the inequality

$$1+\tau(t)<a_{k+1}$$

holds.

Denote

$$\tau_k(t)=\begin{cases}\tau(t)+\frac{1}{k} & \text{for } \ 0\leq t\leq a_k,\\ a_{k+1} & \text{for } \ a_k<t\leq a_{k+1},\end{cases}$$

and for each k in the interval $[0,a_{k+1}]$ consider the Cauchy problem

$$u'(t)=f(t,u(\tau_k(t))), \tag{10}$$

$$u(a_{k+1})=x, \tag{11}$$

where $x\in\mathbb{R}_+$.

Based on condition (8), it can be proved that for every $x\in\mathbb{R}_+$ problem (10), (11) in the interval $[0,a_{k+1}]$ has a unique solution $u(\,\cdot\,;x)$ which continuously depends on the parameter x. Also,

$$u(t;0)\equiv 0,$$

and

$$u(t,x)\geq x \ \text{ for } \ 0\leq t\leq a_{k+1},\quad x>0.$$

Since

$$u(0;0)=0,\quad \lim_{x\to+\infty} u(0;x)=+\infty,$$

there exists a positive number x_k such that

$$u(0;x_k)=c_0.$$

Therefore, for every natural k problem (10), (2) has a solution u_k such that

$$0<u_k(t)\leq c_0 \ \text{ for } \ 0\leq t\leq a_{k+1},$$

$$|u_k'(t)|\leq f^*(t,c_0) \ \text{ for almost all } \ t\in(0,a_{k+1}).$$

According to these last two inequalities and the Arzelà-Ascoli lemma, the sequence $(u_k)_{k=1}^{+\infty}$ contains a subsequence $(u_{k_m})_{m=1}^{+\infty}$ which is uniformly converging on every finite interval contained in $\mathbb{R}_+$. Evidently, the function

$$u(t) = \lim_{m\to+\infty} u_{k_m}(t) \;\text{ for }\; t \in \mathbb{R}_+$$

is a nonnegagive, nonincreasing solution to problem (1), (2). In addition,

$$0 \le u(t) \le c_0 - \int\limits_0^t f_*(s,\delta)\,ds \;\text{ for }\; t \in \mathbb{R}_+,$$

where

$$\delta = \lim_{t\to+\infty} u(t).$$

From the last inequality it follows that if condition (9) is satisfied, then $\delta = 0$, i.e. the solution u is vanishing at infinity. □

Remark 1. If condition (8) holds and the function τ satisfies a more stringent condition than (3)

$$\operatorname{ess\,inf}\Big\{\tau(s) - s :\; 0 \le s \le t\Big\} > 0 \;\text{ for }\; t > 0, \tag{12}$$

then every nonnegative solution to problem (1), (2) is positive. It should be noted that condition (12) cannot be replaced by the condition

$$\operatorname{ess\,inf}\Big\{\tau(s) - s :\; t_0 \le s \le t\Big\} > 0 \;\text{ for }\; t \ge t_0,$$

no matter how small the positive number t_0 is. Indeed, if

$$0 < \lambda < 1, \;\; \alpha = (1-\lambda)^{-1}, \;\; p = \alpha\, c_0^{1-\lambda}/t_0,$$
$$\tau(t) = \begin{cases} t & \text{for } \; 0 \le t < t_0, \\ t+1 & \text{for } \; t \ge t_0, \end{cases}$$

then the function

$$u(t) = \begin{cases} c_0(1 - t/t_0)^\alpha & \text{for } \; 0 \le t < t_0, \\ 0 & \text{for } \; t \ge t_0 \end{cases}$$

is a nonnegative but not positive solution to the differential equation

$$u'(t) = -p|u(\tau(t))|^\lambda \operatorname{sgn}(u(\tau(t)))$$

under the initial condition (2).

Remark 2. According to Proposition 1, condition (8) in Theorem 1 cannot be replaced by the condition

$$f(t,x) \le 0 \;\text{ for }\; t \in \mathbb{R}_+, \;\; x \in \mathbb{R}_+,$$

i.e. the requirement

$$f(t,0) \equiv 0$$

cannot be removed from (8).

As an example, we consider the differential equation

$$u'(t)=-\sum_{i=1}^{n}p_i(t)f_i(u(\tau(t))), \tag{13}$$

where $p_i\in L_{loc}(\mathbb{R}_+)$ $(i=1,\dots,n)$, and $f_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ are continuous functions.

Theorem 1 implies the following corollary.

Corollary 2. *Let the functions p_i and f_i $(i=1,\dots,n)$ be nonnegative in $\mathbb{R}_+$, and*

$$f_i(0)=0\ \ (i=1,\dots,n). \tag{14}$$

Then problem (13), (2) *has at least one nonnegative solution. And if along with the above conditions the following conditions*

$$\int_0^{+\infty}p_m(t)\,dt=+\infty,\ \ f_m(x)>0\ \text{ for }\ x>0 \tag{15}$$

are satisfied for some $m\in\{1,\dots,n\}$, then that solution is vanishing at infinity.

So far we have been able to prove the unique solvability of problem (1), (2) only in the case where τ is a step function of the type

$$\tau(t)=t_k\ \text{ for }\ t_{k-1}<t\le t_k\ \ (k=1,2,\dots), \tag{16}$$

where $t_0=0$, and $(t_k)_{k=1}^{+\infty}$ is some increasing and unbounded sequence of positive numbers.

In particular, the following theorem is proved.

Theorem 2. *Let the function τ have the form* (16), *and let the function f be nonincreasing in the second argument and satisfy the equality*

$$f(t,0)=0\ \text{ for }\ t\in\mathbb{R}_+.$$

Then problem (1), (2) *has a unique solution, admitting the representation*

$$u(t)=c_k-\int_t^{t_k}f(s,c_k)\,ds\ \text{ for }\ t_{k-1}\le t\le t_k\ \ (k=1,2,\dots),$$

where $(c_k)_{k=1}^{+\infty}$ is a sequence of positive numbers such that

$$c_k-\int_{t_{k-1}}^{t_k}f(s,c_k)\,ds=c_{k-1}\ \ (k=1,2,\dots).$$

Corollary 3. *Let the function τ have the form* (16), *let the functions p_i $(i=1,\dots,n)$ be nonnegative, and let the functions f_i $(i=1,\dots,n)$ be nonnegative and satisfy equalities* (14). *Then problem* (13), (2) *has a unique solution, admitting the representation*

$$u(t)=c_k+\sum_{i=1}^{n}f_i(c_k)\int_t^{t_k}p_i(s)\,ds\ \text{ for }\ t_{k-1}\le t\le t_k\ \ (k=1,2,\dots),$$

where $(c_k)_{k=1}^{+\infty}$ is a sequence of positive numbers such that

$$c_k+\sum_{i=1}^{n}f_i(c_k)\int_{t_{k-1}}^{t_k}p_i(s)\,ds=c_{k-1}\ \ (k=1,2,\dots).$$

Remark 3. It is evident that if the conditions of Theorem 2 (of Corollary 3) are satisfied, then a solution to problem (1), (2) (to problem (13), (2)) is positive and vanishes at infinity if

$$\int_0^{+\infty} f(t,y)\,dt = -\infty \ \text{ for } \ y > 0$$

(if for some $m \in \{1,\dots,n\}$ conditions (15) are satisfied).

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Approximation of Stochastic Systems with Integral-Type Delay by Stochastic Systems without Delay

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Abstract

This work generalizes an approach for approximating stochastic delay systems of integral type by stochastic systems without delay. The proposed scheme is based on expanding the solution using Taylor's formula with respect to the delay parameter h, where $[-h,0]$ is the delay interval, and provides convergence results in the mean square metric.

Introduction

Stochastic delay differential equations are mathematical models of real-world processes in the natural sciences that evolve under the influence of random factors and whose future behavior depends on past states. It is well known [2] that continuous (or integrable) functions serve as initial data here, making the phase space of such equations infinite-dimensional, which significantly complicates their study. One possible approach to investigating these equations is the scheme proposed in [1], which approximates the initial problem for systems with delay by a Cauchy problem for systems of ordinary differential equations (ODEs). As the dimension of such systems increases, their solutions approach the solutions of the original initial problem for the delayed system in the uniform metric. This scheme is based on an old idea by M . M. Krasovskii, related to expanding the solution of the delayed system using Taylor's formula with respect to h, where $[-h,0]$ is the delay interval.

This work generalizes such an approach to stochastic systems.

1 Problem statement and the main result

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$, $t\geq 0$, relative to which a scalar Wiener process $W(t)$, $t\geq 0$, is adapted. Without loss of generality, and to simplify the exposition, we will assume it is one-dimensional.

Let $h > 0$ represent the delay interval, on which a continuous deterministic initial function $\phi(t)$ is defined. Denote by $C = C([-h, 0],\ \mathbb{R}^d)$ the class of continuous d-dimensional vector functions $\phi : [-h, 0] \to \mathbb{R}^d$ with the supremum norm

$$\|\phi\| = \sup_{t\in[-h,0]} |\phi(t)|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

We consider the following initial value problem for a system of stochastic functional-differential equations:

$$\begin{cases} dx(t) = f\left(t, x(t), \int\limits_{-h}^{0} x(t+\theta)\, d\theta\right) dt + \sigma\left(t, x(t), \int\limits_{-h}^{0} x(t+\theta)\, d\theta\right) dW(t), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1.1}$$

where the functions $f, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are defined, continuous in all variables, and satisfy the following conditions: there exists a constant $L > 0$ such that:

(1) Linear Growth Condition:

$$|f(t, x, y)|^2 + |\sigma(t, x, y)|^2 \le L\big(1 + |x|^2 + |y|^2\big),$$

for any $t \in [0, T]$, $x, y \in \mathbb{R}^d$.

(2) Lipschitz Condition:

$$\big|f(t, x_1, y_1) - f(t, x_2, y_2)\big|^2 + \big|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\big|^2 \le L\big(|x_1 - x_2|^2 + |y_1 - y_2|^2\big).$$

We will understand the solution to the initial value problem (1.1) in the standard sense [3, p. 61].

Definition 1.1. An $\mathcal{F}_t$-adapted stochastic process with continuous trajectories is called a strong solution to the initial value problem (1.1) on $[0, T]$ if:

1. $x(t) = \phi(t)$, $t \in [-h, 0]$;
2. $x(t) = \phi(0) + \int\limits_0^t f\left(s, x(s), \int\limits_{-h}^{0} x(s+\theta)\, d\theta\right) ds + \int\limits_0^t \sigma\left(s, x(s), \int\limits_{-h}^{0} x(s+\theta)\, d\theta\right) dW(s),$

with probability 1.

Note that equation (1.1) induces abstract mappings from the space C to $\mathbb{R}^d$ of the following form:

$$f_1(t, \phi) = f\left(t, \phi(0), \int\limits_{-h}^{0} \phi(\theta)\, d\theta\right), \quad \sigma_1(t, \phi) = \sigma\left(t, \phi(0), \int\limits_{-h}^{0} \phi(\theta)\, d\theta\right).$$

From conditions (1) and (2), we have:

$$|f_1(t, \phi)|^2 + |\sigma_1(t, \phi)|^2 \le L\big(1 + (1 + h^2)\|\phi\|^2\big),$$
$$|f_1(t, \phi) - f_1(t, \psi)|^2 + |\sigma_1(t, \phi) - \sigma_1(t, \psi)|^2 \le L\big((1 + h^2)\|\phi - \psi\|^2\big).$$

Therefore, the conditions of the existence and uniqueness theorem for a strong continuous solution of problem (1.1) on $[0,T]$ are satisfied, and $\sup\limits_{t\in[0,T]} \mathbb{E}\,|x(t)|^2 < \infty$.

Based on the system of stochastic functional differential equations (1.1), we construct a system of stochastic differential equations without delay, which we call the approximating system, as follows. Fix $m \in \mathbb{N}$ and partition the interval $[-h,0]$ with points $-\frac{hj}{m}$, $j=\overline{0,m}$, into m parts.

Define functions $z_j(t)\in\mathbb{R}^d$ on $[0,T]$ as solutions to the following Cauchy problems:

$$\begin{cases} dz_0(t) = f\Big(t, z_0(t), \dfrac{h}{m}\displaystyle\sum_{j=1}^m z_j(t)\Big)\,dt + \sigma\Big(t, z_0(t), \dfrac{h}{m}\displaystyle\sum_{j=1}^m z_j(t)\Big)\,dW(t), \\ dz_j(t) = \dfrac{m}{h}\,\big[z_{j-1}(t) - z_j(t)\big], \ \ j=\overline{1,m}, \\ z_j(0) = \phi\Big(-\dfrac{hj}{m}\Big), \ \ j=\overline{0,m}. \end{cases} \tag{1.2}$$

Definition 1.2. System (1.2) is called an approximating system for system (1.1) in the mean square sense on $[0,T]$ if

$$\sup_{t\in[0,T]} \mathbb{E}\,\Big|x\Big(t-\frac{hj}{m}\Big) - z_j(t)\Big|^2 \longrightarrow 0, \ \ m\to\infty, \ \ j=\overline{0,m}.$$

The main result of this work is the following theorem.

Theorem 1.1. *Under conditions* (1) *and* (2) *system* (1.2) *is an approximating system in the mean square sense for the initial problem* (1.1)*, uniformly over $j=\overline{0,m}$, i.e.,*

$$\sup_{j=\overline{0,m}}\ \sup_{t\in[0,T]} \mathbb{E}\,\Big|x\Big(t-\frac{hj}{m}\Big) - z_j(t)\Big|^2 \longrightarrow 0, \ \ m\to\infty.$$

2 Proof of the main result

To prove the theorem, we need a lemma about estimating the mean square modulus of continuity of the solution to problem (1.1).

Lemma 2.1 (On the Modulus of Continuity)**.** *Under conditions* (1) *and* (2)*, for the solution of the initial problem* (1.1)*, the following relation holds:*

$$\sup_{t_1\in[-h,T]} \mathbb{E} \sup_{t_2\in[t_1,t_1+l]} |x(t_2)-x(t_1)|^2 \le C\big(T, \|\phi\|, h,\big) \longrightarrow 0, \ \ l\to 0.$$

Proof. Since the solution to the initial problem (1.1) exists on $[0,T]$ and has a bounded second moment, by the linear growth condition, we have

$$|x(t)|^2 \le 3\Bigg(|\phi(0)|^2 + \bigg|\int_0^t f\Big(s, x(s), \int_{-h}^0 x(s+\theta)\,d\theta\Big)\,ds\bigg|^2 + \bigg|\int_0^t \sigma\Big(s, x(s), \int_{-h}^0 x(s+\theta)\,d\theta\Big)\,dW(s)\bigg|^2\Bigg). \tag{2.1}$$

Next, note the inequality

$$\sup_{t\in[0,T]}\ \sup_{\theta\in[-h,0]} |x(t+\theta)|^2 \le \|\phi\|^2 + \sup_{t\in[0,T]} |x(t)|^2. \tag{2.2}$$

Considering (2.2), the Cauchy–Bunyakovsky inequality and using maximal inequality for stochastic integrals, from (2.1) we get

$$\mathbb{E}\sup_{s\in[0,t]}|x(s)|^2 \le 3|\phi(0)|^2 + 3T^2L\|\phi\|^2h^2 + 3T^2L + 6TL\int_0^t \mathbb{E}\sup_{\tau\in[0,s]}|x(\tau)|^2\,d\tau + 12L\int_0^t \Big(1 + h^2\|\phi\|^2 + h^2\mathbb{E}\sup_{\tau\in[0,s]}|x(\tau)|^2\Big)\,d\tau.$$

Applying Gronwall's inequality, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|x(s)|^2 \le C_3\big(T,\|\phi\|,h\big). \tag{2.3}$$

Next, if $t_1 \ge 0$, we have

$$\mathbb{E}\sup_{t\in[t_1,t_1+l]}|x(t)-x(t_1)|^2 \le 2\Bigg(l\int_{t_1}^{t_1+l} L\Bigg(1 + \mathbb{E}\,|x(t)|^2 + \mathbb{E}\,\bigg|\int_{-h}^{0} x(t+\theta)\,d\theta\bigg|^2\Bigg)\,dt + \mathbb{E}\sup_{t\in[t_1,t_1+l]}\bigg|\int_{t_1}^{t}\sigma\Big(s, x(s), \int_{-h}^{0} x(s+\theta)\,d\theta\Big)\,dW(s)\bigg|^2\Bigg).$$

Using (2.3) and the previous inequality, and considering (2.2), we obtain

$$\mathbb{E}\sup_{t\in[t_1,t_1+l]}|x(t_2)-x(t_1)| \le C\big(T,\|\phi\|,h,l\big) \longrightarrow 0, \quad l\to 0.$$

If $t_1, t_1+l \in [-h,0]$, then, by the definition of the solution, we have

$$\mathbb{E}\sup_{t_2\in[t_1,t_1+l]}|x(t_2)-x(t_1)|^2 = \sup_{t_2\in[t_1,t_1+l]}|\phi(t_2)-\phi(t_1)|^2 \longrightarrow 0, \quad l\to 0,$$

due to the uniform continuity of the function $\phi(t)$, which completes the proof of the lemma. □

Continuation of the Proof of Theorem 1.1: Let us proceed with the proof of the main theorem. It is well known that the trajectories of the solution to (1.1) are continuous but nowhere differentiable functions, so we smooth the solution as follows. For any sufficiently small $\mu > 0$, we set

$$x_\mu(t) = \frac{1}{\mu}\int_t^{t+h} x(s)\,ds, \quad t\in[-h,T],$$

where, for $t \ge T$, we extend the process $x(s)$ by a constant random variable due to continuity. It is obvious that the process $x_\mu(t)$ has smooth trajectories with probability 1, and

$$\dot{x}_\mu(t) = \frac{1}{\mu}\big[x(t+h)-x(t)\big].$$

Using the mean value theorem, we have

$$\sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x_\mu(t)|^2 = \sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x(\theta)|^2,$$

where $\theta=\theta(\omega)$ is a random variable with $\theta\in[t,t+\mu]$. Therefore,

$$\sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x(\theta)|^2\le\sup_{t\in[-h,T]}\mathbb{E}\sup_{s\in[t,t+\mu]}|x(t)-x(s)|^2\le C\big(T,\|\phi\|,h,\mu\big)\longrightarrow 0,\ \ \mu\to 0,$$

by the lemma on the modulus of continuity. Let $y_j(t)=x(t-\frac{hj}{m})$, and introduce the differences

$$N_j(t)=\mathbb{E}\,|y_j(t)-z_j(t)|^2,\ \ j=\overline{0,m},$$

where $z_j(t)$ are solutions of system (1.2). Note that by the classical existence and uniqueness theorems for the Cauchy problem for systems of stochastic equations without delay, considering conditions (1) and (2), we obtain that system (1.2) for each natural m has a unique strong solution defined on $[0,T]$. The proof now proceeds through several steps.

Step 1. We decompose (1.2) into two systems and represent its solution as a sum:

$$z_j(t)=z_j^{(1)}(t)+z_j^{(2)}(t),$$

where $z_j^{(1)}$ is the solution of the system

$$\begin{cases}\dfrac{h}{m}\,\dot z_0^{(1)}=x(t)-z_1^{(1)}(t),\\ \dfrac{h}{m}\,\dot z_j^{(1)}=z_{j-1}^{(1)}(t)-z_j^{(1)}(t),\ \ j=\overline{1,m},\\ z_j^{(1)}(0)=x\Big(-\dfrac{hj}{m}\Big),\end{cases}$$

and $z_j^{(2)}$ is the corresponding solution of

$$\begin{cases}\dfrac{h}{m}\,\dot z_1^{(2)}=-z_1^{(2)}(t)+z_0(t)-x(t),\\ \dfrac{h}{m}\,\dot z_j^{(2)}=z_{j-1}^{(2)}(t)-z_j^{(2)}(t),\ \ j=\overline{1,m},\\ z_j^{(2)}(0)=0.\end{cases}$$

For brevity, denote the norm

$$\|\xi\|_2=\sqrt{\mathbb{E}\,\xi^2}\,.$$

Then,

$$\sup_{t\in[0,T]}\Big\|x\Big(t-\frac{hj}{m}\Big)-z_j(t)\Big\|_2\le\sup_{t\in[0,T]}\|y_j(t)-z_j^{(1)}(t)\|_2+\sup_{t\in[0,T]}\|z_j^{(2)}(t)\|_2. \tag{2.4}$$

Step 2. At this step, we estimate the first term in (2.4). We show that the following inequality holds:

$$\sup_{t\in[0,T]}\|y_j(t)-z_j^{(1)}(t)\|_2\le\alpha\Big(T,\|\phi\|,h,\frac{h}{m}\Big)\longrightarrow 0,\ \ m\to\infty. \tag{2.5}$$

Step 3. To estimate the second term in (2.4), using the method of variation of constants, we obtain the inequality

$$\mathbb{E}\,z_1^{(2)}(t)\le\sup_{t\in[0,T]}\mathbb{E}\,|z_0(t)-x(t)|^2=\mathbb{E}\,N_0(t).$$

Step 4. We estimate $z_0(t) - x(t)$. From the Lipschitz condition (2), we have

$$\mathbb{E}\, N_0(t) \le 2(T+1) \int_0^t \left[\mathbb{E}\,|x(s) - z_0(s)|^2\right] + \mathbb{E}\left|\int_{-h}^0 x(s+\theta)\, d\theta - \frac{h}{m}\sum_{j=0}^{m-1} z_j(s)\right|^2 dt.$$

However,

$$\int_{-h}^0 x(s+\theta)\, d\theta - \frac{h}{m}\sum_{j=1}^{m} z_j(s) = \sum_{j=0}^{m-1} \frac{h}{m}\, x(s+\rho_j) - \frac{h}{m}\sum_{j=1}^{m} z_j(s),$$

where $\rho_j(\omega) \in (-\frac{h}{m}(j+1), -\frac{h}{m}j)$ by the mean value theorem. Then

$$\frac{h}{m}\sum_{j=0}^{m-1}\Big(s(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big) + \frac{h}{m}\sum_{j=1}^{m}\Big(x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big),$$

so

$$\mathbb{E}\, N_0(t) \le 2(T+1) \times \int_0^t \left[\mathbb{E}\, N_0(s) + \frac{2h^2}{m^2}\left(\mathbb{E}\left(\sum_{j=1}^m \Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|\right)^2 + \mathbb{E}\left(\sum_{j=1}^m \Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|\right)^2\right)\right] ds. \tag{2.6}$$

Let us estimate the sums in inequality (2.6). For the first of them, by the lemma on the modulus of continuity, we have

$$\mathbb{E}\left(\sum_{j=1}^m \Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|\right)^2 \le m\sum_{j=1}^m \mathbb{E}\,\Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|^2 \le m^2 C\Big(T, \|\phi\|, l, \frac{h}{m}\Big). \tag{2.7}$$

For the second sum, we have the estimate

$$\mathbb{E}\left(\sum_{j=1}^m \Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|\right)^2 \le m\sum_{j=1}^m \mathbb{E}\,\Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|^2 \le m^2\alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big), \tag{2.8}$$

under estimate (2.5). Then, from (2.6)–(2.8), we get

$$\mathbb{E}\, N_0(t) \le 2(T+1)L\int_0^t \mathbb{E}\, N_0(s)\, ds + 2(T+1)T2h^2\Big(C\Big(T, \|\phi\|, l, \frac{h}{m}\Big) + \alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big)\Big).$$

From this, using Gronwall's lemma, we obtain the estimate

$$\mathbb{E}\, N_0(t) \le 2(T+1)T2h^2\Big(C\Big(T, \|\phi\|, l, \frac{h}{m}\Big) + \alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big)\Big)e^{2(T+1)LT}.$$

This last estimate proves the theorem. □

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On the Baire Classification of the Weak Oscillation Exponents of Roots and Hypermultiple Roots

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Along with estimates of the growth with an unlimited increase in time of solutions to differential equations and systems, their oscillatory properties are of great theoretical and practical interest. It is all the more surprising that their respective quantitative Lyapunov-type asymptotic characteristics were introduced quite recently.

Let us recall the definition of the *characteristic frequencies* [6,8] of a scalar function $y:\mathbb{R}_+\to\mathbb{R}$ (these are also called *Sergeev frequencies*); here and subsequently $\mathbb{R}_+$ stands for $[0,+\infty)$.

Let $\varkappa\in\{-,0,+,*\}$. For a given number $t>0$, we will denote by $\nu^\varkappa(y,t)$ depending on the value of $\varkappa$:

- the number of sign changes of y on the half-interval $[0,t)$, if $\varkappa=-$;
- the number of zeros of y on the half-interval $[0,t)$, if $\varkappa=0$;
- the number of roots (i.e. the number of zeros taking into account their multiplicity) of y on the half-interval $[0,t)$, if $\varkappa=+$;
- the number of hypermultiple roots of y on the half-interval $[0,t)$, if $\varkappa=*$. In this case, a simple zero of the function y is counted once, and a multiple one is counted infinitely many times regardless of its actual multiplicity.

Definition 1. The *upper* (*lower*) *characteristic frequency of signs, zeros, roots, and hyperroots* of a scalar function $y:\mathbb{R}_+\to\mathbb{R}$ is the quantity

$$\widehat{\nu}^\varkappa[y]=\varlimsup_{t\to+\infty}\frac{\pi}{t}\,\nu^\varkappa(y,t)\quad\Big(\check{\nu}^\varkappa[y]=\varliminf_{t\to+\infty}\frac{\pi}{t}\,\nu^\varkappa(y,t)\Big),$$

for $\varkappa=-,0,+,*$, respectively.

A generalization of the concept of characteristic frequency to the case of a vector function, i.e. for solutions of differential systems, was given in [9, 10], where the *oscillation exponents* (called there the *total* and *vector frequencies*) were introduced and their basic properties were established. Let us recall their definition.

Definition 2. The *weak upper (lower) oscillation exponent of signs, zeros, roots, or hyperroots* of a vector function $x:\mathbb{R}_+\to\mathbb{R}^n$ is the quantity

$$\widehat{\nu}^\varkappa_\circ(x)=\varlimsup_{t\to+\infty}\inf_{a\in\mathbb{R}^n_*}\frac{\pi}{t}\,\nu^\varkappa(\langle x,a\rangle,t)\quad\Big(\check{\nu}^\varkappa_\circ(x)=\varliminf_{t\to+\infty}\inf_{a\in\mathbb{R}^n_*}\frac{\pi}{t}\,\nu^\varkappa(\langle x,a\rangle,t)\Big),$$

for $\varkappa=-,0,+,*$, respectively, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the scalar product and the asterisk in the subscript denotes the removal of zero.

Note that the quantities defined above are, generally speaking, points of the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$, which we endow with the standard order and the order topology.

To describe the dependence (usually discontinuous) of various characteristics of solutions to differential equations and systems on their right-hand sides, V. M. Millionshchikov proposed using the Baire classification of discontinuous functions and obtained a number of key results in this direction (see, for example, series of papers [4, 5]).

For a given $n \in \mathbb{N}$, denote by $\widetilde{\mathscr{M}}^n$ the set of systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{1}$$

with continuous functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ (which we identify with the systems they define) and the *compact-open* topology induced by the metric

$$\rho_0(A, B) = \sup_{t\in\mathbb{R}_+} \min\bigl\{|A(t) - B(t)|, (t + 1)^{-1}\bigr\}, \quad A, B \in \widetilde{\mathscr{M}}^n.$$

The subset of $\widetilde{\mathscr{M}}^n$ consisting of systems corresponding to linear homogeneous nth-order equations

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)\dot{y} + a_n(t)y = 0, \quad t \in \mathbb{R}_+,$$

will be denoted by $\widetilde{\mathscr{E}}^n$. Further, the subspaces of $\widetilde{\mathscr{M}}^n$ and $\widetilde{\mathscr{E}}^n$ singled out by the condition of boundedness on the half-line $\mathbb{R}_+$ will be denoted by $\mathscr{M}^n$ and $\mathscr{E}^n$, respectively.

I. N. Sergeev established [7] that the least upper characteristic frequency of signs of an equation $a \in \widetilde{\mathscr{E}}^n$ given by the equality $\omega_1(a) = \inf\limits_{y\in\mathscr{S}_*(a)} \widehat{\nu}^-[y]$, $a \in \widetilde{\mathscr{E}}^n$, where $\mathscr{S}_*(A)$ is the set of non-zero solutions to the system A, belongs to the second Baire class. In [8], the characteristic frequencies are considered as functionals on the Cartesian product of the space of equations $\widetilde{\mathscr{E}}^n$ and the space $\mathbb{R}^n_*$ of initial conditions. It is shown that the upper and lower frequencies of zeros and the upper frequency of roots belong to the third Baire class, and the lower frequency of roots belongs to the second one. These results are strengthened and refined in the paper [9], where it is proved, in particular, that the upper and lower frequencies of signs belong to the second and third Baire classes, respectively.

I. N. Sergeev proved [11] that the largest lower oscillation exponent of hyperroots given by the equality $\widehat{\zeta}^*(A) = \sup\limits_{x\in\mathscr{S}_*(A)} \check{\nu}^*[x]$, $A \in \widetilde{\mathscr{M}}^n$, on each of the spaces $\mathscr{M}^n$ and $\mathscr{E}^n$ is exactly of the second Baire class.

The question on the Baire classification of other oscillation characteristics of solutions to linear differential equations and systems has up till now remained open. The present paper contains some results in this direction. To state these we need some more notation.

Let $\mathscr{M}$ and $\mathscr{N}$ be classes of subsets of a metric space M. We say [11, [3, pp. 266–267] that a function $f : M \to \overline{\mathbb{R}}$ belongs to the class $(\mathscr{M},^*)$ (or $(^*, \mathscr{N})$), if for every $r \in \mathbb{R}$, the inclusion $f^{-1}((r, +\infty]) \in \mathscr{M}$ holds (respectively, $f^{-1}([r, +\infty]) \in \mathscr{N}$). Recall that F_σ denotes the class of all countable unions of closed sets, and $F_{\sigma\delta}$ denotes the class of countable intersections of sets from the class F_σ. Additionally, we denote by $X_A(\,\cdot\,, \cdot\,)$ the Cauchy operator of the system A.

Theorem 1. *For any $n \geq 2$, the following statements hold:*

(1) *the functional $\widetilde{\mathscr{M}}^n \times \mathbb{R}^n_* \to \overline{\mathbb{R}}$ defined by $(A, \xi) \mapsto \check{\nu}^*(X_A(\,\cdot\,, 0)\xi)$ is of the class $(F_\sigma,^*)$ and, in particular, belongs to the second Baire class;*

(2) *the functional $\widetilde{\mathscr{M}}^n \times \mathbb{R}^n_* \to \overline{\mathbb{R}}$ defined by $(A, \xi) \mapsto \widehat{\nu}^*(X_A(\,\cdot\,, 0)\xi)$ is of the class $(^*, F_{\sigma\delta})$ and, in particular, belongs to the third Baire class.*

Corollary 1. *For any system* $A \in \widetilde{\mathscr{M}}^n$*, its spectra of the weak oscillation exponents of hyperroots, i.e. the sets* $\{\check{\nu}^*_\circ(X_A(\,\cdot\,,0)\xi) :\ \xi \in \mathbb{R}^n_*\}$ *and* $\{\widehat{\nu}^*_\circ(X_A(\,\cdot\,,0)\xi) : \xi \in \mathbb{R}^n_*\}$ *are Suslin [3, p. 204] subsets of the extended real line* $\overline{\mathbb{R}}$*.*

When studying the oscillation exponents of roots, it is natural to require that solutions be infinitely differentiable and to endow the space of systems with the corresponding topology. Let us denote by $\mathscr{C}^\infty\widetilde{\mathscr{M}}^n$ the set of systems (1) with infinitely differentiable coefficients. We equip the set $\mathscr{C}^\infty\widetilde{\mathscr{M}}^n$ with the metric

$$\rho_\infty(A,B) = \sum_{k=0}^{\infty} 2^{-k} \sup_{t\in\mathbb{R}_+} \min\big\{|A^{(k)}(t) - B^{(k)}(t)|, (t+1)^{-1}\big\},\ \ A,B \in \mathscr{C}^\infty\widetilde{\mathscr{M}}^n,$$

inducing the C^∞*-compact-open* topology.

Theorem 2. *For any* $n \geq 2$*, the following statements hold:*

(1) *the functional* $\mathscr{C}^\infty\widetilde{\mathscr{M}}^n \times \mathbb{R}^n_* \to \overline{\mathbb{R}}$ *defined by* $(A,\xi) \mapsto \check{\nu}^+_\circ(X_A(\,\cdot\,,0)\xi)$ *is of the class* $(F_\sigma,^*)$ *and, in particular, belongs to the second Baire class;*

(2) *the functional* $\mathscr{C}^\infty\widetilde{\mathscr{M}}^n \times \mathbb{R}^n_* \to \overline{\mathbb{R}}$ *defined by* $(A,\xi) \mapsto \widehat{\nu}^+_\circ(X_A(\,\cdot\,,0)\xi)$ *is of the class* $(^*,F_{\sigma\delta})$ *and, in particular, belongs to the third Baire class;*

(3) *the functional* $\mathscr{C}^\infty\widetilde{\mathscr{M}}^n \to \overline{\mathbb{R}}$ *defined by* $A \mapsto \sup_{x\in\mathscr{S}_*(A)} \check{\nu}^+_\circ(x)$ *is of the class* $(F_\sigma,^*)$ *and, in particular, belongs to the second Baire class.*

Corollary 2. *For any system* $A \in \mathscr{C}^\infty\widetilde{\mathscr{M}}^n$*, its spectra of the weak oscillation exponents of roots, i.e. the sets* $\{\check{\nu}^+_\circ(X_A(\,\cdot\,,0)\xi) :\ \xi \in \mathbb{R}^n_*\}$ *and* $\{\widehat{\nu}^+_\circ(X_A(\,\cdot\,,0)\xi) :\ \xi \in \mathbb{R}^n_*\}$ *are Suslin subsets of the extended real line* $\overline{\mathbb{R}}$*.*

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Limiting Behavior of Invariant Measures of Stochastic Functional-Differential Neutral Equations in Hilbert Space

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We consider the following stochastic functional-differential neutral equation on Hilbert space with delay parameter $h \in (0,1]$:

$$d(u(t) + g(u(t-h), u(t))) = (f(u(t-h), u(t)) + Au(t))\,dt + \sigma(u(t-h), u(t))\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t) = \phi(t), \quad t \in [-h, 0]. \tag{0.2}$$

Here A is an inifinitesimal generator of a strong continuous semigroup $\{S(t), t \geq 0\}$ of bounded linear operators in real separable Hilbert space H. The noise $W(t)$ is a Q-Wiener process on separable Hilbert space K. For any $h \in (0,1)$ denote $C_h := C([-h,0], H)$, a space of continuous H-valued functions with a norm.

$$\|\phi\|_{C_h} := \sup_{t \in [-h,0]} \|\phi(t)\|_H.$$

Below we denote $\|\cdot\|_H$ as $\|\cdot\|$. The functions f and g map $H \times H$ into H and $\sigma : H \times H \to L_2^0$, where $L_2^0 = L(Q^{1/2}K, H)$ is the space of Hilbert–Schmidt operators from $Q^{1/2}K$ to a H. Finally, $\phi : [-h,0] \times \Omega \to H$ is the initial condition on probability space $(\Omega, \mathcal{F}, P)$.

We consider the limiting behavior of invariant measures of equation (0.1), (0.2) when delay parameter h converges to zero.

1 Preliminaries

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $\{F_t; t \geq 0\}$ generated by the Q-Wiener process W on (Ω, F, P) with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete orthonormal system e_k in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \dots$, and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion $W(t)=\sum_{k=1}^{\infty}\lambda_k\beta_k(t)e_k$, where $\beta_k(t)$ are real valued Brownian motions mutually independent on (Ω,F,P).

Let $U_0=Q^{\frac{1}{2}}(U)$ and $L_0^2=L_2(U_0,H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi,\Psi)L_0^2=\operatorname{tr}[\Phi Q\Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

Lemma 1.1 (Stochastic Gronwall Lemma [5, 10])**.** *Let Z, H be nonnegative stochastic processes adapted to filtration, M be a continuous local martingale. Then:*

1. *If $M(0)=0$ and there exist $K,C\geq 0$ such that*

$$Z(t)\leq K\int_0^t\sup_{u\in[0,s]}(Z(u))\,ds+M(t)+C,$$

then for all $0<\alpha<1$ there exist $C_1,C_2>0$ such that

$$\mathbf{E}\Big(\sup_{t\in[0,T]}(Z(t))^\alpha\Big)\leq C^\alpha C_1e^{C_2KT}.$$

2. *If $M(0)=0$, $H(0)=0$ and there exists $K\geq 0$ such that*

$$Z(t)\leq K\int_0^t\sup_{u\in[0,s]}(Z(u))ds+M(t)+H(t),$$

then for all $0<\alpha<1$ and $\beta>\frac{1+\alpha}{1-\alpha}$ there exist $C_3,C_4>0$ such that

$$\mathbf{E}\Big(\sup_{t\in[0,T]}(Z(t))^\alpha\Big)\leq C_3e^{C_4KT}\Big(\mathbf{E}\Big(\sup_{t\in[0,T]}H(t)\Big)^\beta\Big)^{\alpha/\beta}.$$

3. *If $H(t)$ is non-negative, then for all $0<\alpha<1$ there exists $C_\alpha\geq 0$ such that*

$$\mathbf{E}\Big(\sup_{t\in[0,T]}(Z(t))^\alpha\Big)\leq (C_\alpha+1)e^{\alpha KT}\Big(\mathbf{E}\Big(\sup_{t\in[0,T]}H(t)\Big)^\alpha\Big).$$

Definition 1.1 (Mild solution)**.** A continuous $\mathcal{F}_t$ adapted stochastic process $u:[-h,T]\times\Omega\to H$ is a mild solution for (0.1), (0.2) for $t\in[0,T]$ if it satisfies the integral equation

$$u(t)=S(t)(\phi(0)+g(\phi(-h),\phi(0)))-g(u(t-h),u(t))-\int_0^t AS(t-s)g(u(s-h),u(s))\,ds$$
$$+\int_0^t S(t-s)f(u(s-h),u(s))\,ds+\int_0^t S(t-s)\sigma(u(s-h),u(h))\,dW(s),$$

and $u(t)=\phi(t)$ a.s. for $t\in[-h,0]$.

A non-delay equation will look as follows

$$d\big(u(t)+g(u(t),u(t))\big)=\big(f(u(t),u(t))+Au(t)\big)\,dt+\sigma(u(t),u(t))\,dW(t),\quad t\geq 0, \tag{1.1}$$
$$u(0)=\phi(0). \tag{1.2}$$

For starting function ϕ and delay parameter $h \in (0,1)$ we denote mild solution of equation (0.1), (0.2) as $u^h(t,\phi)$. For starting point v we denote mild solution of equation (1.1), (1.2) as $u^0(t,v)$.

Let (Z,d) be a polish space with metric d. Suppose for every $\rho \in (0,1]$ and $\phi \in C([-\rho,0],Z)$, $\{X^\rho(t,0,\phi), t \geq 0\}$ is a stochastic process in the state space $C([-\rho,0],Z)$ with initial value ϕ at initial time 0. Similarly, assume for every $x \in Z$, $\{X^0(t,0,x), t \geq 0\}$ is a stochastic process in the state space Z with initial value x at initial time 0. We also assume that the probability transition operators of X^0 are Feller.

$UC_b(Z)$ is the Banach space of all bounded uniformly continuous functions defined on Z with uniform norm.

Given $\rho \in (0,1]$, define an operator $T_\rho : C([-\rho,0],Z) \to Z$ by $T_\rho\phi = \phi(0)$, and $\mathcal{T}_\rho : C([-1,0],Z) \to C([-\rho,0],Z)$ by $\mathcal{T}_\rho\phi(s) = \phi(s)$.

Condition (C1). For every compact set $K \subset C([-1,0],Z)$, $t \geq 0$, and $\eta > 0$,

$$\lim_{\rho\to 0} \sup_{\phi\in\mathcal{T}_\rho K} P\Big(d\big(X^\rho(t,0,\phi)(0), X^0(t,0,T_\rho\phi)\big) \geq \eta\Big) = 0.$$

Theorem 1.1 ([4]). *Assume* (C1) *holds true and* $\rho_n \in (0,1]$. *Let* μ^{ρ_n} *be an invariant measure of* X^{ρ_n} *in* $C([-\rho_n,0],Z)$ *for all* $n \in \mathbb{N}$. *Suppose* $\{\mu^{\rho_n}\}_{n=1}^\infty$ *is tight in a sense that for every* $\varepsilon > 0$ *there exists compact set* $K_1 \subset C([-1,0],Z)$ *such that*

$$\mu^{\rho_n}(\mathcal{T}_{\rho_n}K_1) > 1 - \varepsilon, \tag{1.3}$$

for all $n \in \mathbb{N}$. *Then we have:*

1. *The sequence* $\{\mu^{\rho_n} \circ T_{\rho_n}^{-1}\}_{n=1}^\infty$ *is tight;*
2. *If* $\rho_n \to 0$ *and* μ *is a probability measure in* Z *such that* $\mu^{\rho_n} \circ T_{\rho_n}^{-1} \to \mu$ *weakly, then* μ *must be an invariant measure of* X^0.

Proof.
1. Given $\varepsilon > 0$, let $K_1 \subset C([-1,0],Z)$ be the compact set satisfying (1.3). Denote by $K_0 = \{\phi(0) :\ \phi \in K_1\}$. Then K_0 is a compact subset of Z and for all $n \in \mathbb{N}$,

$$\mu^{\rho_n} \circ T_{\rho_n}^{-1}(K_0) \geq \mu^{\rho_n}(\mathcal{T}_{\rho_n}K_1) > 1 - \varepsilon, \tag{1.4}$$

which shows that $\{\mu^{\rho_n} \circ T_{\rho_n}^{-1}\}$ is tight.

2. We need to prove that for all $\psi \in UC_b(Z)$ and $t > 0$,

$$\int_Z \mathbf{E}\psi(X^0(t,0,x))\mu\,(dx) = \int_Z \psi(x)\mu\,(dx). \tag{1.5}$$

One can notice that

$$\begin{aligned}\int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}\,(dx) &= \int_{C([-\rho_n,0],Z)} \psi(T_{\rho_n}\xi)\mu^{\rho_n}\,d\xi \\ &= \int_{C([-\rho_n,0],Z)} \psi\big(T_{\rho_n}X^{\rho_n}(t,0,\xi)\big)\mu^{\rho_n}\,d\xi = \int_{C([-\rho_n,0],Z)} \psi\big(X^{\rho_n}(t,0,\xi)(0)\big)\mu^{\rho_n}\,d\xi,\end{aligned}$$

which with (1.4) yields that

$$\begin{aligned}
&\left| \int_Z \mathbf{E}\psi(X^0(t,0,x))\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) - \int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) \right| \\
&\qquad \le \int_{C([-\rho_n,0],Z)} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad \le \int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad\qquad + \int_{C([-\rho_n,0],Z)\setminus\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad \le \int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) + 2\varepsilon \sup_{x\in Z}|\psi(x)|. \qquad (1.6)
\end{aligned}$$

Since $\psi \in UC_b(Z)$ and for $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\psi(u) - \psi(v)| < \varepsilon,$$

if $d(u,v) < \eta$. Then we get

$$\begin{aligned}
&\int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&= \int_{\mathcal{T}_{\rho_n}K_1} \left(\int_{\{d(\psi(X^0(t,0,T_{\rho_n}\xi)),\psi(X^{\rho_n}(t,0,\xi)(0)))\ge\eta\}} \big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n} P(d\omega) \right)(d\xi) \\
&+ \int_{\mathcal{T}_{\rho_n}K_1} \left(\int_{\{d(\psi(X^0(t,0,T_{\rho_n}\xi)),\psi(X^{\rho_n}(t,0,\xi)(0)))<\eta\}} \big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n} P(d\omega) \right)(d\xi) \\
&\qquad \le 2\sup_{x\in Z}|\psi(x)| \cdot \sup_{\xi\in\mathcal{T}_{\rho_n}K_1} P\Big(\Big\{ d\Big(\psi(X^0(t,0,T_{\rho_n}\xi)), \psi(X^{\rho_n}(t,0,\xi)(0))\Big) \ge \eta \Big\}\Big) + \varepsilon. \qquad (1.7)
\end{aligned}$$

Then, from (C1) and (1.6), (1.7) we can deduce that

$$\left| \int_Z \mathbf{E}\psi(X^0(t,0,x))\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) - \int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) \right| \le \varepsilon + 2\varepsilon \sup_{x\in Z}|\psi(x)|,$$

and since $\varepsilon > 0$ is arbitrary and $\mu^{\rho_n} \circ T_{\rho_n}^{-1} \to \mu$ weakly, we get that μ is an invariant measure for X^0 by (1.5). □

2 Conditions on functions

Condition (H1). If $\sigma(-A)$ is the spectrum of $(-A)$, we have

$$\operatorname{Re}\sigma(-A) > \delta > 0,$$

and A^{-1} is compact in H.

It follows from [6] that for $0 \le \alpha \le 1$ one can define fractional power $(-A)^\alpha$, which is closed linear operator with domain $D(-A)^\alpha$. We denote H_α to be a Banach space $D(-A)^\alpha$ with the norm

$$\|u\|_\alpha := \|(-A)^\alpha u\|,$$

which is equivalent to the graph norm of $(-A)^\alpha$. This way $H_0 = H$. It follows from [3, Section 1.4] that if A^{-1} is compact, then $S(t)$ is compact for $t > 0$. Next, it follows from [6, Theorem 3.2, p. 48] that under assumption (H1) semigroup $S(t)$ is continuous with respect to uniform operator topology for $t > 0$. Thus, using [6, Theorem 3.3, p. 48], we may conclude that the operator A has a compact resolvent. Consequently, from [3, Theorem 1.4.8], we have the following result.

Proposition 2.1. *Under condition* (H1) *the embedding $H_\alpha \subset H_\beta$ is compact if $0 \le \beta < \alpha \le 1$.*

Proposition 2.2 ([3, Theorem 1.4.3])**.** *Under condition* (H1)*, for every $\alpha \ge 0$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \le C_\alpha t^{-\alpha} e^{-\delta t},$$

for $t > 0$. In particular,

$$\|S(t)\| \le C_0 e^{-\delta t},$$

for $t > 0$.

Proposition 2.3 ([1])**.** *Let $p > 2$, $T > 0$ and let Φ be an L_2^0 valued, predictable process such that*

$$\mathbf{E}\int_0^T \|\Phi(t)\|_{L_2^0}^p \, dt < \infty.$$

Then there is a constant $M_T > 0$ such that

$$\mathbf{E}\sup_{t\in[0,T]} \bigg\| \int_0^t S(t-s)\Phi(s)\, dW(s) \bigg\| \le M_T \mathbf{E}\int_0^T \|\Phi(s)\|_{L_2^0}^p \, ds.$$

Condition (H2). The mappings $f : H \times H \to H$ and $\sigma : H \times H \to L_2^0$ are continuous and satisfy:

1. There exist a positive constant $K > 0$ such that

   $$\|f(u,v)\| + \|\sigma(u,v)\|_{L_2^0} \le K\big(1 + \|u\| + \|v\|\big)$$

   for all $u, v \in H$.

2. There exist a positive constant $L > 0$ such that

   $$\|f(u,v) - f(u_1,v_1)\|^2 + \|\sigma(u,v) - \sigma(u_1,v_1)\|_{L_2^0}^2 \le L\big(1 + \|u-u_1\|^2 + \|v-v_1\|^2\big)$$

   for all $u, v, u_1, v_1 \in H$.

Condition (H3). There exist positive constants $\alpha \in (0,1)$ and $M_g \in (0,1)$ such that for all $u, v, u_1, v_1 \in H$ the function $g : H \times H \to H_\alpha$ satisfies

$$\|g(u,v) - g(u_1,v_1)\|_{H_\alpha}^2 \le M_g\big(\|u-u_1\|^2 + \|v-v_1\|^2\big).$$

Condition (H4). The initial condition $\phi : [-h,0] \times \Omega \to H$ is an $\mathcal{F}_0$-measurable random variable, independent of W, which has continuous trajectories.

Remark. It is easy to see from [9], that under conditions above equation (0.1), (0.2) have unique mild solution, and this solution have an invariant measure.

3 Main results

Lemma 3.1. *Suppose that* (H1)–(H4) *hold. Then for every compact set K in C_h, $t > 0$ and $\eta > 0$ the following holds*

$$\lim_{h\to 0} \sup_{\xi \in \mathcal{T}_h K} P\Big(\|u^h(t,\xi) - u^0(t, T_h\xi)\| \geq \eta\Big) = 0.$$

Sketch of the proof.

Step 1. Rewrite solutions $u^h(t,\xi)$ and $u^0(t, T_h\xi)$ using Definition 1 as follows

$$\begin{aligned} u^h(t,\xi) &= S(t)\big(\xi(0) - g(\xi(-h), \xi(0))\big) \\ &\quad + g\big(u^h(t-h,\xi), u^h(t,\xi)\big) - \int_0^t AS(t-s) g\big(u^h(s-h,\xi), u^h(s,\xi)\big)\, ds \\ &\quad + \int_0^t S(t-s) f\big(u^h(s-h,\xi), u^h(s,\xi)\big)\, ds + \int_0^t S(t-s)\sigma\big(u^h(s-h,\xi), u^h(s,\xi)\big)\, dW(s), \end{aligned}$$

and

$$\begin{aligned} u^0(t, T_h\xi) &= S(t)(T_h\xi - g(T_h\xi, T_h\xi)) \\ &\quad + g\big(u^0(t, T_h\xi), u^0(t, T_h\xi)\big) - \int_0^t AS(t-s) g\big(u^0(s, T_h\xi), u^0(s, T_h\xi)\big)\, ds \\ &\quad + \int_0^t S(t-s) f\big(u^0(s, T_h\xi), u^0(s, T_h\xi)\big)\, ds + \int_0^t S(t-s)\sigma\big(u^0(s, T_h\xi), u^0(s, T_h\xi)\big)\, dW(s). \end{aligned}$$

Step 2. Estimate $\mathbf{E}\|u^h(t,\xi) - u^0(t, T_h\xi)\|^2$ from conditions (H1)–(H4) and Propositions 2.1–2.3 and using Lemma 1.1 (Stochastic Gronwall Lemma).

Step 3. Proposition of the lemma is a direct consequence of Chebyshev inequality. □

Given $h \in [0,1]$, let $p^h(r, \xi; t, \cdot\,)$ be the transition probability function of $u^h(t,\xi)$ with $0 \leq r \leq t$ and $\xi \in C_h$. Denote by $\mathcal{M}^h$ –collection of all limit points of probability measure

$$\frac{1}{n}\int_0^n p^h(0, 0; t, \cdot\,)\, dt.$$

Then we have the following result.

Theorem 3.1. *Suppose that* (H1)–(H4) *hold. Then:*

1. *The union* $\bigcup_{h\in[0,1]} \mathcal{M}^h$ *is tight;*

2. *If $h_n \to 0$ and $\mu^{h_n} \in \mathcal{M}^{h_n}$, then there exist a subsequence $h_{k(n)}$ and an invariant measure $\mu^0 \in \mathcal{M}^0$ such that $\mu^{h_{k(n)}} \circ T^{-1}_{h_{k(n)}} \to \mu^0$ weakly.*

Proof.

1. Direct consequence of [9].

2. By first item we know that $\{\mu^{h_n}\}$ is tight and hence by Theorem 1.1 and Lemma 3.1 we infer that sequence $\{\mu^{h_n} \circ T_{h_n}^{-1}\}_{n=1}^{\infty}$ is also tight. Consequently, there exists a subsequence h_{n_k} and a probability measure μ^* such that $\mu^{h_{n_k}} \circ T_{h_{n_k}}^{-1} \to \mu^*$ weakly. By Theorem 1.1 and Lemma 3.1 we find that μ^* is invariant and $\mu^* \in \mathcal{M}^0$. □

As an immediate corollary of Theorem 3.1, we have the following result.

Theorem 3.2. *Suppose that* (H1)–(H4) *hold and* $h_n \to 0$. *Then, if* μ^{h_n} *and* μ^0 *are the unique invariant measures of equations* (0.1), (0.2) *and* (1.1), (1.2) *correspondingly, then* $\mu^{h_n} \to \mu^0$ *weakly.*

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On Investigation of Non-Linear Differential Systems with Mixed Boundary Conditions

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We study a non-linear differential system with mixed boundary conditions on a compact interval $[a,b]$

$$\begin{aligned} x'(t) &= f_1(t,x(t),y(t)), \quad t\in[a,b],\\ y'(t) &= f_2(t,x(t),y(t)), \quad t\in[a,b], \end{aligned} \tag{1}$$

$$x(a)=x(b), \tag{2}$$

$$\phi(y)=d, \tag{3}$$

where $x:[a,b]\to\mathbb{R}^p$, $y:[a,b]\to\mathbb{R}^q$, $d\in\mathbb{R}^q$. It is supposed that f_1, f_2 are continuous as functions $f_1:[a,b]\times U\times V\to\mathbb{R}^p$, $f_2:[a,b]\times U\times V\to\mathbb{R}^q$, where bounded sets $U\subset\mathbb{R}^p$, $V\subset\mathbb{R}^q$ are specified later (see (4)). We also assume the continuity of $\phi:V\to\mathbb{R}^q$. Continuously differentiable solutions of problem (1)–(3) are considered. For problem (1)–(3) we will use an approach similar to that of [2,3].

For vectors $x=\operatorname{col}(x_1,\dots,x_n)\in\mathbb{R}^n$ the notation $|x|=\operatorname{col}(|x_1|,\dots,|x_n|)$ is used and the inequalities between vectors are understood componentwise; the operations max and min for vectors are understood similarly. I denotes the identity matrix. For a non-negative vector ϱ, we define the componentwise ϱ-neighbourhood of a point z by putting

$$\mathscr{O}_\varrho(z)=\big\{\xi\in\mathbb{R}^n: \ |\xi-z|\le\varrho\big\}.$$

The ϱ-neighbourhood of a set $\Omega\subset\mathbb{R}^n$ is then defined as $\mathscr{O}_\varrho(\Omega)=\bigcup\limits_{z\in\Omega}\mathscr{O}_\varrho(z)$. The particular sets Ω and values of ϱ used in the assumptions are specified below in (4), (5).

We will use a reduction of the given problem to a family of simpler auxiliary boundary value problems [2]. Let us fix certain compact convex sets $\mathscr{V}_a \subset \mathbb{R}^q$, $\mathscr{V}_b \subset \mathbb{R}^q$ and $\mathscr{U} \subset \mathbb{R}^p$, take some positive vectors $\varrho_U \in \mathbb{R}^p$, $\varrho_V \in \mathbb{R}^q$ and put

$$U = \mathscr{O}_{\varrho_U}(\mathscr{U}), \quad V = \mathscr{O}_{\varrho_V}(\mathscr{C}(\mathscr{V}_a, \mathscr{V}_b)), \tag{4}$$

where

$$\mathscr{C}(X,Y) = \big\{(1-\theta)x + \theta y : \ x \in X, \ y \in Y, \ 0 \le \theta \le 1\big\}.$$

It is convenient to choose the sets $\mathscr{U}$, $\mathscr{V}_a$, $\mathscr{V}_b$ as some parallelepipeds. We will consider solutions (x,y) of problem (1)–(3) with values $x(a) = x(b) \in \mathscr{U}$, $y(a) \in \mathscr{V}_a$, $y(b) \in \mathscr{V}_b$ and range in $U \times V$.

Introduce the notation

$$\delta_{U\times V}(f_k) = \frac{1}{2}\Big(\max_{[a,b]\times U\times V} f_k - \min_{[a,b]\times U\times V} f_k\Big), \ \ k = 1,2,$$

and assume that the positive vectors ϱ_U, ϱ_V can be chosen so that

$$\varrho_U \ge \frac{b-a}{2}\,\delta_{U\times V}(f_1), \quad \varrho_V \ge \frac{b-a}{2}\,\delta_{U\times V}(f_2). \tag{5}$$

Let f_1, f_2 satisfy the Lipchitz condition on U, V:

$$\big|f_k(t,x,y) - f_k(t,\widetilde{x},\widetilde{y})\big| \le K_{k1}|x-\widetilde{x}| + K_{k2}|y-\widetilde{y}|, \ \ k = 1,2, \tag{6}$$

for $t \in [a,b]$, $\{x,\widetilde{x}\} \subset U$, $\{y,\widetilde{y}\} \subset V$, where K_{11}, K_{12}, K_{21}, K_{22} are positive matrices of dimensions $p\times p$, $p\times q$, $q\times p$, $q\times q$. We assume that the maximal in modulus eigenvalue of the matrix $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ is small enough:

$$r(Q) < 1, \tag{7}$$

where $Q = \frac{3}{10}\,(b-a)K$.

We introduce the vectors of parameters $z \in \mathbb{R}^p$, $\gamma \in \mathbb{R}^q$, $\lambda \in \mathbb{R}^q$ by formally putting

$$z = x(a) = x(b), \quad \gamma = y(a), \quad \lambda = y(b)$$

and, instead of problem (1)–(3), consider the following two auxiliary boundary value problems with periodic and two-point linear separated conditions at a and b:

$$x'(t) = f_1(t,x,y), \ \ t \in [a,b], \tag{8}$$

$$x(a) = z, \ \ x(b) = z \tag{9}$$

and

$$y'(t) = f_2(t,x,y), \ \ t \in [a,b], \tag{10}$$

$$y(a) = \gamma, \ \ y(b) = \lambda. \tag{11}$$

As will be seen from statements below, there is a certain relation to the original problem depending on the choice of the values of z, γ and λ. Let us relate problems (8), (9) and (10), (11) to the sequences of functions

$$x_{m+1}(t,z,\gamma,\lambda) = z + \int_a^t f_1\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds$$

$$- \frac{t-a}{b-a}\int_a^b f_1\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds \tag{12}$$

and

$$y_{m+1}(t,z,\gamma,\lambda) = \gamma + \int_a^t f_2\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds$$

$$- \frac{t-a}{b-a}\int_a^b f_2\big(s, x_m(s,z,\gamma,\lambda), y_m(s,z,\gamma,\lambda)\big)\,ds + \frac{t-a}{b-a}\,(\lambda-\gamma), \quad (13)$$

where $t \in [a,b]$, $m = 0, 1, \ldots,$

$$x_0(t,z) = z, \quad y_0(t,\gamma,\lambda) = \gamma + \frac{t-a}{b-a}\,(\lambda-\gamma).$$

Theorem 1. *Let conditions* (5), (6), (7) *be fulfilled. Then, for all fixed* $z \in \mathscr{U}$, $\gamma \in \mathscr{V}_a$, $\lambda \in \mathscr{V}_b$:

1. *Each of the functions of sequence* (12) *has range in* U, *is continuosly differentiable on* $[a,b]$, *and satisfies conditions* (9). *The limit*

$$x_\infty(t,z,\gamma,\lambda) = \lim_{m\to\infty} x_m(t,z,\gamma,\lambda) \quad (14)$$

exists uniformly in $(t,z,\gamma,\lambda) \in [a,b] \times \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b$. *Function* (14) *satisfies the boundary condition* (9).

2. *Each of the functions of sequence* (13) *has range in* V, *is continuosly differentiable on* $[a,b]$, *and satisfies conditions* (11). *The limit*

$$y_\infty(t,z,\gamma,\lambda) = \lim_{m\to\infty} y_m(t,z,\gamma,\lambda) \quad (15)$$

exists uniformly in $(t,z,\gamma,\lambda) \in [a,b] \times \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b$. *Function* (15) *satisfies the boundary condition* (11).

3. *The functions* $x_\infty(\,\cdot\,,z,\gamma,\lambda)$, $y_\infty(\,\cdot\,,z,\gamma,\lambda)$ *form the unique continuously differentiable solution of the system of integral equations*

$$x(t) = z + \int_a^t f_1(s,x(s),y(s))\,ds - \frac{t-a}{b-a}\int_a^b f_1(s,x(s),y(s))\,ds,$$

$$y(t) = \gamma + \frac{t-a}{b-a}\,(\lambda-\gamma) + \int_a^t f_2(s,x(s),y(s))\,ds - \frac{t-a}{b-a}\int_a^b f_2(s,x(s),y(s))\,ds.$$

4. *The following error estimate holds:*

$$\big|x_\infty(t,z,\gamma,\lambda) - x_m(t,z,\gamma,\lambda)\big| \le \frac{10}{9}\,\alpha_1(t)\left\{Q^m(I_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{U\times V}(f_1)\\ \delta_{U\times V}(f_2)\end{pmatrix}\right\}_1^p,$$

$$\big|y_\infty(t,z,\gamma,\lambda) - y_m(t,z,\gamma,\lambda)\big| \le \frac{10}{9}\,\alpha_1(t)\left\{Q^m(I_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{U\times V}(f_1)\\ \delta_{U\times V}(f_2)\end{pmatrix}\right\}_{p+1}^{p+q},$$

where

$$\alpha_1(t) = 2(t-a)\Big(1 - \frac{t-a}{b-a}\Big), \quad t \in [a,b],$$

and $\{u\}_1^p = \operatorname{col}(u_1,u_2,\ldots,u_p)$, $\{u\}_{p+1}^{p+q} = \operatorname{col}(u_{p+1},u_{p+2},\ldots,u_{p+q})$ *for a vector* $u \in \mathbb{R}^n$.

The idea of proof is to show that (12), (13) are Cauchy sequences in the Banach spaces $C([a,b],\mathbb{R}^p)$ and $C([a,b],\mathbb{R}^q)$, respectively.

Under conditions of Theorem 1, functions (14), (15) are solutions of the Cauchy problems for the forced systems

$$x'(t) = f_1(t, x(t), y(t)) + \Delta_U(z,\gamma,\lambda), \ \ x(a) = z,$$
$$y'(t) = f_2(t, x(t), y(t)) + \Delta_V(z,\gamma,\lambda), \ \ x(a) = \gamma,$$

where $\Delta_U : \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b \to \mathbb{R}^p$ and $\Delta_V : \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b \to \mathbb{R}^p$ are the mappings given by the formulas

$$\Delta_U(z,\gamma,\lambda) = -\frac{1}{b-a}\int_a^b f_1\big(s, x_\infty(s,z,\gamma,\lambda), y_\infty(s,z,\gamma,\lambda)\big)\,ds,$$

$$\Delta_V(z,\gamma,\lambda) = \frac{1}{b-a}\,(\lambda-\gamma) - \frac{1}{b-a}\int_a^b f_2\big(s, x_\infty(s,z,\gamma,\lambda), y_\infty(s,z,\gamma,\lambda)\big)\,ds.$$

Theorem 2. *Under the assumptions of Theorem 1, the limit functions* (14), (15) *of sequences* (12), (13) *form a solution of the boundary value problem* (1)–(3) *if and only if the parameters* (z,γ,λ) *satisfy the system of* $p+2q$ *equations*

$$\Delta_U(z,\gamma,\lambda) = 0, \quad \Delta_V(z,\gamma,\lambda) = 0, \quad \Lambda(z,\gamma,\lambda) = 0, \tag{16}$$

where

$$\Lambda(z,\gamma,\lambda) = \phi(y_\infty(\,\cdot\,,\gamma,\lambda)) - d. \tag{17}$$

The proof can be carried out similarly to [1,2]. The next statement shows that the system of determining equations (16) determines all possible solutions of the original non-linear boundary value problem (1)–(3) having range in $U \times V$.

Theorem 3. *Let the assumptions of Theorem 1 hold.*

1. *If there exist some* $(z_*,\gamma_*,\lambda_*) \in \mathscr{U} \times \mathscr{V}_a \times \mathscr{V}_b$ *satisfying the system of determining equations* (16)*, then problem* (1)–(3) *has a solution* (x_*,y_*) *such that*

   $$x_*(a) = x_*(b) = z_*, \quad y_*(a) = \gamma_*, \quad y_*(b) = \lambda_*$$

   and, moreover,

   $$x_*(\,\cdot\,) = x_\infty(\,\cdot\,,z_*,\gamma_*,\lambda_*), \quad y_*(\,\cdot\,) = y_\infty(\,\cdot\,,z_*,\gamma_*,\lambda_*).$$

2. *If the boundary value problem* (1)–(3) *has a solution* (x_*,y_*) *with the range in* $U \times V$*, then the system of determining equations* (16) *is satisfied with*

   $$z = x_*(a), \quad \gamma = y_*(a), \quad \lambda = y_*(b).$$

The proof can be carried out by analogy to [1,2].

The solvability of system (16), under additional conditions, can be proved if a solution of an approximate determining system

$$\Delta_{U,m}(z,\gamma,\lambda) = 0, \quad \Delta_{V,m}(z,\gamma,\lambda) = 0, \quad \Lambda_m(z,\gamma,\lambda) = 0,$$

has been found, where m is fixed and $\Delta_{U,m}$, $\Delta_{V,m}$, Λ_m are defined similarly to (17) with x_∞, y_∞ replaced by x_m, y_m. Practical calculations using Maple confirm the constructiveness of the proposed approach.

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On Lyapunov and Krasovskii Stability/Instability of Pendulum Equation with Non-Constant Coefficients

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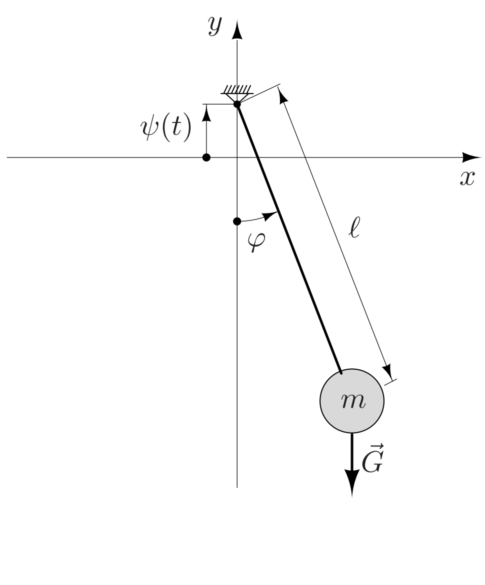
In this note, which is based on the paper [12], we discuss stability of two geometrically distinct equilibria $u_* = 0$ and $u^* = \pi$ of the equation

$$u'' + q(t)u' + r(t)\sin u = 0, \tag{0.1}$$

where $q, r : \mathbb{R} \to \mathbb{R}$ are continuous T-periodic functions. This equation can be understood as a generalisation of the equation of motion of free damped pendulum consisting of a point mass m attached to the massless rod of the length ℓ, whose suspension point oscillates vertically. The pendulum is a system with one degree of freedom, described by the coordinate φ, and its equation of motion is of the form



$$\varphi'' + \frac{b}{m}\,\varphi' + \Big(\frac{\mathrm{g}}{\ell} + \frac{\psi''(t)}{\ell}\Big)\sin\varphi = 0, \tag{0.2}$$

where $b \geq 0$ is the damping coefficient, g denotes the gravitational acceleration, and $\psi \in C^2(\mathbb{R})$ is a T-periodic function determining the oscillations of the suspension point.

If the suspension point is fixed, i.e., if $\psi(t) \equiv const.$ on $\mathbb{R}$, then the situation is simple, because equation (0.2) becomes autonomous (with constant coefficients). One can apply a standard technique of the dynamical systems theory to show that the lower equilibrium $\varphi_* = 0$ is stable and the the upper equilibrium $\varphi^* = \pi$ is unstable.

If $\psi(t) \not\equiv const.$ on $\mathbb{R}$, then the situation is much more complicated. The complications are not connected with the linearizations of (0.2) at its equilibria, but with the use of stability/instability criteria. The linearizations obtained are non-autonomous and thus, verifying all the hypotheses of the stability/instability criteria is far from being trivial. It is well known that, without some additional assumptions, stability/instability of the linearized equation does not guarantee stability/instability of the corresponding solution to the original non-linear equation (see, e. g., [10, Section 3]).

1 History of the problem

The history of the problem goes back to the beginning of the 20th century to works of Stephenson (see, e.g., the paper [2] for overview and historical background). Stephenson considered "small" deflections from the upper equilibrium of the pendulum with harmonic oscillations of the suspension point. Therefore, he studied stability and the approximate solutions to the linear second-order differential equations with a non-constant coefficient

$$\theta'' \pm \Big(\frac{\mathrm{g}}{\ell} - \frac{A}{\ell}\,\Omega^2\sin(\Omega t)\Big)\theta = 0,$$

which are known as Mathieu equations. In the first half of the 20th century, these problems were studied by many mathematicians such as Van der Pol, Strutt, Hirsh, Erdleyi, Lowenstern, etc. However, the works of Bogolyubov (published in 1950) and Kapitza (published in 1951) became really important for the field of non-linear dynamics.

In the paper [3], N. N. Bogolyubov applied the method of averaging to the study of stability of the upper equilibrium $\varphi^* = \pi$ of the pendulum with a harmonically oscillating suspension point, namely, of the equation

$$\varphi'' + \frac{b}{m}\,\varphi' + \Big(\frac{\mathrm{g}}{\ell} - \frac{A}{\ell}\,\Omega^2\sin(\Omega t)\big)\sin\varphi = 0, \tag{1.1}$$

where $A > 0$ is the amplitude of oscillations and $\Omega > 0$ is their angular frequency. Bogolyubov shows in [3] that if

$$\frac{A}{\ell} \ll 1 \;\text{ and }\; \Omega > \sqrt{2}\,\frac{\ell}{A}\sqrt{\frac{\mathrm{g}}{\ell}}\,, \tag{1.2}$$

then the equilibrium $\varphi^* = \pi$ of the pendulum equation (1.1) is stable. Moreover, it follows from his results that if (1.2) is satisfied, then the lower equilibrium $\varphi_* = 0$ of the pendulum equation is also stable and equation (1.1) possesses approximate "quasistatic solutions"

$$\varphi_{1,2}(t) \approx \alpha_{1,2} - \frac{A}{\ell}\,\sin(\alpha_{1,2})\sin(\Omega t),$$

where $\alpha_{1,2} = \pm\arccos\big(-\frac{2\mathrm{g}\ell^2}{A^2\Omega^2}\big)$.

P. L. Kapitza studied in [7] stability of the upper equilibrium of the undamped pendulum with a harmonically oscillating suspension point. His approach is based on the physical reasoning together with the method of averaging, whereas he derived the same condition for stability of the upper equilibrium of the pendulum as Bogolyubov. Denote by θ the angle between the rod of the pendulum and the positive semi-axis of the y-axis (tj. $\theta = \pi - \varphi$). Kapitza claims in [7] (see also [8]) that if (1.2) is satisfied, then the upper equilibrium $\theta^* = 0$ and the lower equilibrium $\theta_* = \pi$ of the pendulum are stable and, moreover, there are two the so-called "quasistatic balances" of the pendulum given by the formula

$$\theta_{1,2}(t) \approx \lambda_{1,2} + \frac{A}{\ell}\,\sin(\lambda_{1,2})\sin(\Omega t),$$

where $\lambda_{1,2} = \pm\arccos\big(\frac{2\mathrm{g}\ell}{A^2\Omega^2}\big)$.

2 Basic Notions

Consider the second-order differential equation

$$u'' = h(t, u, u'), \tag{2.1}$$

where $h : [a, \infty[\, \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

Definition 2.1. A point $\widetilde{u} \in \mathbb{R}$ is called an equilibrium of equation (2.1) if $h(t, \widetilde{u}, 0) \equiv 0$ on $[a, \infty[$.

As usual, we reduce the question of stability/instability of the equilibria of (2.1) into the question of stability/instability of constant solutions to the planar system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= h(t, y_1, y_2). \end{aligned} \tag{2.2}$$

Definition 2.2. The solution $y_0 : [a, \infty[\, \to \mathbb{R}^2$ to system (2.2) is said to be:

(1) Lyapunov stable if for any $\varepsilon > 0$ and any $t_0 \geq a$, there exists $\delta(\varepsilon, t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(\varepsilon, t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\|y(t) - y_0(t)\| \leq \varepsilon \ \text{ for } \ t \geq t_0.$$

Otherwise, it is said to be Lyapunov unstable.

(2) Krasovskii stable if for any $t_0 \geq a$, there exist $\delta(t_0) > 0$ and $R(t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\|y(t) - y_0(t)\| \leq R(t_0)\|y(t_0) - y_0(t_0)\| \ \text{ for } \ t \geq t_0.$$

Otherwise, it is said to be Krasovskii unstable.

(3) Attractive if for any $t_0 \geq a$, there exists $\delta(t_0) > 0$ such that every solution y to system (2.2) with $\|y(t_0) - y_0(t_0)\| \leq \delta(t_0)$ both exists for all $t \geq t_0$ and satisfies

$$\lim_{t \to \infty} \|y(t) - y_0(t)\| = 0.$$

(4) Asymptotically stable, if it is both stable and attractive.

It is clear that if the solution $y_0 : [a, \infty[\, \to \mathbb{R}^n$ to system (2.2) is Krasovskii stable, then it is Lyapunov stable as well. The converse implication does not hold in general.

Definition 2.3. A solution $u_0 : [a, \infty[\, \to \mathbb{R}$ to equation (2.1) is said to be Lyapunov (Krasovskii) stable (resp. attractive) if the corresponding solution $y_0 = (u_0, u_0')$ to system (2.2) is Lyapunov (Krasovskii) stable (resp. attractive).

By Lyapunov (Krasovskii) stability (resp. attractivity) of an equilibrium $\widetilde{u}$ of equation (2.1) we understand Lyapunov (Krasovskii) stability (resp. attractivity) of the corresponding constant solution $u_0(t) := \widetilde{u}$ to equation (2.1).

Since the coefficients r, q in pendulum-like equation (0.1) are supposed to be T-periodic, the linearizations of (0.1) along its equilibria are second-order ODEs with periodic coefficients Therefore, it is not surprising that, in the proofs of stability/instability of the equilibria of (0.1), we apply Floquet theory for the linear equation

$$u'' + q(t)u' + p(t)u = 0 \tag{2.3}$$

in which a T-periodic coefficient $p : \mathbb{R} \to \mathbb{R}$ depends on the given equilibrium. It is well-known that stability criteria for equation (2.3) can be formulated in terms of Floquet multipliers but, unfortunately, the those criteria are not effective. In our results, we formulate stability criteria for the equilibria of (0.1) in terms of the classes $\mathbb{V}^+(T)$ and $\mathbb{V}^-(T)$ introduced for the the linear periodic problem

$$u'' + q(t)u' + p(t)u = 0; \quad u(0) = u(T), \;\; u'(0) = u'(T). \tag{2.4}$$

Definition 2.4. We say that a pair of functions $(p, q) \in L([0,T]) \times L([0,T])$ belongs to the set $\mathbb{V}^+(T)$ (resp. $\mathbb{V}^-(T)$) if for any function $v : [0,T] \to \mathbb{R}$, which is absolutely continuous together with its first derivative and satisfies

$$v''(t) + q(t)v'(t) + p(t)v(t) \geq 0 \;\text{ for a.e. }\; t \in [0,T], \quad v(0) = v(T), \quad v'(0) = v'(T),$$

the inequality $v(t) \geq 0$ $t \in [0,T]$ (resp. $v(t) \leq 0$ for $t \in [0,T]$).

In other words, $(p, q) \in \mathbb{V}^+(T)$ (resp. $(p, q) \in \mathbb{V}^-(T)$) if and only if Green's function of the periodic problem (2.4) exists and is positive (resp. negative). In another terminology, $(p, q) \in \mathbb{V}^+(T)$ (resp. $(p, q) \in \mathbb{V}^-(T)$) if and only if the anti-maximum principle (resp. maximum principle) holds for periodic problem (2.4).

3 Main results

We first formulate two general results for pendulum-like equation (0.1) in terms of Floquet multipliers of the linear equations

$$u'' + q(t)u' + r(t)u = 0 \tag{3.1$_0$}$$

and

$$u'' + q(t)u' - r(t)u = 0, \tag{3.1$_\pi$}$$

which are in fact linearizations of (0.1) at its equilibria $u_* = 0$ and $u^* = 0$, respectively.

Proposition 3.1. *Let $\varrho_1, \varrho_2 \in \mathbb{C}$ be Floquet multipliers of equation* (3.1$_0$) *(resp. equation* (3.1$_\pi$)*). Then:*

(1) *If $|\varrho_1| < 1$ and $|\varrho_2| < 1$, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(12) *If $|\varrho_k| > 1$ for some $k \in \{1, 2\}$, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation* (0.1) *is Krasovskii unstable.*

Remark 3.1. Since equations (3.1$_0$) and (3.1$_\pi$) are equations with periodic coefficients, Proposition 3.1 can be easily reformulated in terms of Lyapunov exponents as well as in terms of stability of linearized equations as follows:

(A) If $\int_0^T q(s)\,\mathrm{d}s > 0$ and the linear equation (3.1$_0$) (resp. equation (3.1$_\pi$)) is asymptotically Lyapunov stable, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation (0.1) is asymptotically Lyapunov stable.

(B) If $\int_0^T q(s)\,\mathrm{d}s > 0$ and the linear equation (3.1$_0$) (resp. equation (3.1$_\pi$)) is Lyapunov unstable, then the equilibrium $u_* = 0$ (resp. the equilibrium $u^* = \pi$) of equation (0.1) is Krasovskii unstable.

One can see that asymptotic Krasovskii stability/instability of the equilibria of (0.1) is more or less completely described in terms of Floquet multipliers of their linearizations. However, it is a far more delicate question to guarantee Lyapunov instability of the equilibria of equation (0.1).

Proposition 3.2. *Let $q(t) \equiv Const.$, equation (3.1_π) be disconjugate on $\mathbb{R}$ and have a real Floquet multiplier ϱ satisfying $\varrho > 1$. Then, the equilibrium $u^* = \pi$ of equation* (0.1) *is Lyapunov unstable.*

Remark 3.2. Similarly as in Remark 3.1, Proposition 3.2 can be reformulated in terms of instability of the linearized equation:

(C) If $q(t) \equiv q_0 > 0$ and the linear equation (3.1_π) is disconjugate on $\mathbb{R}$ and Lyapunov unstable, then the equilibrium $u^* = \pi$ of equation (0.1) is Lyapunov unstable.

Now, we provide stability criteria for the equilibria of (0.1) in terms of the classes $\mathbb{V}^+(T)$ and $\mathbb{V}^-(T)$.

Theorem 3.1. *The following conclusions hold:*

(1) *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and $(r,q) \in \operatorname{Int}\mathbb{V}^+(T)$, then the equilibrium $u_* = 0$ of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(2) *If $q(t) \equiv q_0 > 0$ and $(r,q_0) \in \mathbb{V}^-(T)$, then the equilibrium $u_* = 0$ of equation* (0.1) *is Lyapunov unstable and consequently, Krasovskii unstable.*

(3) *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and $(-r,q) \in \operatorname{Int}\mathbb{V}^+(T)$, then the equilibrium $u^* = \pi$ of equation* (0.1) *is asymptotically Krasovskii stable and consequently, asymptotically Lyapunov stable.*

(4) *If $q(t) \equiv q_0 > 0$ and $(-r,q_0) \in \mathbb{V}^-(T)$, then the equilibrium $u^* = \pi$ of equation* (0.1) *is Lyapunov unstable and consequently, Krasovskii unstable.*

Let us mention here that some effective conditions for the inclusions $(p,q) \in \mathbb{V}^+(T)$ and $(p,q) \in \mathbb{V}^-(T)$ are derived, e.g., in [1,6,13] (see also [4,11] for the case of $q(t) \equiv 0$).

Theorem 3.2. *If $\int\limits_0^T q(s)\,\mathrm{d}s > 0$ and*

$$(r,q) \in \operatorname{Int}\mathbb{V}^+(T), \ \ (-r,q) \in \operatorname{Int}\mathbb{V}^+(T),$$

then both equilibria $u_* = 0$ and $u^* = \pi$ of equation (0.1) *are asymptotically Lyapunov stable and there exists a T-periodic solution $u_{\rm per}$ to equation* (0.1) *such that*

$$0 < u_{\rm per}(t) < \pi \ \ \text{for} \ \ t \in \mathbb{R}. \tag{3.2}$$

Remark 3.3. It seems that under the hypotheses of the previous theorem, the solution $u_{\rm per}$ should be unique. Unfortunately, we cannot prove this fact.

However, it follows from the proof of Theorem 3.2 and [5, Proposition 3.1] that if $u_{\rm per}$ in the previous theorem is a unique T-periodic solution satisfying (3.2), then it is Lyapunov unstable.

Applying the results of [9,11,13], we can derive from Theorem 3.1(1,3,4) the following effective criteria for equation (1.1), i.e., for pendulum equation (0.2) with $\psi(t) := A\sin(\Omega t)$.

Corollary 3.1. *Let $b > 0$ and either*

$$\left(\frac{A}{\ell}\right)^2\Omega^2\Big[1-\Big(\frac{\pi}{e^{\frac{2\pi A}{\ell}}-1}\Big)^2\Big] \le -\frac{g}{\ell}+\left(\frac{b}{2m}\right)^2 \le 0 \tag{3.3}$$

or

$$\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi), \quad 0 \le -\frac{g}{\ell}+\left(\frac{b}{2m}\right)^2 \le \frac{4\ln(1+\pi)}{\pi^3}\left(\frac{A}{\ell}\right)^2\Omega^2.$$

Then, the equilibrium $\varphi_* = 0$ of equation (1.1) is asymptotically Lyapunov stable.

Corollary 3.2. *Let $b > 0$ and*

$$\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi), \quad \frac{g}{\ell}+\left(\frac{b}{2m}\right)^2 \le \frac{4\ln(1+\pi)}{\pi^3}\left(\frac{A}{\ell}\right)^2\Omega^2. \tag{3.4}$$

Then, the equilibrium $\varphi^* = \pi$ of equation (1.1) is asymptotically Lyapunov stable.

Remark 3.4. The second inequality in (3.4) can be rewritten into the form

$$\Omega \ge \sqrt{\frac{\pi^3}{4\ln(1+\pi)}}\,\frac{\ell}{A}\sqrt{\frac{g}{\ell}+\left(\frac{b}{2m}\right)^2}.$$

It is clear that the requirement on Ω in Corollary 3.2 is stronger than condition (1.2) derived by Bogolyubov and Kapitza under the assumption $\frac{A}{\ell} \ll 1$. On the other hand, in their approaches, the assumption $\frac{A}{\ell} \ll 1$ is essential (Kapitza chose $\frac{A}{\ell} = 0.05$ in his example), but we require quite weaker and the explicit assumption $\frac{A}{\ell} \le \frac{1}{2\pi}\ln(1+\pi) \approx 0.22$.

Corollary 3.3. *If $b \ge 0$ and $\Omega^2 \le \frac{g}{A}$, then the equilibrium $\varphi^* = \pi$ of equation (1.1) is Lyapunov unstable.*

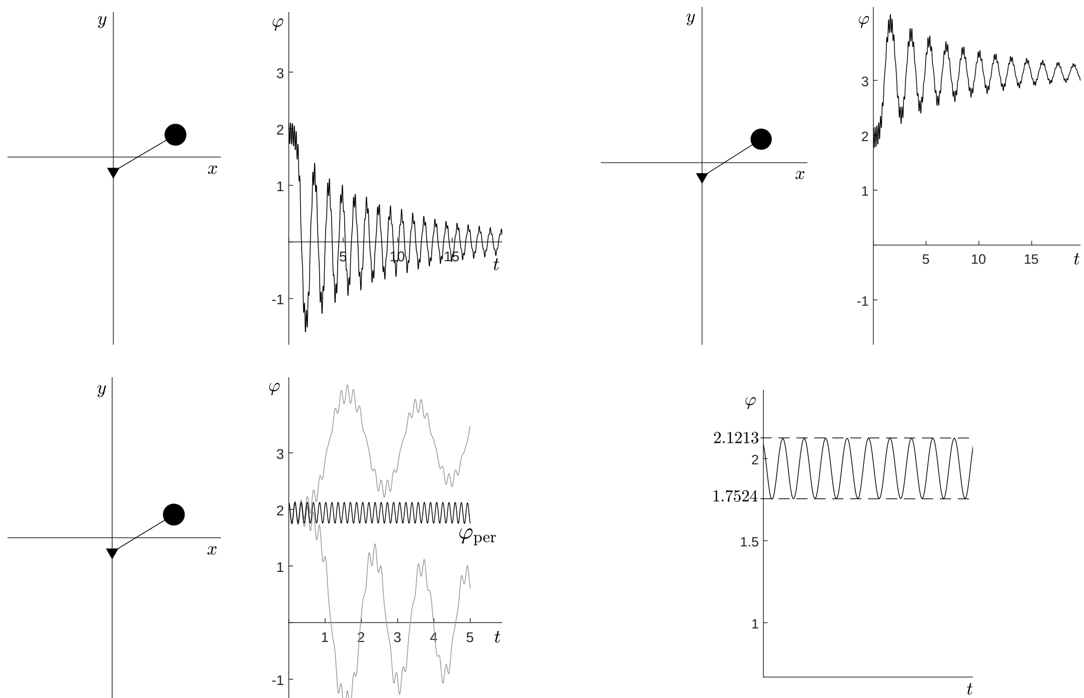
Corollary 3.4. *If $\frac{A}{\ell} < 1$ and $(\frac{A}{\ell})^2\Omega^2 \le \frac{g}{\ell}$, then there exists $b_0 > 0$ such that for any $b \in [0, b_0[$, the equilibrium $\varphi^* = \pi$ of equation (1.1) is Lyapunov unstable.*

Remark 3.5. Instability criterion provided in Theorem 3.13.1 cannot be applied to equation (1.1). If $q(t) := \frac{b}{m} > 0$ and $r(t) := \frac{g}{\ell} - \frac{A}{\ell}\Omega^2\sin(\Omega t)$, then (q, r) cannot belong to the class $\mathbb{V}^-(T)$, because the inequality $\int_0^T r(s)\,ds < 0$ is a necessary condition for the validity of the inclusion $(q, r) \in \mathbb{V}^-(T)$ with $q(t) := Const$.

Corollary 3.5. *Let $b > 0$ and conditions (3.3) and (3.4) hold. Then, both equilibria $\varphi_* = 0$ and $\varphi^* = \pi$ of equation (1.1) are asymptotically Lyapunov stable and, moreover, there exists a $\frac{2\pi}{\Omega}$-periodic solution $\varphi_{\rm per}$ to equation (1.1) satisfying*

$$0 < \varphi_{\rm per}(t) < \pi \quad \text{for } t \in \mathbb{R}.$$

The solution $\varphi_{\rm per}$ in the previous corollary corresponds to the "quasistatic solution" of Bogolyubov as well as to the "quasistatic balance" of Kapitza described in Section 1. On pictures below, there are the results of some numerical simulations showing that free damped pendulum with periodically oscillating suspension point can actually move periodically if its both lower and upper equilibria are asymptotically stable.



Detail of φ_{per}

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Dirichlet Problem for Singular Fractional Differential Equations with Given Maximal Value for Positive Solutions

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1 Introduction

Let $J=[0,1]$, $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$, while $\|x\|_{L^1}=\int\limits_0^1|x(t)|\,\mathrm{d}t$ is the norm in $L^1(J)$.

We discuss the singular fractional differential equation

$$D^\alpha x(t)+\mu\, f\big(t,x(t),D^\beta x(t)\big)=0, \tag{1.1}$$

depending on the real parameter μ. Here $\alpha\in(1,2]$, $\beta\in(0,\alpha-1]$, f satisfies the local Carathéodory conditions on $J\times(0,\infty)\times\mathbb{R}$, $\lim\limits_{x\to 0+}f(t,x,y)=\infty$ for a.e. $t\in J$ and $y\in\mathbb{R}$, and D^γ is the Riemann–Liouville fractional derivative of order γ.

Together with equation (1.1) the boundary conditions

$$x(0)=0,\quad x(1)=0, \tag{1.2}$$

$$\max\big\{x(t):\ t\in J\big\}=A \tag{1.3}$$

are considered, where $A>0$ is given.

We are looking for a value of the parameter μ in (1.1) for which problem (1.1)–(1.3) has a positive solution.

Definition. We say that $x:J\to\mathbb{R}$ is *a positive solution of problem* (1.1)–(1.3) if

(a) $x, D^\beta x\in C(J)$, $D^\alpha x\in L^1(J)$, $x>0$ on $(0,1)$,

(b) x satisfies the boundary conditions (1.2), (1.3),

(c) there exists $\mu_*>0$ such that (1.1) for $\mu=\mu_*$ holds for a.e. $t\in J$.

The special case of (1.1) (for $\alpha=2$, $\beta=1$) is the differential equation

$$x''(t)=\mu f(t,x(t),x'(t)).$$

The existence result for solutions of this equation satisfying the boundary conditions (1.2), (1.3) was given in [1].

The Riemann–Liouville fractional derivative $D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is defined as [3, 4]

$$D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\,x(s)\,\mathrm{d}s,$$

where $n = [\gamma] + 1$ and $[\gamma]$ means the integral part of γ. If $\gamma \in \mathbb{N}$, then $D^\gamma x(t) = x^{(\gamma)}(t)$. The Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is given as

$$I^\gamma x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\,\mathrm{d}s$$

and I^0 is the identity operator. Γ is the Euler gamma function.

We work with the following growth conditions for the function f in (1.1):

(H_1) There exists $m > 0$ such that

$$f(t,x,y) \ge m(1-t)^{2-\alpha} \ \text{ for a.e. } \ t \in J \ \text{ and all } \ (x,y) \in (0,\infty) \times \mathbb{R}.$$

(H_2) For a.e. $t \in J$ and all $(x,y) \in (0,\infty) \times \mathbb{R}$,

$$f(t,x,y) \le \phi(t)g(x) + \rho(t)\big(p(x) + w(|y|)\big),$$

where $\phi, \rho \in L^1(J)$, $g \in C(0,\infty)$, $p, w \in C[0,\infty)$ are positive, g is nonincreasing, p, w are nondecreasing and

$$\lim_{\kappa \to 0+} \kappa \int_0^1 \phi(t) g(\kappa t(1-t))\,\mathrm{d}t = 0, \qquad \lim_{v \to \infty} \frac{w(v)}{v} = 0.$$

The existence results for problem (1.1)–(1.3) are proved by the combination of the regularization and sequential techniques with the Leray–Schauder degree method.

2 Preliminaries

Let α, β be from (1.1) and

$$G(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \ 0 \le s \le t \le 1, \\[2mm] \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \ 0 \le t \le s \le 1. \end{cases}$$

Then for $h \in L^1(J)$

$$\int_0^1 G(t,s)h(s)\,\mathrm{d}s = -I^\alpha h(t) + I^\alpha h(t)\big|_{t=1} t^{\alpha-1},$$

and

$$D^\beta \int_0^1 G(t,s)h(s)\,\mathrm{d}s = -I^{\alpha-\beta}h(t) + t^{\alpha-\beta-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\beta)}\, h(s)\,\mathrm{d}s \tag{2.1}$$

since

$$D^\beta t^{\alpha-1} = \frac{t^{\alpha-\beta-1}\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\,.$$

Let $X = \{x \in C(J) : \ D^\beta x \in C(J)\}$. X is a Banach space equipped with the norm

$$\|x\|_* = \max\big\{\|x\|, \|D^\beta x\|\big\}.$$

Lemma 2.1. *Let $h \in L^1(J)$. Then*

$$x(t) = \int\limits_0^1 G(t,s)h(s)\,\mathrm{d}s \tag{2.2}$$

is the unique solution in X of the equation

$$D^\alpha x(t) + h(t) = 0, \tag{2.3}$$

satisfying the Dirichlet condition (1.2).

Proof. By [2, Lemma 2.2], x is the unique solution of problem (2.3), (1.2) in $C(J)$. Since $\alpha - \beta \geq 1$, $I^{\alpha-\beta}h \in C(J)$. Hence (2.1) and (2.2) give $D^\beta x \in C(J)$. Consequently, $x \in X$. □

Lemma 2.2. *Let $m > 0$ be from (H_1), $h \in L^1(J)$, $h(t) \geq m(1-t)^{2-\alpha}$ for a.e. $t \in J$ and let*

$$K = \frac{m}{2\Gamma(\alpha-1)}\,.$$

Then

$$\int\limits_0^1 G(t,s)h(s)\,\mathrm{d}s \geq Kt(1-t) \ \text{ for } \ t \in J.$$

3 Regular problems

For $n \in \mathbb{N}$, let

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{if } \ x \geq \dfrac{1}{n}\,, \\ f\Big(t, \dfrac{1}{n}, y\Big) & \text{if } \ x < \dfrac{1}{n} \end{cases}$$

for a.e. $t \in J$ and $y \in \mathbb{R}$. Then f_n satisfies the local Carathéodory conditions on $J \times \mathbb{R}^2$.

We now discuss the regular fractional differential equation

$$D^\alpha x(t) + \mu f_n\big(t, x(t), D^\beta x(t)\big) = 0 \tag{3.1}$$

together with the boundary conditions (1.2) and

$$\max\big\{x(t) : \ t \in J\big\} = \lambda A, \tag{3.2}$$

where $A > 0$ is from (1.3) and $\lambda \in (0,1]$.

Lemma 3.1. *Let (H_1) and (H_2) hold and let $K > 0$ be from Lemma 2.2. Then there exists a positive constant P independent of $\lambda \in (0,1]$ such that for solutions x of problem* (3.1), (1.2), (3.2) *with $\mu = \mu_x$ in* (3.1) *the estimates*

$$\|D^\beta x\| < P, \ \ 0 < \mu_x \leq \frac{4A}{K}$$

hold and $x > 0$ on $(0,1)$.

Let $Y = X \times \mathbb{R}$ and an operator $\mathcal{L}$ acting on $Y \times [0,1]$ be given by the formula

$$\mathcal{L}(x,\mu,\lambda)(t) = \mu \int_0^1 G(t,s)\Big(m(1-\lambda)(1-s)^{2-\alpha} + \lambda f_n\big(s,x(s),D^\beta x(s)\big)\Big)\,\mathrm{d}s,$$

where $m > 0$ is from (H_1).

Lemma 3.2. *Let (H_1) hold. Then $\mathcal{L} : Y \times [0,1] \to X$ and $\mathcal{L}$ is a completely continuous operator.*

Let $A > 0$ be from (1.3), $K > 0$ from Lemma 2.2 and $P > 0$ from Lemma 3.1. Let

$$\Omega = \Big\{(x,\mu) \in Y : \ \ \|x\| < A+1, \ \ \|D^\beta x\| < P, \ \ |\mu| < \frac{4A}{K} + 1\Big\}$$

and "deg" stand for the Leray–Schauder degree, $\mathcal{I}$ be the identical operator on Y, $\theta = (0,0) \in Y$.

Lemma 3.3. *Let (H_1) and (H_2) hold. Then problem (3.1), (1.2), (1.3) has at least one positive solution.*

Sketch of the proof.

Step 1. Let $\mathcal{K} : \overline{\Omega} \times [0,1] \to Y$,

$$\mathcal{K}(x,\mu,\lambda) = \big(\mathcal{L}(x,\mu,\lambda), \Lambda(x,\mu,\lambda)\big),$$

where $\Lambda : Y \times [0,1] \to \mathbb{R}$,

$$\Lambda(x,\mu,\lambda) = \lambda\Big(\max\big\{x(t) : \ t \in J\big\} + \min\big\{x(t) : \ t \in J\big\}\Big) + (1-\lambda)x\Big(\frac{1}{2}\Big) + \mu.$$

$\mathcal{K}$ is a compact operator. Since $\mathcal{K}(x,\mu,\lambda) \neq (x,\mu)$ for $(x,\mu) \in \partial\Omega$, $\lambda \in [0,1]$ and $\mathcal{K}(\,\cdot\,,\,\cdot\,,0)$ is an odd operator, we conclude from the Borsuk antipodal theorem and the homotopy property that $\deg(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,0),\Omega,\theta) \neq 0$ and

$$\deg\big(\mathcal{I} - \mathcal{K}(\,\cdot\,,\,\cdot\,,1),\Omega,\theta\big) \neq 0. \tag{3.3}$$

Step 2. Let $\mathcal{H} : \overline{\Omega} \times [0,1] \to Y$,

$$\mathcal{H}(x,\mu,\lambda) = \big(\mathcal{L}(x,\mu,1),\ \Phi(x,\mu,\lambda)\big),$$

where $\Phi : \overline{\Omega} \times [0,1] \to \mathbb{R}$,

$$\Phi(x,\mu,\lambda) = \max\big\{x(t) : \ t \in J\big\} + \min\big\{x(t) : \ t \in J\big\} - \lambda A + \mu,$$

and $A > 0$ is from (1.3). $\mathcal{H}$ is a compact operator and if $\mathcal{H}(x_*,\mu_*,1) = (x_*,\mu_*)$ for some $(x_*,\mu_*) \in \overline{\Omega}$, then x_* is a positive solution of problem (3.1), (1.2), (1.3) for $\mu = \mu_*$ in (3.1). Since $\mathcal{H}(x,\mu,\lambda) \neq (x,\mu)$ for $(x,\mu) \in \partial\Omega$ and $\lambda \in [0,1]$, we conclude from $\mathcal{H}(\,\cdot\,,\,\cdot\,,0) = \mathcal{K}(\,\cdot\,,\,\cdot\,,1)$, the homotopy property and (3.3) that

$$\deg\big(\mathcal{I} - \mathcal{H}(\,\cdot\,,\,\cdot\,,1),\Omega,\theta\big) \neq 0.$$

Hence there exists a fixed point $(x_0,\mu_0) \in \Omega$ of $\mathcal{H}(\,\cdot\,,\,\cdot\,,1)$. Therefore x_0 is a positive solution of problem (3.1), (1.2), (1.3) for $\mu = \mu_0$ in (3.1). □

4 Problem (1.1)–(1.3)

Theorem 4.1. *Let* (H_1) *and* (H_2) *hold. Then problem* (1.1)–(1.3) *has at least one positive solution.*

Sketch of the proof. By Lemmas 3.1 and 3.3, for each $n \in \mathbb{N}$ there exists a positive solution $x_n \in X$ of problem $(3.1), (1.2), (1.3)$ for $\mu = \mu_n$ in (3.1), $x_n > 0$ on $(0,1)$, $\|x_n\| = A$, $\|D^\beta x_n\| < P$ and $0 < \mu_n \le 4A/K$. Hence the sequence $\{(x_n, \mu_n)\}$ is bounded in $X \times \mathbb{R}$. We begin by proving that $\{\mu_n\}$ has a positive lower bound $\Delta > 0$ and the sequences $\{x_n\}$ $\{D^\beta x_n\}$ are equicontinuous on J. Consequently, $\{(x_n, \mu_n)\}$ is relatively compact in Y. Without loss of generality we can assume that $\{(x_n, \mu_n)\}$ is convergent in Y and let $(x, \rho) \in Y$ be its limit. Then $\rho \ge \Delta$, $x(t) \ge \Delta K t(1-t)$, $\|D^\beta x\| \le P$, x satisfies the boundary condition (1.2), (1.3) and

$$\lim_{n\to\infty} f_n\big(t, x_n(t), D^\beta x_n(t)\big) = f\big(t, x(t), D^\beta x(t)\big) \ \text{ for a.e. } \ t \in J.$$

Letting $n \to \infty$ in the equality

$$x_n(t) = \mu_n \int_0^1 G(t,s) f_n\big(s, x_n(s), D^\beta x_n(s)\big)\,\mathrm{d}s,$$

we get

$$x(t) = \rho \int_0^1 G(t,s) f\big(s, x(s), D^\beta x(t)\big)\,\mathrm{d}s, \ \ t \in J,$$

by the Lebesgue dominated convergence theorem. Consequently, x is a positive solution of problem (1.1)–(1.3) for $\mu = \rho$ in (1.1). □

Example. Let $r_1, r_2 \in L^1(J)$ and $q_1 \in C(J)$ be nonnegative, $q_2 \in C(J)$ be positive, $\nu, \tau \in (0,1)$ and

$$f(t,x,y) = r_1(t) + \frac{q_1(t)}{x^\nu} + q_2(t)e^x + r_2(t)|y|^\tau \ \text{ for a.e. } \ t \in J, \ \ x > 0, \ \ y \in \mathbb{R}.$$

Then f satisfies the local Carathéodory conditions on $J \times (0,\infty) \times \mathbb{R}$, and the conditions (H_1), (H_2). Hence, by Theorem 4.1, there exists a positive solution of the problem

$$\left.\begin{array}{c} D^\alpha x(t) + \mu\Big(r_1(t) + \dfrac{q_1(t)}{(x(t))^\nu} + q_2(t)e^{x(t)} + r_2(t)|D^\beta x(t)|^\tau\Big) = 0, \\ x(0) = 0, \ \ x(1) = 0, \ \ \max\big\{x(t) : \ t \in J\big\} = A, \ \ A > 0. \end{array}\right\}$$

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Asymptotic Equivalence of Infinite-Dimensional Linear Stochastic Systems

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1 Problem statement

In this paper, we extend the classical Levinson theorem to infinite-dimensional stochastic systems. Specifically, we analyze differential equations in a Banach space that are asymptotically equivalent to the corresponding stochastic equations.

In particular, we consider the following linear differential equation in a Banach space X:

$$dx = Ax\,dt \tag{1.1}$$

and the equation of stochastic differential equations:

$$dy = Ay\,dt + B(t)y\,dt + D(t)y\,dW(t), \tag{1.2}$$

where A, $B(t)$, $D(t)$ are continuous linear operators in a Banach space $L(X)$. $W(t)$ is a standard scalar Wiener process defined for $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_t, t \geq 0\}$, the process $W(t)$ is $\mathcal{F}_t$-adapted.

We will define asymptotic equivalence in the following mean square sense.

Definition 1.1. If for each solution $y(t)$ of equation (1.2) there exists a corresponding solution $x(t)$ of equation (1.1) such that

$$\lim_{t\to\infty} E\big[\,\|x(t) - y(t)\|^2\,\big] = 0,$$

then system (1.2) is called asymptotically mean square equivalent to system (1.1).

2 Main results

Our main result is a generalization of the classical Levinson's theorem to stochastic systems in Banach spaces.

Theorem 2.1. *Let for operator A from system* (1.1) $\sigma(A) = \sigma_0(A) \cup \sigma_-(A)$, *where* $\sigma_0(A)$, $\sigma_-(A)$ *are spectral sets with* $\sigma_0(A) \subset i\,\Re$ *and* $\sigma_-(A)$ *lies in the left half-plane. For the Riesz* P_0 *projector, corresponding to* $\sigma_0(A)$ *we have*

$$\sup_{-\infty<t<\infty} \|e^{P_0At}\| < \infty.$$

If in addition

$$\int_0^\infty \|B(t)\|\,dt < \infty, \quad \int_0^\infty \|D(t)\|^2\,dt < \infty,$$

then system (1.2) *is asymptotically mean square equivalent to system* (1.1).

Proof.
I) The spectral operators are defined using the Riesz integral:

$$P_0 = -\frac{1}{2\pi i}\oint_{\Gamma_0} R_\mu\,d\mu,$$
$$P_- = -\frac{1}{2\pi i}\oint_{\Gamma_-} R_\mu\,d\mu,$$

where the contours Γ_0 and Γ_- enclose regions with the corresponding spectra, and R_μ is the resolvent of the operator A.

We have a decomposition of the Banach space X into a direct sum of spaces:

$$X = X_0 \oplus X_-.$$

Therefore, we obtain the decomposition of the operator A:

$$A = P_0 A + P_- A.$$

Additionally, using the property of the operator exponential we get

$$X(t) = e^{P_- At} + e^{P_0 At} = X_1(t) + X_2(t) \tag{2.1}$$

Considering $\sigma(P_- A) = \sigma_-(A)$, and also noting that according to the compactness of the spectral set, $\max\limits_{\mu\in\sigma_-(A)}\{\mathrm{Re}\,(\mu)\} < -\alpha < 0$, we have:

$$\|X_1(t)\| = \|e^{P_- At}\| \le ae^{-\alpha t}, \ \ t \ge t_0 \ge 0.$$

The estimate on the norm of $X_2(t)$ is obtained directly from the next condition:

$$\|X_2(t)\| \le b, \ \ t \in R.$$

II) Let $X(t) = e^{At}$. Then, using the method of variation of an arbitrary constant (see [4, p. 234]), we obtain that the initial system is equivalent to:

$$y(t) = X(t-t_0)y(t_0) + \int_{t_0}^t X(t-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^t X(t-\tau)D(\tau)y(\tau)\,dW(\tau).$$

Taking into account the decomposition (2.1), we get:

$$y(t) = X(t-t_0)y(t_0) + \int_{t_0}^t X_1(t-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^t X_2(t-\tau)B(\tau)y(\tau)\,d\tau$$
$$+ \int_{t_0}^t X_1(t-\tau)D(\tau)y(\tau)\,dW(\tau) + \int_{t_0}^t X_2(t-\tau)D(\tau)y(\tau)\,dW(\tau).$$

Considering the relation:

$$X_2(t-\tau) = e^{P_0 A(t-\tau)} = e^{A(t-t_0)} e^{A(t_0-\tau)} P_0 = e^{A(t-t_0)} e^{P_0 A(t_0-\tau)} = X(t-t_0) X_2(t_0-\tau),$$

we obtain

$$\begin{aligned} y(t) = X(t-t_0)\Big[y(t_0) &+ \int_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)y(\tau)\,dW(\tau)\Big] \\ &+ \int_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)\,dW(\tau) \\ &- \int_{t}^{\infty} X_2(t-\tau)D(\tau)y(\tau)\,dW(\tau) - \int_{t}^{\infty} X_2(t-\tau)B(\tau)y(\tau)\,d\tau. \end{aligned}$$

To this solution we compare the solution of equation (1.1) with the initial conditions:

$$x(t_0) = y(t_0) + \int_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)y(\tau)\,d\tau + \int_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)y(\tau)\,dW(\tau).$$

We need to prove the correctness of such correspondence, as well as the asymptotic equivalence (in the mean square sense).

III) Correctness of the definition of a solution of the similar system (convergence in the mean square sense).

For the expectation, we have the following estimate:

$$\begin{aligned} E\big(\|y(t)\|^2\big) &\le 3\|X(t-t_0)\|^2 E\big(\|y(t_0)\|^2\big) \\ &+ 3E\bigg(\Big\|\int_{t_0}^{t} X(t-\tau)B(\tau)y(\tau)\,d\tau\Big\|^2\bigg) + 3E\bigg(\Big\|\int_{t_0}^{t} X(t-\tau)D(\tau)y(\tau)\,dW(\tau)\Big\|^2\bigg). \end{aligned}$$

Using the Cauchy–Schwarz inequality, we get:

$$\begin{aligned} E\bigg(\Big\|\int_{t_0}^{t} X(t-\tau)B(\tau)y(\tau)\,d\tau\Big\|^2\bigg) &\le E\bigg(\Big(\int_{t_0}^{t} \|y(\tau)\|\,\|X(t-\tau)\|\,\|B(\tau)\|\,d\tau\Big)^2\bigg) \\ &= E\bigg(\Big(\int_{t_0}^{t} \|y(\tau)\|\sqrt{\|X(t-\tau)\|}\,\sqrt{\|B(\tau)\|}\sqrt{\|X(t-\tau)\|}\,\sqrt{\|B(\tau)\|}\,d\tau\Big)^2\bigg) \\ &\le E\bigg(\int_{t_0}^{t} \|X(t-\tau)\|\,\|B(\tau)\|\,d\tau \int_{t_0}^{t} \|X(t-\tau)\|\,\|B(\tau)\|\,\|y(\tau)\|^2\,d\tau\bigg) \\ &= \int_{t_0}^{t} \|X(t-\tau)\|\,\|B(\tau)\|\,d\tau \int_{t_0}^{t} \|X(t-\tau)\|\,\|B(\tau)\| E\big(\|y(\tau)\|^2\big)\,d\tau \\ &\le \max_{t>0}\|X(t)\|^2 \int_{0}^{+\infty} \|B(\tau)\|\,d\tau \int_{t_0}^{t} \|X(t-\tau)\|\,\|B(\tau)\| E\big(\|y(\tau)\|^2\big)\,d\tau. \end{aligned}$$

And also, for the estimate of the stochastic part we have:

$$E\bigg(\bigg\|\int\limits_{t_0}^{t} X(t-\tau)D(\tau)y(\tau)\,dW(\tau)\bigg\|^2\bigg) \leq \int\limits_{t_0}^{t} E\Big(\big\|X(t-\tau)D(\tau)y(\tau)\big\|^2\Big)\,d\tau \leq \int\limits_{t_0}^{t} \|X(t-\tau)\|^2\,\|D(\tau)\|^2 E\big(\|y(\tau)\|^2\big)\,d\tau.$$

Taking into account that

$$\max_{t>0}\|X(t)\|^2 \leq \max_{t>0}\big(\|X_1(t)\|+\|X_2(t)\|\big)^2 \leq 2(a^2+b^2) \leq 4\max\{a^2,b^2\},$$

we obtain the estimate:

$$\begin{aligned} E\big(\|y(t)\|^2\big) &\leq 12\max\{a^2,b^2\}\big(E\big(\|y(t_0)\|^2\big)\big) \\ &\quad + 12\max\{a^2,b^2\}\bigg(\int\limits_{0}^{+\infty}\|B(\tau)\|\,d\tau\int\limits_{t_0}^{t}\|B(\tau)\|E\big(\|y(\tau)\|^2\big)\,d\tau\bigg) \\ &\quad + 12\max\{a^2,b^2\}\bigg(\int\limits_{t_0}^{t}\|D(\tau)\|^2 E\big(\|y(\tau)\|^2\big)\,d\tau\bigg). \end{aligned}$$

From Gronwall's inequality we get

$$\begin{aligned} E\big(\|y(t)\|^2\big) &\leq 12\max\{a^2,b^2\}E\big(\|y(t_0)\|^2\big)e^{12\max\{a^2,b^2\}\int\limits_{t_0}^{t}(K_1\|B(\tau)\|+\|D(\tau)\|^2)\,d\tau} \\ &\leq 12\max\{a^2,b^2\}E\big(\|y(t_0)\|^2\big)e^{12\max\{a^2,b^2\}\int\limits_{t_0}^{\infty}(K_1\|B(\tau)\|+\|D(\tau)\|^2)\,d\tau} \\ &\leq \widehat{K}E\big(\|y(t_0)\|^2\big), \end{aligned}$$

where

$$\widehat{K} = 12\max\{a^2,b^2\}e^{12\max\{a^2,b^2\}(K_1^2+K_1)}.$$

Thus, we can also conclude that the integrals

$$\int\limits_{t_0}^{\infty} X_2(t_0-\tau)B(\tau)y(\tau)\,d\tau, \quad \int\limits_{t_0}^{\infty} X_2(t_0-\tau)D(\tau)y(\tau)\,dW(\tau)$$

converge in the mean square sense.

IV) We will estimate in mean square the norm of the difference between the corresponding solutions $x(t)$ and $y(t)$. Since

$$x(t) = X(t-t_0)x(t_0),$$

with the given initial condition $x(t_0)$, we have

$$E\big(\|y(t)-x(t)\|^2\big) = E\bigg(\bigg\|\int\limits_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)\,d\tau + \int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)\,dW(\tau)$$

$$
- \int\limits_t^{\infty} X_2(t-\tau)D(\tau)y(\tau)dW(\tau) - \int\limits_t^{\infty} X_2(t-\tau)B(\tau)y(\tau)\,d\tau\Big\|^2\Big)
$$

$$
\le 4E\Big(\Big\|\int\limits_{t_0}^{t} X_1(t-\tau)B(\tau)y(\tau)\,d\tau\Big\|^2\Big) + 4E\Big(\Big\|\int\limits_{t_0}^{t} X_1(t-\tau)D(\tau)y(\tau)\,dW(\tau)\Big\|^2\Big)
$$

$$
+ 4E\Big(\Big\|\int\limits_{t}^{\infty} X_2(t-\tau)B(\tau)y(\tau)\,d\tau\Big\|^2\Big) + 4E\Big(\Big\|\int\limits_{t}^{\infty} X_2(t-\tau)D(\tau)y(\tau)\,dW(\tau)\Big\|^2\Big). \quad (2.2)
$$

The third and fourth terms in (2.2) can be easily upper-bounded as

$$
\widehat{K}E(\|y(t_0)\|^2)b^2\Big(\int\limits_t^{\infty}\|B(\tau)\|\,d\tau\Big)^2 \quad \text{and} \quad \widehat{K}E\big(\|y(t_0)\|^2\big)b^2\int\limits_t^{\infty}\|D(\tau)\|^2\,d\tau,
$$

respectively. It is obvious that these expressions tend to zero as $t\to\infty$. The first term in (2.2) can be estimated using the Cauchy–Schwarz inequality as

$$
\widehat{K}E\big(\|y(t_0)\|^2\big)\Big(\int\limits_{t_0}^{t} ae^{-\alpha(t-\tau)}\|B(\tau)\|\,d\tau\Big)^2.
$$

Taking into account the absolute integrability of the operator $B(t)$, we derive the following result:

$$
\int\limits_{t_0}^{t} e^{-\alpha(t-\tau)}\|B(\tau)\|\,d\tau = \int\limits_{t_0}^{t/2} e^{-\alpha(t-\tau)}\|B(\tau)\|\,d\tau + \int\limits_{t/2}^{t} e^{-\alpha(t-\tau)}\|B(\tau)\|\,d\tau
$$

$$
\le e^{-\alpha t/2}\int\limits_{t_0}^{t/2}\|B(\tau)\|\,d\tau + \int\limits_{t/2}^{t}\|B(\tau)\|\,d\tau
$$

$$
\le e^{-\alpha t/2}\int\limits_{0}^{\infty}\|B(\tau)\|d\tau + \int\limits_{t/2}^{t}\|B(\tau)\|\,d\tau.
$$

Since the last expression tends to zero as $t\to\infty$, the first term in (2.2) also tends to zero as $t\to\infty$.

By a similar splitting of the integral into intervals $[t_0, t/2]$ and $[t/2, t]$, we will obtain an estimate for the second term in (2.2), which will also tend to zero.

Proof is completed. □

3 Example

Let's consider the integro-differential equation

$$
dx = (-\lambda x(t) + \int\limits_{t_0}^{t} K(t,s)x(s)\,ds)\,dt
$$

in the Hilbert space $L_2[0,+\infty)$ (where $K(t,s)$ self-adjoint kernel such that $K(t,s)\in L_2[[0,+\infty)^2]$, $\mathrm{Re}\,(\lambda)>0$).

This equation is asymptotically mean square equivalent to the equation

$$dx = \Big(-\lambda x(t) + b(t)x(t) + \int\limits_{t_0}^{t} K(t,s)x(s)\,ds \Big)\,dt + D(t)x(t)\,dW(t),$$

provided $b(t) \in L_1[0,+\infty)$, $D(t) \in L_2[0,+\infty)$, $\operatorname{Re}(\lambda) \geq \sup\limits_{\nu\in\sigma(K)} Re(\nu)$.

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On the Optimization Problem for the Quasi-Linear Neutral Functional-Differential Equation with the Discontinuous Initial Condition

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In the paper, necessary conditions of optimality of the delay parameter containing in the phase coordinates, the initial vector and initial function, the control function are obtained for the quasi-linear neutral optimization problem with the discontinuous initial condition.

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$; let $I=[t_0,t_1]$ be a fixed interval and let $\sigma>0$, $\tau_2>\tau_1>0$ be given numbers,with $t_0+\max\{\sigma,\tau_2\}<t_1$. Suppose that $O\subset\mathbb{R}^n$, $U\subset\mathbb{R}^r$ are compact and convex sets. Further, the $n\times n$-dimensional matrix function $A(t,x)$ is continuous on the set $I\times O$ and continuously differentiable with respect to $x^i, i=1,2,\dots,n$; the n-dimensional function $f(t,x,y,u)$ is continuous on the set $I\times O^2\times U$ and continuously differentiable with respect to x, y, u. We denote by Φ and Ω the sets of continuously differentiable initial functions $\varphi(t)\in O$, $t\in I_1=[\widehat{\tau},t_0]$, where $\widehat{\tau}=t_0-\max\{\sigma,\tau_2\}$ and measurable control functions $u(t)\in U$, $t\in I$, respectively.

To each element

$$w=(\tau,x_0,\varphi(t),u(t))\in W=(\tau_1,\tau_2)\times O\times\Phi\times\Omega$$

we assign the quasi-linear controlled neutral functional-differential equation

$$\dot{x}(t)=A(t,x(t))\dot{x}(t-\sigma)+f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in I \tag{1}$$

with the discontinuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0),\quad x(t_0)=x_0. \tag{2}$$

The condition (2) is called the discontinuous initial condition because in general $\varphi(t_0)\neq x_0$. Discontinuity at the initial moment may be related to the instant change in a dynamical process (changes of investment, environment and so on).

Definition 1. Let $w\in W$, a function $x(t)=x(t;w)\in O$, $t\in[\widehat{\tau},t_1]$ is called a solution of equation (1) with condition (2) or a solution corresponding to the element w if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Let the scalar-valued functions $q^i(\tau, x_0, x)$, $i = 0, 1, \dots, l$, be continuously differentiable on $[\tau_1, \tau_2] \times O^2$.

Definition 2. An element $w = (\tau, x_0, \varphi(t), u(t)) \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$, satisfying the conditions

$$q^i(\tau, x_0, x(t_1)) = 0, \quad i = 1, 2, \dots, l. \tag{3}$$

By W_0 we denote the set of admissible elements.

Definition 3. An element $w_0 = (\tau_0, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$q^0(\tau_0, x_{00}, x_0(t_1)) \le q^0(\tau, x_0, x(t_1)) \tag{4}$$

holds, where $x_0(t) = x(t; w_0)$.

(1)–(4) is called the quasi-linear neutral optimization problem with the discontinuous initial condition.

Theorem 1. *Let $w_0 = (\tau_0, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ be an optimal element and let $x_0(t)$ be the corresponding solution, with*

$$t_0 + \tau_0 \notin \{t_1 - \sigma, t_1 - 2\sigma, \dots\}$$

and the function $u_0(t)$ is continuous at the point $t_0 + \tau_0$. Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, with $\pi_0 \le 0$, and a solution $(\chi(t), \psi(t))$ of the system

$$\begin{cases} \dot\chi(t) = -\psi(t)\Big\{\dfrac{\partial}{\partial x}\big[A[t]\dot x_0(t-\sigma)\big] + f_x[t]\Big\} - \psi(t+\tau_0) f_y[t+\tau_0], \\ \psi(t) = \chi(t) + \psi(t+\sigma)A[t+\sigma], \quad t \in I \end{cases}$$

with the initial condition

$$\chi(t_1) = \psi(t_1) = \pi Q_{0x}, \quad \chi(t) = \psi(t) = 0, \quad t > t_1,$$

where

$$Q = (q^0, q^1, \dots, q^l)^T, \quad Q_{0x} = \frac{\partial Q(\tau_0, x_{00}, x_0(t_1))}{\partial x},$$

$$\frac{\partial}{\partial x}\Big[A[t]\dot x_0(t-\sigma)\Big] = \frac{\partial}{\partial x}\big[A(t,x)\dot x_0(t-\sigma)\big]_{x=x_0(t)}, \quad A[t] = A(t, x_0(t)),$$

$$f_y[t] = f_y\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big),$$

such that the following conditions hold:

– *the condition for the delay τ_0*

$$\pi Q_{0\tau} = \psi(t_0+\tau_0)\Big[f\big(t_0+\tau_0, x_0(t_0+\tau_0), x_{00}, u_0(t_0+\tau_0)\big) - f\big(t_0+\tau_0, x_0(t_0+\tau_0), \varphi_0(t_0), u_0(t_0+\tau_0)\big)\Big] + \int\limits_{t_0}^{t_1} \psi(t) f_y[t]\dot x_0(t-\tau_0)\,dt;$$

– *the condition for the initial vector x_{00}*

$$\big(\pi Q_{0x_0} + \chi(t_0)\big)x_{00} = \max_{x_0 \in O}\big(\pi Q_{0x_0} + \chi(t_0)\big)x_0;$$

– *the condition for the initial function* $\varphi_0(t)$

$$\int\limits_{t_0-\sigma}^{t_0}\psi(t+\sigma)A[t+\sigma]\dot\varphi_0(t)\,dt+\int\limits_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)f_y[t+\tau_0]\varphi_0(t)\,dt$$
$$=\max_{\varphi(t)\in\Phi}\int\limits_{t_0-\sigma}^{t_0}\psi(t+\sigma)A[t+\sigma]\dot\varphi(t)\,dt+\int\limits_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)f_y[t+\tau_0]\varphi(t)\,dt;$$

– *the condition for the control function* $u_0(t)$

$$\int\limits_{t_0}^{t_1}\psi(t)f_u[t]u_0(t)\,dt=\max_{u(t)\in\Omega}\int\limits_{t_0}^{t_1}\psi(t)f_u[t]u(t)\,dt.$$

Theorem 1 is proved by the scheme given in [2] on the basis of the representation formula of a solution [1]. The case when $A(t,x)=A(t)$ and Q does not depend on the parameter τ is considered in [3]. Now we consider a particular case of the problem (1)–(4):

$$\dot x(t)=A(t)\dot x(t-\sigma)+B(t)x(t)+C(t)x(t-\tau)+D(t)u(t),\ \ t\in I, \tag{5}$$
$$x(t)=\varphi(t),\quad x(t_0)=x_0, \tag{6}$$
$$q^i(\tau,x(t_1))=0,\ \ i=1,2,\dots,l, \tag{7}$$
$$q^0(\tau,x(t_1))\to\min. \tag{8}$$

Here $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are the continuous matrix functions with dimensions $n\times n$ and $n\times r$, respectively; $\varphi(t)$ is a fixed initial function; x_0 is a fixed initial vector. In this case we have $w=(\tau,u(t))\in W=(\tau_1,\tau_2)\times\Omega$ and $w_0=(\tau_0,u_0(t))$;

$$Q(\tau,x)=(q^0(\tau,x),\dots,q^l(\tau,x))^T,\quad Q_{0x}=\frac{\partial Q(\tau_0,x_0(t_1))}{\partial x}\,.$$

Theorem 2. *Let* $w_0=(\tau_0,u_0(t))$ *be an optimal element for problem* (5)–(8) *and let* $x_0(t)$ *be the corresponding solution, with*

$$t_0+\tau_0\notin\{t_1-\sigma,t_1-2\sigma,\dots\}$$

and the function $u_0(t)$ *is continuous at the point* $t_0+\tau_0$. *Then there exist a vector* $\pi=(\pi_0,\dots,\pi_l)\neq 0$ *with* $\pi_0\le 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system*

$$\begin{cases}\dot\chi(t)=-\psi(t)B(t)-\psi(t+\tau_0)C(t+\tau_0),\\ \psi(t)=\chi(t)+\psi(t+\sigma)A(t+\sigma),t\in I\end{cases}$$

with the initial condition

$$\chi(t_1)=\psi(t_1)=\pi Q_{0x},\quad \chi(t)=\psi(t)=0,\ \ t>t_1,$$

such that the following conditions hold:

– *the condition for the delay* τ_0

$$\pi Q_{0\tau}=\psi(t_0+\tau_0)C(t_0+\tau_0)[x_0-\varphi(t_0)]+\int\limits_{t_0}^{t_1}\psi(t)C(t)\dot x_0(t-\tau_0)\,dt;$$

– *the condition for the control function* $u_0(t)$

$$\int_{t_0}^{t_1} \psi(t)D(t)u_0(t)\,dt = \max_{u(t)\in\Omega} \int_{t_0}^{t_1} \psi(t)D(t)u(t)\,dt.$$

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On a Control-Volume Model for Fiber Coating

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1 Introduction

The study of liquid films flowing down vertical fibers is a topic of significant interest in fluid dynamics due to its wide range of industrial applications, such as in heat exchangers, desalination processes, and fiber coating technologies, one can find a review of industrial applications in [1]. The dynamics of such films are influenced by various factors including gravity, viscosity, surface tension, and the geometry of the fiber. This introduction aims to provide an overview of the current research in this area, highlighting key contributions from recent and well-cited studies.

Ruan et al. (2021) presented a comprehensive framework using control-volume methods to model liquid films on vertical fibers. Their work included both numerical simulations and experimental validations, demonstrating the formation of traveling wave solutions and droplet patterns on fibers [8]. Similarly, Kalliadasis and Chang (2020) focused on direct numerical simulations of thin film flows, employing domain mapping techniques to solve the Navier–Stokes equations [4].

Quere (2003) offered an extensive review of the fluid dynamics involved in coating fibers, highlighting various flow regimes observed in experiments [7]. Kalliadasis and Chang (1994) provided a detailed analysis of droplet formation processes during the coating of vertical fibers, using both experimental and theoretical approaches to elucidate the underlying mechanisms [4]. This foundational work is supported by later studies such as those by Ji and Witelski (2017), who examined the three-dimensional dynamics of thin liquid films under various conditions, revealing complex behaviours observed in experiments [3].

2 Setting of the problem and the main results

Following the framework of Ruan et al. [2, 8] which is based on a control-volume approach, we expresses the conservation of mass and axial momentum via a coupled system for the fluid film radius $h(x,t)$ and the mean axial velocity $u(x,t)$:

$$u_t + a\Big(\frac{u^2}{2}\Big)_x + b\kappa_x = c\,\frac{[(h^2-1)u_x]_x}{h^2-1} + 1 - \frac{u}{g(h)} \ \text{ in } \Omega_T, \tag{2.1}$$

$$2hh_t + a[u(h^2-1)]_x = 0 \ \text{ in } \Omega_T, \tag{2.2}$$

$$u = h_x = 0 \ \text{ on } \partial\Omega \times (0,T), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad h(x,0) = h_0(x), \tag{2.4}$$

where $\Omega \subset \mathbb{R}^1$ is an open interval, $\Omega_T := \Omega \times (0,T)$. Here, the dimensionless parameter a represents the square of the Froude number, b is the reciprocal of the Bond number, c represents the ratio of axial viscous to gravitational forces, and $g(h) = h^2 - 1$ represents the axial velocity profile. The film thickness is given by $h(x,t) - 1$, and κ represents the combined azimuthal and streamwise curvatures of the free surface:

$$\kappa = \Big(1 - \frac{1}{2}\, h_x^2\Big) h^{-1} - h_{xx}.$$

Time evolution of solutions for thin liquid film models with full and approximated curvature terms was compared for example in [6], where the authors showed that the qualitative behaviour of solutions (with periodic boundary conditions) is almost the same.

Let us denote by

$$v = h^2 - 1.$$

Then we can rewrite (2.1) and (2.2) in the following form:

$$u_t + a\Big(\frac{u^2}{2}\Big)_x + b\kappa_x = c\,\frac{(v\,u_x)_x}{v} + 1 - \frac{u}{v} \ \text{ in } \Omega_T, \tag{2.5}$$

$$v_t + a(uv)_x = 0 \ \text{ in } \Omega_T. \tag{2.6}$$

Integrating (2.6) in Ω_t, we find that $v(x,t)$ satisfies

$$\int\limits_{\Omega} v(x,t)\,dx = \int\limits_{\Omega} v_0(x)\,dx := M > 0 \ \ \forall\, t \geqslant 0. \tag{2.7}$$

Furthermore, we assume that the initial data (v_0, u_0) satisfy

$$\begin{gathered} h_0 \geqslant 1, \ \text{ i.e. } \ v_0 := h_0^2 - 1 \geqslant 0 \ \text{ for all } \ x \in \overline{\Omega}, \\ \sqrt{v_0} \in H^1(\Omega), \ \ h_0 h_{0,x}^2, v_0 u_0^2, -\log(v_0) \in L^1(\Omega). \end{gathered} \tag{2.8}$$

Definition 2.1. A pair (h,u) is a weak solution to (2.5), (2.6) with the boundary conditions (2.3) and the initial conditions (h_0, u_0) if $1 \leqslant h \in C(\overline{Q}_T)$, $v = h^2 - 1$, and u satisfy the regularity properties

$$\begin{gathered} \sqrt{v} \in L^\infty(0,T; H^1(\Omega)), \ \ -\log(v), vu^2 \in L^\infty(0,T; L^1(\Omega)), \\ hh_x^2 \in L^\infty(0,T; L^1(\Omega)), \ \ h^{-\frac{1}{4}} h_x \in L^4(\Omega_T), \\ \sqrt{h}\, h_{xx}, \chi_{\{v>0\}} \sqrt{v}\, u_x, u \in L^2(\Omega_T), \end{gathered}$$

and the following holds

$$\iint\limits_{\Omega_T} v\phi_t \,dx\,dt + \int\limits_{\Omega} v_0 \phi(x,0)\,dx + a \iint\limits_{\Omega_T} uv\phi_x \,dx\,dt = 0,$$

$$\begin{aligned} \iint\limits_{\Omega_T} uv\psi_t \,dx\,dt + \int\limits_{\Omega} u_0 v_0 \psi(x,0)\,dx + \frac{a}{2} \iint\limits_{\Omega_T} \chi_{\{v>0\}} vu^2 \psi_x \,dx\,dt \\ + 2b \iint\limits_{\Omega_T} \Big(1 - \frac{1}{2}\, h_x^2 - hh_{xx}\Big) h_x \psi \,dx\,dt + b \iint\limits_{\Omega_T} \Big(\Big(1 - \frac{1}{2}\, h_x^2\Big) h^{-1} - h_{xx}\Big) v\psi_x \,dx\,dt \\ - c \iint\limits_{\Omega_T} \chi_{\{v>0\}} vu_x \psi_x \,dx\,dt + \iint\limits_{\Omega_T} (v-u)\psi \,dx\,dt = 0 \end{aligned}$$

for all $\phi \in C_c^\infty(\overline{\Omega}_T)$ and $\psi \in C_c^\infty(\overline{\Omega}_T)$ such that $\phi(x,T) = \psi(x,T) = 0$.

We note that the set $v = 0$ coincides with the set $h = 1$. Based on Definition 2.1, we will establish the existence of weak solutions to the problem and prove the following theorem.

Theorem. *Let the initial data* (h_0, u_0) *satisfy* (2.7), (2.8) *and* $T > 0$. *Then there exists a weak solution* (h, u) *in the sense of Definition 2.1, where* $v = h^2 - 1$. *Moreover, the set* $\{v(\cdot, t) = 0\}$ *has Lebesgue measure zero for any* $t \in [0, T]$.

The proof of Theorem is based on the method of energy-entropy a priori estimates which was active developed in [5, 9].

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On the Solvability of a Family of Nonlinear Two-Point Boundary Value Problems for Systems of Integro-Differential Equations and its Application

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1 Introduction

Conditions for the existence of solutions of systems of integro-differential equations and boundary-value problems for these systems are considered in [8, 9]. A general theory for these systems is developed and effective methods for solving them are proposed. In this paper, we study the x-parametric family of nonlinear boundary-value problems for integro-differential equations

$$\frac{\partial V}{\partial t} = f\bigg(x, t, \psi(t) + \int\limits_0^x V(\xi, t)\, d\xi, V\bigg), \ \ V \in \mathbb{R}^n, \ \ (x, t) \in [0, \omega] \times [0, T], \tag{1.1}$$

$$g\big(x, V(x, 0), V(x, T)\big) = 0, \ \ x \in [0, \omega], \tag{1.2}$$

where the functions $f : [0, \omega] \times [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $g : [0, \omega] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

Our study is based on the parametrization method [10, 12] proposed by Dzhumabaev. The parametrization method was developed to various boundary-value problems for some types of differential equations [2, 13, 15], such that Fredholm integro-differential equations [3–7, 17], delay differential equations [16], hyperbolic equations [1, 14] etc. The application of this method made it possible to derive the solvability conditions for the above problems.

This approach can also be applied to study the nonlinear nonlocal boundary value problem for the system of partial differential equations ($m = 1, 2, \dots$)

$$\frac{\partial^{m+1} u}{\partial t \partial x^m} = f\Big(x, t, \frac{\partial^{m-1} u}{\partial x^{m-1}}, \frac{\partial^m u}{\partial x^m}\Big), \ \ u \in \mathbb{R}^n, \ \ (x, t) \in [0, \omega] \times (0, T),$$
$$\frac{\partial^k u}{\partial x^k}\Big|_{x=0} = \psi_k(t), \ \ t \in [0, T], \ \ k = 0, 1, \dots, m-1,$$
$$g\bigg(x, \frac{\partial^m u(x,t)}{\partial x^m}\Big|_{t=0}, \frac{\partial^m u(x,t)}{\partial x^m}\Big|_{t=T}\bigg) = 0.$$

Therefore, the study of problem (1.1), (1.2) is of interest from the point of view of its application to nonlocal boundary value problems for a class of partial differential equations.

In the present paper, we propose a modified algorithm of the parametrization method for finding an isolated solution of problem (1.1), (1.2) and derive sufficient conditions for the existence of such a solution.

2 Setting of the problem and the main results

We consider the nonlinear nonlocal boundary value problem (1.1), (1.2) for the x-parametric family of integro-differential equations.

A solution of problem (1.1), (1.2) is a function $V(x,t) \in C([0,\omega] \times [0,T]\mathbb{R}^n)$, which is continuously differentiable on $[0,T]$ (at fixed $x \in [0,\omega]$) and satisfies the system of integro-differential equations (1.1) and the boundary conditions (1.2).

We take $h > 0$, $Nh = T$ ($N \in \mathbb{N}$), and make the partition $[0,\omega] \times [0,T) = \bigcup\limits_{r=1}^{N} \Omega_r$, where $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $r = \overline{1,N}$.

We will use the following notations.

- $C([0,\omega], \mathbb{R}^{n(N+1)})$ is the space of systems of functions $\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_{N+1}(x))$ with the norm

$$\|\lambda\|_1 = \max_{x\in[0,\omega]} \max_{r=\overline{1,(N+1)}} \|\lambda_r(x)\|,$$

  here the functions $\lambda_r : [0,\omega] \to \mathbb{R}^n$ are continuous, $r = \overline{1,(N+1)}$;

- $C([0,\omega] \times [0,T], \Omega_r, R^{nN})$ is the space of systems of functions

$$V[x,t] = \big(V_1(x,t), V_2(x,t), \ldots, V_N(x,t)\big)$$

  with the norm

$$\big\|V[\,\cdot\,]\big\|_2 = \max_{r=\overline{1,N}} \max_{x\in[0,\omega]} \sup_{t\in[t_{r-1},t_r)} \|V_r(x,t)\|,$$

  where the functions $V_r(x,t) \in C(\Omega_r)$ have finite limits $\lim\limits_{t\to t_r-0} V_r(x,t)$ uniform in x, $x \in [0,\omega]$ ($r = \overline{1,N}$);

The restriction of the function $V(x,t)$ into Ω_r is denoted by $V_r(x,t)$, i.e. $V_r(x,t) = V(x,t)$, $(x,t) \in \Omega_r$, $r = \overline{1,N}$.

Let us set additional parameters

$$\lambda_r(x) = V_r(x,(r-1)h), \quad r = \overline{1,N}$$

and

$$\lambda_{N+1}(x) = \lim_{t\to T-0} V_N(x,t), \quad x \in [0,\omega],$$

and introduce the functions

$$\widetilde{V}_r(x,t) = V_r(x,t) - \lambda_r(x) \ \text{ on } \Omega_r, \quad r = \overline{1,N}.$$

We then obtain the family of multipoint nonlinear boundary value problems for integro-differential equations with parameters

$$\frac{\partial \widetilde{V}_r}{\partial t} = f\bigg(x,t,\psi(t) + \int_0^x \lambda_r(\xi)\,d\xi + \int_0^x \widetilde{V}_r(\xi,t)\,d\xi, \lambda_r(x) + \widetilde{V}_r\bigg), \quad (x,t) \in \overline{\Omega}_r, \quad r = \overline{1,N}, \tag{2.1}$$

$$\widetilde{V}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \tag{2.2}$$

$$g\big(x, \lambda_1(x), \lambda_{N+1}(x)\big) = 0, \quad x \in [0,\omega], \tag{2.3}$$

$$\lambda_r(x) + \lim_{t\to rh-0} \widetilde{V}_r(x,t) - \lambda_{r+1}(x) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}. \tag{2.4}$$

It can be easily shown that the families of problems (1.1), (1.2) and (2.1)–(2.4) are equivalent.

Suppose that for all $r = \overline{1, N+1}$ and $x \in [0,\omega]$ the family of parameters $\lambda_r(x)$ is known. Then the functions $\widetilde{V}_r(x,t)$, $(x,t) \in \Omega_r$ $(r = \overline{1,N})$, can be determined from the Cauchy problem (2.1), (2.2). For a fixed $x \in [0,\omega]$, this problem is equivalent to the family of mixed type systems of integral equations

$$\widetilde{V}_r(x,t) = \int\limits_{(r-1)h}^{t} f\bigg(x, \tau, \psi(t) + \int\limits_0^x \lambda_r(\xi)\,d\xi + \int\limits_0^x \widetilde{V}_r(\xi,\tau)\,d\xi, \lambda_r(x) + \widetilde{V}_r(x,\tau)\bigg)\,d\tau, \tag{2.5}$$

$$t \in [(r-1)h, rh], \ \ r = \overline{1,N}.$$

By substituting the values of $\lim\limits_{t\to rh-0} \widetilde{V}_r(x,t)$, found from (2.5), into (2.3) and (2.4), we obtain

$$g\big(x, \lambda_1(x), \lambda_{N+1}(x)\big) = 0,$$

$$\lambda_r(x) + \int\limits_{(r-1)h}^{rh} f\bigg(x, t, \psi(t) + \int\limits_0^x \big(\lambda_r(\xi) + \widetilde{V}_r(\xi,\tau)\big)\,d\xi, \lambda_r(x) + \widetilde{V}_r(x,t)\bigg)\,dt - \lambda_{r+1}(x) = 0.$$

This is a system of nonlinear functional equations in parameters $\lambda_r(x)$, $x \in [0,\omega]$, $r = \overline{1,N+1}$. We rewrite this system in the form

$$Q_{1,h}\bigg(x, \lambda(x), \int\limits_0^x \lambda(\xi)\,d\xi, \widetilde{V}\bigg) = 0, \ \ \lambda(x) \in \mathbb{R}^{n(N+1)}, \ \ x \in [0,\omega]. \tag{2.6}$$

Condition 2.1. There exists $h > 0 : Nh = T$ $(N \in \mathbb{N})$, such that the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V} = 0$, has a solution $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), \dots, \lambda_{N+1}^{(0)}(x)) \in C([0,\omega], \mathbb{R}^{n(N+1)})$.

Let Condition 2.1 be met. We denote the solution of the Cauchy problem (2.1), (2.2), corresponding to $\lambda_r(x) = \lambda_r^{(0)}(x)$, by $\widetilde{V}_r^{(0)}(x,t)$. Let us define the function

$$V^{(0)}(x,t) = \begin{cases} \lambda_r^{(0)}(x) + \widetilde{V}_r^{(0)}(x,t) & \text{for } \ (x,t) \in \Omega_r, \ \ r = \overline{1,N}, \\ \lambda_{N+1}^{(0)}(x) & \text{for } \ (x,t) \in [0,\omega] \cup \{T\}. \end{cases}$$

We choose some numbers $\rho_\lambda > 0$, $\rho_{\widetilde{V}} > 0$, $\rho_V > 0$ and define the following sets:

$$S(\lambda^{(0)}(x), \rho_\lambda) = \Big\{\lambda(x) \in C([0,\omega], \mathbb{R}^{n(N+1)}) : \ \ \|\lambda - \lambda^{(0)}\|_1 < \rho_\lambda\Big\},$$

$$S(\widetilde{V}^{(0)}(x,[t]), \rho_{\widetilde{V}}) = \Big\{\widetilde{V}(x,[t]) \in C(\overline{\Omega}, \Omega_r, \mathbb{R}^{nN}) : \ \ \big\|(\widetilde{V} - \widetilde{V}^{(0)})[\,\cdot\,]\big\|_2 < \rho_{\widetilde{V}}\Big\},$$

$$S(V^{(0)}(x,t), \rho_V) = \Big\{V(x,t) \in C(\overline{\Omega}, \mathbb{R}^n) : \ \ \max_{(x,t)\in\overline{\Omega}} \|V(x,t) - V^{(0)}(x,t)\| < \rho_V\Big\},$$

$$G_f(x, \rho_V) = \Big\{(x,t,u,v) \in \overline{\Omega} \times \mathbb{R}^{2n} : \ \ (x,t) \in \overline{\Omega}, \ \ \|u - u^{(0)}(x,t)\| < \omega \cdot \rho_V, \ \ \|v - v^{(0)}(x,t)\| < \rho_V\Big\},$$

$$G_g(x, \rho_\lambda) = \Big\{(x, w_1, w_2) \in [0,\omega] \times \mathbb{R}^{2n} : \ \ \|w_1 - V^{(0)}(x,0)\| < \rho_\lambda, \ \ \|w_2 - V^{(0)}(x,T)\| < \rho_\lambda\Big\}.$$

Condition 2.2. The function $f(x,t,u,v)$ has uniformly continuous partial derivatives f'_u, f'_v in $G_f(x, \rho_u, \rho_v)$ and the following inequalities hold:

$$\|f'_u(x,t,u,v)\| \le L_1, \quad \|f'_v(x,t,u,v)\| \le L_2 \ \ \forall\,(x,t,u,v) \in G_f(x, \rho_u, \rho_v).$$

The function $g(x, w_1, w_2)$ has uniformly continuous partial derivatives g'_{w_1}, g'_{w_2} in $G_g(x, \rho_\lambda)$ and the following inequalities hold:

$$\|g'_{w_1}(x, w_1, w_2)\| \leq L_3, \quad \|g'_{w_2}(x, w_1, w_2)\| \leq L_4, \ \ (x, w_1, w_2) \in G_g(x, \rho_\lambda).$$

Here L_i $(i = \overline{1,4})$ are some constants.

Let Condition 2.2 be met. We take the pair $(\lambda^{(0)}(x), \widetilde{V}^{(0)}(x, [t]))$ and determine the sequence $(\lambda^{(k)}(x), \widetilde{V}^{(k)}(x, [t]))$, $k = 1, 2, \dots$, by the following algorithm.

Step 1.

(i) Find $\lambda^{(1)}(x) = (\lambda_1^{(1)}(x), \lambda_2^{(1)}(x), \dots, \lambda_{N+1}^{(1)}(x)) \in C([0, \omega], \mathbb{R}^{n(N+1)})$ by solving the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V} = \widetilde{V}^{(0)}$.

(ii) By solving the family of Cauchy problems (2.1), (2.2), where $\lambda(x) = \lambda^1(x)$, find the system of functions $\widetilde{V}^{(1)}(x, [t])$.

(iii) Define the function

$$V^{(1)}(x, t) = \begin{cases} \lambda_r^{(1)}(x) + \widetilde{V}_r^{(1)}(x, t) & \text{for } \ (x, t) \in \Omega_r, \ \ r = \overline{1, N}, \\ \lambda_{N+1}^{(1)}(x) & \text{for } \ (x, t) \in [0, \omega] \cup \{T\}. \end{cases}$$

Step k.

(i) Find $\lambda^{(k)}(x) = (\lambda_1^{(k)}(x), \lambda_2^{(k)}(x), \dots, \lambda_{N+1}^{(k)}(x)) \in C([0, \omega], \mathbb{R}^{n(N+1)})$ by solving the family of systems of implicit nonlinear Fredholm integral equations (2.6), where $\widetilde{V} = \widetilde{V}^{(k-1)}$.

(ii) By solving the family of Cauchy problems (2.1), (2.2), where $\lambda(x) = \lambda^2(x)$, find the system of functions $\widetilde{V}^{(2)}(x, [t])$.

(iii) Define the function

$$V^{(k)}(x, t) = \begin{cases} \lambda_r^{(k)}(x) + \widetilde{V}_r^{(k)}(x, t) & \text{for } \ (x, t) \in \Omega_r, \ \ r = \overline{1, N}, \\ \lambda_{N+1}^{(k)}(x) & \text{for } \ (x, t) \in [0, \omega] \cup \{T\}. \end{cases}$$

The following statement represents sufficient conditions for the existence of an isolated solution of the family of boundary value problems with parameters (2.1)–(2.4).

Theorem 2.1. *Let for some $h > 0 : Nh = T$ $(N = 1, 2, \dots)$, $\rho_\lambda > 0$, $\rho_{\widetilde{V}} > 0$, $\rho_V > 0$ fulfill Condition 2.1 and Condition 2.2 are met, the Jacobi matrix $\frac{\partial Q_{1,h}(x, \widetilde{w}_1, \widetilde{w}_2, \widetilde{V})}{\partial \widetilde{w}_1}$ has an inverse for $x \in [0, \omega]$ $(\widetilde{w}_1 = \lambda(x)$, $\widetilde{w}_2 = \int_0^x \lambda(\xi)\, d\xi)$ and for all $(\lambda(x), \widetilde{V}(x, [t])) \in S(\lambda^{(0)}(x), \rho_\lambda) \times S(\widetilde{V}^{(0)}(x, [t]), \rho_{\widetilde{V}})$, and let the following inequalities hold:*

(1) $$\left\| \left(\frac{\partial}{\partial \widetilde{w}_1} Q_{1,h}\left(x, \lambda(x), \int_0^x \lambda(\xi)\, d\xi, \widetilde{V}\right)\right)^{-1} \right\| \leq \gamma_1(h), \ x \in [0, \omega], \ \gamma_1(h) \ \text{– const;}$$

(2) $$q_1(h) = \gamma_1(h) e^{h \cdot \gamma_1(h) L_1 \omega} \, \frac{(L_1 \omega + L_2)^2 h^2}{1 - (L_1 \omega + L_2) h} < 1;$$

(3) $$\frac{\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\max_{x\in[0,\omega]}\Big\|Q_{1,h}\Big(x,\lambda^{(0)}(x),\int\limits_0^x\lambda^{(0)}(\xi)\,d\xi,\widetilde{V}^{(0)}\Big)\Big\|<\rho_\lambda;$$

(4) $$\frac{\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\cdot\frac{(L_1\omega+L_2)h}{1-(L_1\omega+L_2)h}\max_{x\in[0,\omega]}\Big\|Q_{1,h}\Big(x,\lambda^{(0)}(x),\int\limits_0^x\lambda^{(0)}(\xi)\,d\xi,\widetilde{V}^{(0)}\Big)\Big\|<\rho_{\widetilde{V}};$$

(5) $\rho_\lambda+\rho_{\widetilde{V}}<\rho_V$.

Then for any $x\in[0,\omega]$ the sequence of pairs

$$\big(\lambda^{(k)}(x),\widetilde{V}^{(k)}(x,[t])\big)\in S(\lambda^{(0)}(x),\rho_\lambda)\times S(\widetilde{V}^{(0)}(x,[t]),\rho_{\widetilde{V}})$$

converges to $(\lambda^*(x),\widetilde{V}^*(x,[t]))$; an isolated solution of problem (2.1)–(2.4) *in $S(\lambda^{(0)}(x),\rho_\lambda)\times S(\widetilde{V}^{(0)}(x,[t]),\rho_{\widetilde{V}})$. Moreover, the following estimates hold:*

$$\|\lambda^*-\lambda^{(0)}\|_1\le\frac{h\cdot\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\,\frac{(L_1\omega+L_2)h}{1-h(L_1\omega+L_2)}\max_{r=\overline{1,N}}\widetilde{K}_r,$$
$$\|\widetilde{V}^*-\widetilde{V}^{(0)}\|_2\le\frac{(L_1\omega+L_2)h}{1-(L_1\omega+L_2)h}\,\|\lambda^*-\lambda^{(0)}\|_1,$$

where

$$\widetilde{K}_r=\sup_{(x,t)\in\Omega_r}\Big\|f\Big(x,t,\int\limits_0^x\lambda_r^{(0)}(\xi)\,d\xi,\lambda_r^{(0)}(x)\Big)\Big\|,\ \ r=\overline{1,N}.$$

The proof of Theorem 2.1 is based on the sequential implementation of the steps of the proposed algorithm. To find the solution of the nonlinear operator equation with respect to the family of parameters for each fixed $x\in[0,\omega]$, a sharper version of the local Hadamard theorem [12, p. 41] is used.

Remark. The conditions of Theorem 2.1 are sufficient for the feasibility and convergence of the proposed algorithm.

Due to the equivalence of problems (2.1)–(2.4) and problems (1.1), (1.2), the following statement is true.

Theorem 2.2. *Let for some $h>0$: $Nh=T$ $(N=1,2,\dots)$, $\rho_\lambda>0$, $\rho_{\widetilde{V}}>0$ and $\rho_V>0$ all conditions of Theorem 2.1 are met. Then for any $x\in[0,\omega]$ the sequence of functions $V^{(k)}(x,t)\in S(V^{(0)}(x,t),\rho_V)$ converges to $V^*(x,t)$, an isolated solution of problem* (1.1), (1.2) *in $S(V^{(0)}(x,t),\rho_V)$ and the following estimate holds:*

$$\max_{(x,t)\in\overline{\Omega}}\big\|V^*(x,t)-V^{(0)}(x,t)\big\|\le\frac{h\cdot\gamma_1(h)}{1-q_1(h)}\,e^{h\cdot\gamma_1(h)L_1\omega}\,\frac{h\cdot(L_1\omega+L_2)}{(1-h(L_1\omega+L_2))^2}\cdot K,$$

where

$$K=\max_{r=\overline{1,N}}\sup_{(x,t)\in\Omega_r}\Big\|f\Big(x,t,\int\limits_0^x(V^{(0)}(\xi,t)-V^{(0)}(\xi,(r-1)h))\,d\xi,V^{(0)}(x,t)-V^{(0)}(x,(r-1)h)\Big)\Big\|.$$

Theorem 2.2 is a corollary of Theorem 2.1.

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Stability of Solutions of One-Dimensional Stochastic Differential Equations Controlled by Rough Paths

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Abstract

We prove theorems on the stability of solutions of one-dimensional stochastic differential equations controlled by rough paths with arbitrary positive Holder exponent.

1 Introduction

Consider the one-dimensional stochastic differential equation

$$dY_t = f(Y_t)dX_t, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where X_t is a random process whose paths are a.s. Holder continuous of order $\alpha \in (0, 1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function with bounded continuous derivatives of any order $m \in \{0, \dots, [1/\alpha] + 1\}$.

In the present paper, we prove that the conditions ensuring the existence and uniqueness of solutions of Eq. (1.1) also guarantee the continuous dependence of the solutions on the initial data on any finite interval; the Lyapunov stability of the zero solution of Eq. (1.1) is studied on the basis of the stability of the zero solution of the corresponding ordinary differential equation (ODE) $dZ_t = f(Z_t)dt$. Here a solution of Eq. (1.1) is understood as a solution of a stochastic differential equation weakly controlled by the corresponding rough path [1]. To define solutions, we need a number of notions introduced in the papers [1] and [2].

2 Definition of rough paths

Fix some $T > 0$ and $\alpha \in (0, 1]$. Let V be a finite-dimensional Euclidean space. By $C^\alpha([0, T], V)$ and $C_2^\alpha([0, T], V)$ we denote the sets of functions $f : [0, T] \to V$ and $g : [0, T]^2 \to V$, respectively, with finite norms

$$\|f\|_\alpha := \sup_{s,t \in [0,T],\, s \neq t} \frac{|f_t - f_s|}{|t - s|^\alpha},$$

$$\|g\|_{\alpha,2} := \sup_{s,t \in [0,T], s \neq t} \frac{|g_{s,t}|}{|t - s|^\alpha}.$$

Further, for a function of two variables $g_{s,t}$ we write $\|g\|_\alpha$ instead of $\|g\|_{\alpha,2}$. For a function f_t of one variable, by $f_{s,t}$ we denote the increment $f_t - f_s$.

For an integer non-negative k and finite-dimensional Euclidean spaces V and W, by $C_b^k(V, W)$ we denote the set of functions $h : V \to W$ with finite norm

$$\|h\|_{C_b^k} := \sum_{i=0}^{k} \|D^i h\|_\infty,$$

where

$$\|D^i h\|_\infty = \sup_{t\in[0,T]} |D^i h_t|.$$

Set $n = [1/\alpha]$. By $\mathscr{C}^\alpha([0,T],V)$ we denote the set of Holder α-continuous rough paths, i.e., the set of elements $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ such that $\mathbf{X}^i \in C_2^{i\alpha}([0,T], V^{\otimes i})$ for any $i = 1, \dots, n$, and any $s, u, t \in [0,T]$ there holds the Cheng identity

$$\mathbf{X}_{s,t} = \mathbf{X}_{s,u} \boxplus \mathbf{X}_{u,t},$$

where

$$\big(\mathbf{X}_{s,u} \boxplus \mathbf{X}_{u,t}\big)^i = \sum_{j=0}^{i} \mathbf{X}_{s,u}^j \otimes \mathbf{X}_{u,t}^{i-j}.$$

Note that the operation $\boxplus$ defines multiplication on the tensor algebra $T^{(n)}(V) = \bigoplus\limits_{i=0}^{n} V^{\otimes i}$, where $V^{\otimes 0} = \mathbb{R}$. Thus, an element $\mathbf{X} : [0,T]^2 \to T^{(n)}(V)$ is uniquely determined by the values $\mathbf{X}_{0,t}$, $t \in [0,T]$, because $\mathbf{X}_{s,t} = (\mathbf{X}_{0,s})^{-1} \boxplus \mathbf{X}_{0,t}$. In what follows, we write $\mathbf{X}_t$ instead of $\mathbf{X}_{0,t}$.

A rough path $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ is said to be geometric if

$$\operatorname{Sym}(\mathbf{X}_{s,t}^i) = \frac{1}{i!}\,(\mathbf{X}_{s,t}^1)^{\otimes i} \ \ \forall\, i = 1, \dots, n.$$

The set of geometric rough paths will be denoted by $\mathscr{C}_g^\alpha([0,T],V)$.

We say that an element $\mathbf{X} \in \mathscr{C}^\alpha([0,T],V)$ is a rough path over $X \in C^\alpha([0,T],V)$ if $\mathbf{X}_{0,t}^1 = X_t$ for any $t \in [0,T]$.

Definition of weakly controlled rough paths

Let $X \in C^\alpha([0,T],V)$ and let $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n)$ be a rough path over X. Let W be a finite-dimensional Euclidean space. We say that a function $Y_t \in C^\alpha([0,T],W)$ is weakly controlled by the rough path $\mathbf{X} \in \mathscr{C}^\alpha([0,T],V)$ if there exist functions $Y^{(1)} : [0,T] \to \mathcal{L}(V,W), \dots, Y^{(n-1)} : [0,T] \to \mathcal{L}(V^{\otimes(n-1)}, W)$ such that

$$\begin{aligned}
Y_{s,t} &= Y_s^{(1)}\mathbf{X}_{s,t}^1 + \dots + Y_s^{(n-1)}\mathbf{X}_{s,t}^{n-1} + R_{s,t}^{Y,n},\\
Y_{s,t}^{(1)} &= Y_s^{(2)}\mathbf{X}_{s,t}^1 + \dots + Y_s^{(n-1)}\mathbf{X}_{s,t}^{n-2} + R_{s,t}^{Y,n-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
Y_{s,t}^{(n-2)} &= Y_s^{(n-1)}\mathbf{X}_{s,t}^1 + R_{s,t}^{Y,2},\\
Y_{s,t}^{(n-1)} &= R_{s,t}^{Y,1};
\end{aligned}$$

and the norm $\|R^{Y,i}\|_{i\alpha}$ is finite for each of the remainder terms $R^{Y,i}$, $i = 1, \dots, n$. The function $Y^{(i)}$ will be called the i-th rough derivative of Y.

Define the Banach space

$$\mathcal{D}_{\mathbf{X}}^\alpha([0,T],W) = \Big\{(Y, Y^{(1)}, \dots, Y^{(n-1)}) : \ \ Y \in C^\alpha([0,T],W), \ \ \sum_{i=1}^{n} \|R^{Y,i}\|_{i\alpha} < \infty\Big\}$$

with the seminorm

$$\big\|(Y, Y^{(1)}, \dots, Y^{(n-1)})\big\|_{\mathcal{D}_{\mathbf{X}}^\alpha} = \sum_{i=1}^{n} \|R^{Y,i}\|_{i\alpha}.$$

The norm of an element $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)}) \in \mathcal{D}^{\alpha}_{\mathbf{X}}([0,T], W)$ is defined by the formula

$$\|\mathbf{Y}\|_{\mathcal{D}^{\alpha}_{\mathbf{X}}} := \sum_{i=0}^{n-1} |Y_0^{(i)}| + \big\|(Y, Y^{(1)}, \dots, Y^{(n-1)})\big\|_{\mathcal{D}^{\alpha}_{\mathbf{X}}},$$

where $Y_t^{(0)} = Y_t$.

3 Definition of the integral over rough paths

Let V and W be some finite-dimensional Euclidean spaces,

$$\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n) \in \mathscr{C}^{\alpha}([0,T], V), \quad Y \in C^{\alpha}([0,T], \mathcal{L}(V,W)),$$
$$(Y, Y^{(1)}, \dots, Y^{(n-1)}) \in \mathcal{D}^{\alpha}_{\mathbf{X}}([0,T], \mathcal{L}(V,W)).$$

Take some $s,t \in [0,T]$, $s < t$, and let $\mathcal{P}$ be an arbitrary finite partition of the interval $[s,t]$ by points.

The rough path integral $\int_s^t Y_r \, d\mathbf{X}_r$ is defined as the following limit of integral sums (if the limit exists, then it is finite and does not depend on the choice of partitions of the interval $[s,t]$ by points):

$$\int_s^t Y_r \, d\mathbf{X}_r := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \sum_{i=0}^{n-1} Y_u^{(i)} \mathbf{X}_{u,v}^{i+1}.$$

4 Definition of rough paths on a half-line

Let $X : \mathbb{R}_+ \to \mathbb{R}$; i.e., assume that for each $T > 0$ the restriction $X|_{[0,T]}$ belongs to the space $C^{\beta}([0,T], \mathbb{R})$, $\beta \in (\frac{1}{n+1}, \frac{1}{n}]$. For each $i \in \{1, \dots, n\}$ we define $\mathbf{X}^i_{s,t} = \frac{(X_{s,t})^i}{i!}$, $s,t \in \mathbb{R}_+$. The element $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n) \mathbb{R}^2_+ \to T^{(n)}(\mathbb{R})$ is called a geometric rough path over X. The set of geometric rough paths is denoted by $\mathscr{C}^{\beta}_g(\mathbb{R}_+, \mathbb{R})$.

We say that a function $Y \in C^{\alpha}(\mathbb{R}_+, \mathbb{R})$, $\frac{1}{n+1} < \alpha < \beta$, is weakly controlled by a geometric rough path $\mathbf{X} \in \mathscr{C}^{\beta}_g(\mathbb{R}_+, \mathbb{R})$ if there exist $Y^{(i)} : \mathbb{R}_+ \to \mathbb{R}$, $i \in \{1, \dots, n-1\}$, such that the $i\alpha$-Holder norms of $R^{Y,i}$, $i \in \{1, \dots, n\}$, are finite on each bounded segment $\mathbb{R}_+$. We say that a vector function $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ belongs to the set $\mathcal{D}^{\alpha}_{\mathbf{X}}(\mathbb{R}_+, \mathbb{R})$ if for each $T > 0$ its restriction $\mathbf{Y}|_{[0,T]}$ belongs to the space $\mathcal{D}^{\alpha}_{\mathbf{X}}([0,T], \mathbb{R})$.

5 Stochastic differential equations weakly controlled by rough paths with arbitrary positive Holder exponent

Suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ with a flow $(\mathcal{F}_t)_{t \geq 0}$ of σ-algebras are given an $\mathcal{F}_t$-adapted random process X_t, $t \in \mathbb{R}_+$, such that almost all trajectories of X_t belong to the space $C^{\beta}(\mathbb{R}_+, \mathbb{R})$, $\beta \in (\frac{1}{n+1}, \frac{1}{n}]$. Define a process $\mathbf{X}_\cdot = (1, \mathbf{X}^1_{0,\cdot}, \dots, \mathbf{X}^n_{0,\cdot})$ as a random variable a.s. taking values in $\mathscr{C}^{\beta}_g(\mathbb{R}_+, \mathbb{R})$ a.s., where $\mathbf{X}^i_{s,t} = \frac{(X_{s,t})^i}{i!}$.

Let $Y \in C^{\alpha}([0,T], \mathbb{R})$, $(Y, Y^{(1)}, Y^{(2)}, \dots, Y^{(n-1)}) \in \mathcal{D}^{\alpha}_{\mathbf{X}}([0,T], \mathbb{R})$; $f \in C^n_b(\mathbb{R}, \mathbb{R})$. Define $Z_t = f(Y_t)$. By analogy with the Faà di Bruno's formula, we set

$$Z^{(k)} = \sum_{j=1}^{k} D^j f(Y) B_{k,j}(Y^{(1)}, \dots, Y^{(k-j+1)}), \ \ k = 1, \dots, n-1, \tag{5.1}$$

where the $B_{k,j}(x_1,\dots,x_{k-j+1})$ – are Bell polynomials.

Consider the stochastic differential equation

$$dY_t = f(Y_t)d\mathbf{X}_t, \ \ t \in \mathbb{R}_+. \tag{5.2}$$

Definition 5.1. Let $\xi : \Omega \to \mathbb{R}$ be an $\mathcal{F}_0$-measurable random variable. A solution of Eq. (5.2) with the initial condition $Y_0 = \xi$ is an $\mathcal{F}$-measurable random variable $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ with values in $\mathcal{D}^{\alpha}_{\mathbf{X}}(\mathbb{R}_+, \mathbb{R})$ a.s., $\frac{1}{n+1} < \alpha < \beta$, such that the random process $\mathbf{Y}_t$ is $\mathcal{F}_t$-adapted and a.s. the equality

$$Y_t = \xi + \int\limits_0^t f(Y_s)\, d\mathbf{X}_s$$

holds for all $t \in \mathbb{R}_+$, where the rough derivatives of the function $f(Y)$, occurring in the definition of the integral on the right-hand side, are determined by formulas (5.1). A solution of Eq. (5.2) with the initial condition $Y_0 = \xi$ is said to be unique if for two arbitrary solutions $\mathbf{Y}$, $\overline{\mathbf{Y}}$ of Eq. (5.2) with the initial condition $Y_0 = \xi$ one has the equality $P(\mathbf{Y} = \overline{\mathbf{Y}}) = 1$.

Consider the ODE

$$dZ_t = f(Z_t)dt, \ \ t \in \mathbb{R}. \tag{5.3}$$

Let $S_t = e^{tV_f}$, $t \in \mathbb{R}$, be the flow generated by Eq. (5.3), i.e., $Z_t = S_t Z_0$, where the operator $V_f : C(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ acts according to the rule $(V_f g)_t = f(g_t)$.

The following assertion was proved in [1].

Proposition 5.1 ([1])**.** *Let $\alpha, \beta \in (\frac{1}{n+1}, \frac{1}{n}]$, $\alpha < \beta$, $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n) \in \mathscr{C}^{\beta}_g(\mathbb{R}_+, \mathbb{R})$ a.s. If $f \in C^{n+1}_b(\mathbb{R}, \mathbb{R})$, then for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$ there exists a unique solution $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ of Eq. (1.1) with the initial condition $Y_0 = \xi$, and a.s. one has*

$$Y_t = S_{X_{0,t}}\xi, \ \ Y^{(i)}_t = D^{i-1}_f f(Y_t), \ \ i \in \{1, \dots, n-1\}, \ \ t \in \mathbb{R}_+.$$

6 Continuous dependence of solutions on the initial data

Along with Eq. (5.2), consider the perturbed equation

$$dY_t = \widetilde{f}(Y_t)d\mathbf{X}_t, \ \ t \in \mathbb{R}_+. \tag{6.1}$$

Theorem 6.1. *Let $\alpha, \beta \in (\frac{1}{n+1}, \frac{1}{n}]$, $\alpha < \beta$, $p \geq 1$, $T > 0$, $\mathbf{X} = (1, \mathbf{X}^1, \dots, \mathbf{X}^n) \in \mathscr{C}^{\beta}_g(\mathbb{R}_+, \mathbb{R})$ a.s., $\xi : \Omega \to \mathbb{R}$ be a $\mathcal{F}_0$-measurable random variable; $f \in C^{n+1}_b(\mathbb{R}, \mathbb{R})$. If $\mathbb{E}\|X\|^p_{\alpha,[0,T]} < \infty$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, T)$ such that for any $\widetilde{f} \in C^{n+1}_b(\mathbb{R}, \mathbb{R})$ and $\mathcal{F}_0$-measurable random variable $\widetilde{\xi} : \Omega \to \mathbb{R}$ such that*

$$\|\widetilde{f} - f\|_{C^{n+1}_b} + \mathbb{E}|\widetilde{\xi} - \xi|^p \leq \delta,$$

there holds the inequality

$$\sum_{i=0}^{n-1} \mathbb{E}\|\widetilde{Y}^{(i)} - Y^{(i)}\|^p_{\alpha,[0,T]} \leq \varepsilon,$$

where $\mathbf{Y} = (Y, Y^{(1)}, \dots, Y^{(n-1)})$ is a solution of Eq. (5.2) with the initial condition $Y_0 = \xi$ and $\widetilde{\mathbf{Y}} = (\widetilde{Y}, \widetilde{Y}^{(1)}, \dots, \widetilde{Y}^{(n-1)})$ is a solution of Eq. (6.1) with the initial condition $Y_0 = \widetilde{\xi}$.

Proof. Assume that the assertion in the theorem does not hold; i.e., there exists an $\varepsilon_0 > 0$ such that for any $\delta_k = \frac{1}{k}$, $k \in \mathbb{N}$, there exist $f_k \in C_b^{n+1}(\mathbb{R},\mathbb{R})$ and $\mathcal{F}_0$-measurable random variables $\xi_k : \Omega \to \mathbb{R}$ such that

$$\|f_k - f\|_{C_b^{n+1}} + \mathbb{E}|\xi_k - \xi|^p \le \delta_k,$$
$$\sum_{i=0}^{n-1} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|^p_{\alpha,[0,T]} \ge \varepsilon_0,$$

where $\mathbf{Y}_k = (Y_k, Y_k^{(1)}, \dots, Y_k^{(n-1)})$ is a solution of equation

$$dY_t = f_k(Y_t)d\mathbf{X}_t, \;\; t \in \mathbb{R}_+,$$

with the initial condition

$$\mathbf{Y}_0 = \big(\xi_k, f_k(\xi_k), Df_k(\xi_k)f_k(\xi_k), \dots\big).$$

Let $S_{k,t} = e^{tV_{f_k}}$ be the flow corresponding to the equation $dZ_t = f_k(Z_t)dt$. By Proposition 1 we get

$$Y_t = S_{X_t - X_0}\xi, \quad Y_t^{(i)} = D_f^{i-1}f(Y_t),$$
$$Y_{k,t} = S_{k,X_t - X_0}\xi_k, \quad Y_{k,t}^{(i)} = D_{f_k}^{i-1}f_k(Y_{k,t}).$$

Without loss of generality we may assume that $X_0 = 0$. Set $g(\tau) = S_\tau\xi$, $g_k(\tau) = S_{k,\tau}\xi_k$, $\psi_k(\tau) = g_k(\tau) - g(\tau)$, $\tau \in \mathbb{R}$. Thus,

$$\|Y_k - Y\|_{\alpha,[0,T]} = \sup_{s\ne t} \frac{|\psi_k(X_t) - \psi_k(X_s)|}{|t-s|^\alpha}$$
$$= \sup_{s\ne t} \frac{|(X_t - X_s)D\psi_k(X_s + \theta_k(X_t - X_s))|}{|t-s|^\alpha} \le \|X\|_{\alpha,[0,T]}\|D\psi_k\|_\infty.$$

Since $\mathbb{E}\|X\|^p_{\alpha,[0,T]} < \infty$, we have

$$\lim_{k\to\infty} \mathbb{E}\|Y_k - Y\|^p_{\alpha,[0,T]} = 0.$$

Take arbitrary $i \in \{1, \dots, n-1\}$. Denote

$$h(y) = D_f^{i-1}f(y), \;\; h_k(y) = D_{f_k}^{i-1}f_k(y), \;\; \varphi_k(y) = h_k(y) - h(y), \;\; y \in \mathbb{R}.$$

Then

$$\|Y_k^{(i)} - Y^{(i)}\|_{\alpha,[0,T]} = \sup_{s\ne t} \frac{|h_k(Y_{k,t}) - h_k(Y_{k,s}) - h(Y_t) + h(Y_s)|}{|t-s|^\alpha}$$
$$= \sup_{s\ne t} \frac{|(Y_{k,t} - Y_{k,s})D\varphi_k(Y_{k,s} + \theta_k(Y_{k,t} - Y_{k,s}))|}{|t-s|^\alpha} + \sup_{s\ne t} \frac{|h(Y_{k,t}) - h(Y_{k,s}) - h(Y_t) + h(Y_s)|}{|t-s|^\alpha}$$
$$\le \|Y_k - Y\|_{\alpha,[0,T]}\|D\varphi_k\|_\infty + \|Dh\|_\infty\|Y_k - Y\|_{\alpha,[0,T]} + C\|D^2h\|_\infty\big(\|Y_k - Y\|_{\alpha,[0,T]} + |\xi_k - \xi|\big).$$

Hence,

$$\lim_{k\to\infty} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|^p_{\alpha,[0,T]} = 0.$$

Therefore,

$$\lim_{k\to\infty} \sum_{i=0}^{n-1} \mathbb{E}\|Y_k^{(i)} - Y^{(i)}\|_{\alpha,[0,T]} = 0.$$

The resulting contradiction completes the proof of the theorem. □

7 Lyapunov stability of solutions on the half-line

Let us proceed to the stability analysis of the zero solution of Eq. (5.2) under the assumption that $f(0)=0$. Additionally, we assume that the function $f\in C^{n+1}(\mathbb{R},\mathbb{R})$ is such that no solution Z_t, $t\geq 0$, of Eq. (5.3) has blow-ups. In what follows, the zero solution of Eq. (5.2) is understood as the solution $\mathbf{Y}\equiv 0$ of Eq. (5.2) with the zero initial condition $Y_0=0$.

Definition 7.1. We say that the zero solution of Eq. (5.2) is stable in probability if for any ε_1, $\varepsilon_2>0$ there exists $\delta=\delta(\varepsilon_1,\varepsilon_2)>0$ such that for each $\mathcal{F}_0$-measurable random variable $\xi:\Omega\to\mathbb{R}$, $|\xi|\leq\delta$ a.s., there holds the inequality

$$P\Big(\sup_{t\geq 0}|Y_t|\geq\varepsilon_1\Big)\leq\varepsilon_2,$$

where $\mathbf{Y}=(Y,Y^{(1)},\dots,Y^{(n-1)})$ is the solution of Eq. (5.2) with the initial condition $Y_0=\xi$. We say that the zero solution of Eq. (5.2) is asymptotically stable in probability if it is stable in probability and there exists a $\Delta>0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi:\Omega\to\mathbb{R}$, $|\xi|\leq\Delta$ a.s., one has the convergence in probability $Y_t\xrightarrow[t\to+\infty]{P}0$. Let $p\geq 1$; we say that the zero solution of Eq. (5.2) is p-stable if for each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi:\Omega\to\mathbb{R}$, $|\xi|\leq\delta$ a.s., there holds the inequality $\sup\limits_{t\geq 0}\mathbb{E}|Y_t|^p\leq\varepsilon$.

Theorem 7.1. *Let $X_t\xrightarrow[t\to+\infty]{P}+\infty$ and let the expectation $\mathbb{E}\big(\sup\limits_{t\in[0,T]}|X_t|\big)$ is finite for each $T>0$. If the zero solution of Eq.* (5.3) *is Lyapunov stable (respectively, asymptotically stable) for $t\geq 0$, then the zero solution of Eq.* (5.2) *is stable in probability (respectively, asymptotically stable in probability).*

Proof. Without loss of generality, we can assume that $X_0=0$. Let Z_t be the solution of Eq. (5.3) with the initial condition $Z_0=\xi$, then $Y_t=Z_{X_t}$. Fix arbitrary ε_1, $\varepsilon_2>0$.

Since $X_t\xrightarrow[t\to+\infty]{P}+\infty$, for any $\varepsilon_2>0$ there exists $\tau=\tau(\varepsilon_2)>0$ such that

$$P(X_t\geq 0\ \ \forall\,t>\tau)\geq 1-\frac{\varepsilon_2}{2}\,.$$

Since $\mathbb{E}\big(\sup\limits_{t\in[0,\tau]}|X_t|\big)$ is finite, it follows by the Chebyshev inequality that there exists a constant $M=M(\tau,\varepsilon_2)>0$ such that

$$P\big(|X_t|\leq M\ \forall\,t\in[0,\tau]\big)\geq 1-\frac{\varepsilon_2}{2}\,.$$

Assume that the zero solution of Eq. (5.3) is Lyapunov stable for $t\geq 0$. Then there exists a $\delta=\delta(\varepsilon_1,M)>0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi:\Omega\to\mathbb{R}$, $|\xi|\leq\delta$ a.s., one has the inequality $\sup\limits_{t\geq -M}|Z_t|\leq\varepsilon_1$.

Thus, we have

$$\begin{aligned}P\Big(\sup_{t\geq 0}|Y_t|>\varepsilon_1\Big)=P\Big(\sup_{t\geq 0}|Z_{X_t}|>\varepsilon_1\Big)\leq P\big(\exists\,t\geq 0:\ X_t<-M\big)\\ \leq P\big(\exists\,t\in[0,\tau]:\ X_t<-M\big)+P\big(\exists\,t>\tau:X_t<0\big)\leq\frac{\varepsilon_2}{2}+\frac{\varepsilon_2}{2}=\varepsilon_2.\end{aligned}$$

Thus, the zero solution of Eq. (5.2) is stable in probability.

Consequently, the zero solution of Eq. (5.3) is asymptotically stable for $t \geq 0$. Then there exists a $\Delta > 0$ such that for any $\mathcal{F}_0$-measurable random variable $\xi : \Omega \to \mathbb{R}$, $|\xi| \leq \Delta$ a.s., the solution Z_t of Eq. (5.3) with the initial condition $Z_0 = \xi$ has the following property: the convergence

$$Z_t \xrightarrow[t \to +\infty]{} 0$$

holds with probability 1. Take arbitrary $\varepsilon_1, \varepsilon_2 > 0$. There exists $\delta = \delta(\varepsilon_1)$ such that

$$P\big(|Z_t| \leq \varepsilon_1 \ \forall\, t \geq \delta\big) = 1.$$

Since $X_t \xrightarrow[t \to +\infty]{P} +\infty$, there exists $\delta_1 > 0$ such that

$$P\big(\exists\, t \geq \delta_1 : \ X_t < \delta\big) \leq \varepsilon_2.$$

Thus,

$$\begin{aligned} P\big(|Y_t| \leq \varepsilon_1 \ \forall\, t \geq \delta_1\big) &= P\big(|Z_{X_t}| \leq \varepsilon_1 \ \forall\, t \geq \delta_1\big) \\ &= 1 - P\big(\exists\, t \geq \delta_1 : \ |Z_{X_t}| > \varepsilon_1\big) \geq 1 - P\big(\exists\, t \geq \delta_1 : \ X_t < \delta\big) \geq 1 - \varepsilon_2. \end{aligned}$$

Hence, $Y_t \xrightarrow[t \to +\infty]{P} 0$, therefore, the zero solution of Eq. (5.2) is asymptotically stable in probability.

The proof of the theorem is complete. □

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On Non-Belonging to the Second Baire Class of the Topological Entropy of One Family of Non-Autonomous Dynamical Systems on an Interval Continuously Depending on a Real Parameter

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Let (X,d) be a compact metric space, $f\equiv(f_1,f_2,\dots)$ be a sequence of continuous mappings from X to X. Along with the original metric d, we define on X an additional system of metrics

$$d_n^f(x,y)=\max_{0\le i\le n-1} d\big(f^{\circ i}(x),f^{\circ i}(y)\big),\ \ f^{\circ i}\equiv f_i\circ\cdots\circ f_1\circ \mathrm{id}_X,\ \ x,y\in X,\ \ n\in\mathbb{N}.$$

For any $n\in\mathbb{N}$ and $\varepsilon>0$, let $N(f,\varepsilon,n)$ denote the maximum number of points in X whose pairwise d_n^f-distances are greater than ε. Such a set of points is called (f,ε,n)-separated. Then the *topological entropy* of a nonautonomous dynamical system (X,f) is the quantity

$$h_{\rm top}(f)=\lim_{\varepsilon\to0}\overline{\lim_{n\to\infty}}\,\frac1n\,\ln N(f,\varepsilon,n). \tag{1}$$

Note that the topological entropy does not depend on the choice of a metric generating the given topology on X, so definition (1) is correct.

Given a metric space $\mathcal{M}$ and a sequence of continuous mappings

$$f\equiv(f_1,f_2,\dots),\quad f_k:\mathcal{M}\times X\to X, \tag{2}$$

we form a function

$$\mu\mapsto h_{\rm top}(f(\mu,\,\cdot\,)). \tag{3}$$

For arbitrary $\mathcal{M}$, X and for any sequence of mappings (2) function (3) belongs to the third Baire class [4]. In the case when X is a Cantor perfect set [4] or a segment of the real line [5] and $\mathcal{M}$ is the set of irrational numbers on the segment $[0;1]$ with the metric induced by the standard metric of the real line, there is a sequence of mappings (2) for which function (3) is everywhere discontinuous and does not belong to the second Baire class.

A natural question arises on the smallest Baire class to which function (3) belongs in the case $\mathcal{M}=[0;1]$.

Theorem. *Let $\mathcal{M}=X=[0;1]$, then there exists a sequence of continuous mappings* (2) *such that the function* (3) *is everywhere discontinuous and does not belong to the second Baire class on the space $\mathcal{M}$.*

Proof. Given a continuous function $\alpha:\mathcal{M}\to\mathcal{M}$

$$\alpha(\mu)=\begin{cases}0, & \text{if } \ \mu=0,\\ \mu\Big(1-\sin\dfrac1\mu\Big), & \text{if } \ 0<\mu\leqslant1,\end{cases}$$

we will construct a sequence of functions

$$\alpha_k(\,\cdot\,) = \max\Big\{\frac{1}{k}, \alpha^{\circ[\log_2(k+4)]}(\,\cdot\,)\Big\}, \quad k = 1, 2, \ldots$$

($[\,\cdot\,]$ is the integer part of number) and a sequence of mappings from $[0;1]^2$ to $[0;1]$

$$f \equiv (f_1, f_2, \ldots),$$
$$f_k(\mu, x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant 1 - \alpha_k(\mu), \\ 2x - 1 + \alpha_k(\mu), & \text{if } 1 - \alpha_k(\mu) < x \leqslant 1 - \dfrac{\alpha_k(\mu)}{2}, \\ -2x + 3 - \alpha_k(\mu), & \text{if } 1 - \dfrac{\alpha_k(\mu)}{2} < x \leqslant 1. \end{cases}$$

By definition, the function f_k is continuous on $[0;1]^2$.

Let $\mathbf{E}$ denote the set of those μ from $[0;1]$ for which the equality $\lim\limits_{k\to\infty} \alpha_k(\mu) = 0$ holds. It is not empty because it contains zero.

Lemma 1. *Let $\mu \in \mathbf{E}$, then*

$$h_{\text{top}}(f(\mu, \,\cdot\,)) = 0.$$

Proof. We recall another formula for calculating the topological entropy of a nonautonomous dynamical system [1]. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, denote by $B_f(x, \varepsilon, n)$ the open ball $\{y \in X : d_n^f(x, y) < \varepsilon\}$. A set $U \subset X$ is called an (f, ε, n)-covering if

$$X \subset \bigcup_{x\in U} B_f(x, \varepsilon, n).$$

Let $S(f, \varepsilon, n)$ denote the minimum number of elements of an (f, ε, n)-covering, then the topological entropy can be calculated by the formula

$$h_{\text{top}}(f) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S(f, \varepsilon, n).$$

We fix $\varepsilon > 0$ and $\mu \in \mathbf{E}$, then there is a number k_0 such that $\alpha_{k_0}(\mu) < \frac{1}{2}\varepsilon$ and for any $k \geqslant k_0$ the inequality $\alpha_k(\mu) \leqslant \alpha_{k_0}(\mu)$ holds. Let $U_{k_0} \subset [0;1]$ be a minimal $(f(\mu, \,\cdot\,), \frac{1}{2}\varepsilon, k_0)$-covering of the interval $[0;1]$. The set $U_{k_0} \cup \{x_0\}$, where $f^{\circ k_0}(\mu, x_0) = 1 - \alpha_{k_0}(\mu)$, due to the inclusion

$$f_k\big(\mu, [1 - \alpha_{k_0}(\mu), 1]\big) \subset \big[1 - \alpha_{k_0}(\mu), 1\big],$$

for $k > k_0$ is an $(f(\mu, \,\cdot\,), \frac{1}{2}\varepsilon, k)$-covering of the interval $[0;1]$, therefore

$$h_{\text{top}}(f(\mu, \,\cdot\,)) \leqslant \lim_{k\to\infty} \frac{1}{k} \ln\big(|U_{k_0}| + 1\big) = 0.$$

Lemma 1 is proved. □

Lemma 2. *Let $\mu \notin \mathbf{E}$, then*

$$h_{\text{top}}(f(\mu, \,\cdot\,)) \geqslant \frac{1}{2} \ln 2.$$

Proof. Let $\mu \notin \mathbf{E}$, then there exists a subsequence $(\alpha_{k_j}(\mu))_{j=1}^{\infty} \subset (\alpha_k(\mu))_{k=1}^{\infty}$ and a number $q > 0$ such that $\inf\limits_{j\in\mathbb{N}} \alpha_{k_j}(\mu) = q$.

For all $j \in \mathbb{N}$, $k \in \{2^{k_j}, \dots, 2^{k_j+1} - 1\}$ and $x \in [0;1]$ the equality $f_k(\mu,x) = f_{2^{k_j}}(\mu,x)$ holds. Using the affine transformation φ we map the square $[1-\alpha_{k_j}(\mu), 1]^2$ onto the square $[0,1]^2$, and the mapping $f_{2^{k_j}}(\mu,\cdot)\big|_{[1-\alpha_{k_j}(\mu),1]}$ becomes the mapping

$$g(x) = \begin{cases} 2x, & \text{if } 0 \leqslant x \leqslant \dfrac{1}{2}\,, \\ 2-2x, & \text{if } \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

For any $n \in \mathbb{N}$, consider the set of points of the form

$$\sum_{k=1}^{n} \frac{a_k}{2^k}\,, \quad \text{where } a_k \in \{0,1\}.$$

Using mathematical induction, we prove the equality

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = \begin{cases} 0, & \text{if } a_n = 0, \\ 1, & \text{if } a_n = 1. \end{cases} \tag{4}$$

Indeed, for $n = 1$ we have $g(0) = 0$ and $g(\frac{1}{2}) = 1$.

Let

$$\sum_{k=1}^{n} \frac{a_k}{2^k} < \frac{1}{2}\,,$$

then

$$\sum_{k=1}^{n} \frac{a_k}{2^k} = \sum_{k=2}^{n} \frac{a_k}{2^k}$$

and

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(2\sum_{k=2}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(\sum_{k=1}^{n-1} \frac{a_{k+1}}{2^k}\Big) = \begin{cases} 0, & \text{if } a_n = 0, \\ 1, & \text{if } a_n = 1. \end{cases}$$

Let

$$\sum_{k=1}^{n} \frac{a_k}{2^k} \geqslant \frac{1}{2}\,,$$

then $a_1 = 1$ and

$$g^n\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(2 - 2\sum_{k=1}^{n} \frac{a_k}{2^k}\Big) = g^{(n-1)}\Big(1 - \sum_{k=1}^{n-1} \frac{a_{k+1}}{2^k}\Big) = \begin{cases} 0, & \text{if } a_n = 0, \\ 1, & \text{if } a_n = 1. \end{cases}$$

Thus, equality (4) is proved.

By (4), for $\varepsilon < \frac{1}{q}$ and $n \in \{1, \dots, 2^{k_j+1} - 2^{k_j} - 1\}$, we have that $d_{2^{k_j}+n}^{f(\mu,\cdot)}$-distance between any preimages of two points

$$\varphi^{(-1)}\Big(\sum_{k=1}^{n} \frac{a_k}{2^k}, 0\Big) \ \text{ and } \ \varphi^{(-1)}\Big(\sum_{k=1}^{n} \frac{a_k}{2^k} + \frac{1}{2^n}, 0\Big)$$

under the mapping $f^{\circ(2^{k_j}-2)}(\mu,\,\cdot\,)$ is greater than ε, and therefore

$$N(f(\mu,\,\cdot\,),\varepsilon,2^{k_j+1}) \geqslant 2^{k_j+1}-2^{k_j},$$

whence we get

$$h_{\mathrm{top}}(f(\mu,\,\cdot\,)) \geqslant \lim_{\varepsilon\to 0}\varlimsup_{j\to\infty}\frac{1}{2^{k_j+1}}\,\ln(2^{2^{k_j+1}-2^{k_j}}) = \frac{1}{2}\,\ln 2.$$

Lemma 2 is proved. □

Completion of the proof of the theorem. In the paper [2] it was established that the set $\mathbf{E}$ is an $F_{\sigma\delta}$-set and is not a $G_{\delta\sigma}$-set. We use the following statement from [3]: if a functional $h:\mathcal{M}\to\mathbb{R}$ belongs to the second Baire class, then the intersection of the closures of the sets $h(\mathbf{E})$ and $h(\mathcal{M}\setminus\mathbf{E})$ is nonempty. By Lemmas 1 and 2, we have

$$h_{\mathrm{top}}(f(\mathbf{E},\,\cdot\,)) \leqslant 0 < \frac{1}{2}\,\ln 2 \leqslant h_{\mathrm{top}}(f(\mathcal{B}\setminus\mathbf{E},\,\cdot\,)),$$

therefore, the function $\mu\mapsto h_{\mathrm{top}}(f(\mu,\,\cdot\,))$ does not belong to the second Baire class. Theorem is proved. □

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Regularly Varying Solutions of Differential Equations of the Second Order with Nonlinearities of Exponential Types

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The differential equation

$$y'' = \alpha_0 p(t) f(t, y, y'), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$ is a continuous function, $f : [a, \omega[\times \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ is continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$[1] or $]Y_i, y_i^0]$, is considered. We also suppose that the function f satisfies the conditions

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \cdot \frac{\partial f}{\partial t}(t, v_0, v_1)}{f(t, v_0, v_1)} = \gamma \ \text{ uniformly by } \ v_0 \in \Delta_{Y_0}, \ \ v_1 \in \Delta_{Y_1}, \tag{2}$$

$$\lim_{\substack{y_k \to Y_k \\ y_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(t, v_0, v_1)}{f(t, v_0, v_1)} = \sigma_k \ \text{ uniformly by } \ t \in [a, \omega[, v_j \in \Delta_{Y_j}, \ \ j \neq k, \ \ k \in \{0, 1\}. \tag{3}$$

By conditions (2), (3) the function f is in some sense close to regularly varying function by every variable.

We call the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ a regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma. \tag{4}$$

Here $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [5] that regularly varying functions have the next properties.

- R_1: The function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if the next representation takes place
$$\varphi(z) = z^\sigma \theta(z),$$
where $\theta(z)$ is a slowly varying function as $z \to Y$.

- R_2: If the function $L : \Delta_{Y^0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

- R_3: If the function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition
$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z \varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$
then φ is regularly varying as $z \to Y$of index σ.

[1] As $Y_i = +\infty$ $(Y_i = -\infty)$ assume $y_i^0 > 0$ $(y_i^0 < 0)$.

R_4: For every regularly varying as $z \to Y$ function φ the property (4) takes place uniformly as $\lambda \in [c,d]$ for every segment $[c,d] \subset]0,+\infty[$.

Definition. Solution y of the equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ if it is defined on $[t_0,\omega[\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

For different values of parameter λ_0 the class of such solutions contains regularly slowly and rapidly varying as $t \uparrow \omega$ functions. $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$.

A lot of works (see, for example, [2,3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $f(t,y,y') \equiv \varphi_0(y)\varphi_1(y')$, where φ_0 and φ_1 are regularly varying functions. For more general case as f depends only on y and y' asymptotic properties and necessary and sufficient conditions of existence of such solutions of the equation (1) have been obtained in [1].

We need the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t-\omega & \text{as } \ \omega < +\infty, \end{cases}$$

$$\Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,1),$$

$$J_1(t) = \int_{A_\omega^1}^{t} \left(\alpha_0 p(\tau) |\pi_\omega(\tau)|^{\gamma+\sigma_0} \left| \frac{\lambda_0 - 1}{\lambda_0} \right|^{\sigma_0} \right) d\tau,$$

$$A_\omega^1 = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma+\sigma_0}\, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma+\sigma_0}\, d\tau < +\infty, \end{cases}$$

$$J_2(t) = \left|(1-\sigma_0-\sigma_1)\right|^{\frac{1}{1-\sigma_0-\sigma_1}} \operatorname{sign} y_1^0 \int_{B_\omega^2}^{t} |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\, d\tau,$$

$$B_\omega^2 = \begin{cases} b, & \text{if } \displaystyle\int_b^\omega |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\, d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_b^\omega |J_1(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}\, d\tau < +\infty. \end{cases}$$

The following theorem is obtained for the equation (1).

Theorem 1. *Let in the equation* (1) $\sigma_1 \neq 1$. *Then for the existence of* $P_\omega(Y_0,Y_1,\lambda_0)$*-solutions to the equation* (1) *in cases* $\lambda_0 \in R \setminus \{0,1\}$, *it is necessary and if*

$$\lambda_0 \neq \sigma_1 - 1 \ \ \text{or} \ \ (\sigma_1 - 1)(\sigma_0 + \sigma_1 - 1) > 0,$$

then also sufficient

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t)\alpha_0 y_1^0 \lambda_0(\lambda_0-1) > 0 \ \ \text{as} \ \ t \in [a,\omega[,$$

$$\lim_{t\uparrow\omega} y_0^0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}} = Y_1,$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_2'(t)}{J_2(t)} = \frac{\lambda_0}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_1'(t)}{J_1(t)} = \frac{1-\sigma_0-\sigma_1}{\lambda_0-1}\,.$$

The next result is devoted to research $P_\omega(Y_0,Y_1,\lambda_0)$-solutions in special and most complex case $\lambda_0=0$. In this case, such decisions or their first order derivatives will be slowly changing functions as $t\uparrow\omega$, which significantly complicates the study. Therefore, consider the differential equation

$$y'' = \alpha_0 p(t)|y|^{\sigma_0}y'|^{\sigma_1}\exp\big(R\big(\big|\ln|\pi_\omega(t)yy'|\big|\big)\big), \tag{5}$$

where α_0, p are the same as in a general equation, and R is continuously differentiable, with a monotone derivative, regularly variable at infinity function return order μ, $0<\mu<1$.

Theorem 2. *Let*
$$\lim_{t\uparrow\omega} \frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)} = 0.$$
Then, for the existence of $P_\omega(Y_0,Y_1,0)$-solutions to the equation (5) *for which there is a finite or infinite boundary*
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)}\,,$$
the following conditions and inequalities are sufficient and sufficient
$$\lim_{t\uparrow\omega} y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}} = Y_0, \quad \lim_{t\uparrow\omega} y_1^0|I(t)|^{\frac{1}{1-\sigma_1}} = Y_1,$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \sigma_1-1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = 0,$$
$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0, \quad \frac{y_0^0y_1^0(1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1} > 0 \ \ as \ \ t\in\,]a,\omega[\,.$$

In addition, for each such solution, the following asymptotic representations hold as $t\uparrow\omega$
$$\frac{y(t)}{|\exp(R(|\ln|\pi_\omega(t)y(t)y'(t)||))|y(t)|^{\sigma_0}|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}}J(t)[1+o(1)],$$
$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)J(t)}{(1-\sigma_1)J'(t)}\,[1+o(1)],$$

where
$$I(t) = \alpha_0\int_{A_\omega}^{t} p(\tau)\,d\tau, \quad J(t) = \int_{B_\omega}^{t} |I(\tau)|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

the integration limits A_ω, B_ω are chosen so that the corresponding integrals are either 0, *or* ∞.

For differential equations of more specific type, one can get more detailed information about $P_\omega(Y_0,Y_1,0)$-solutions to the equation (3).

In [4] it was considered the differential equation

$$y'' = mt^{\sigma_1-2}\exp(k\ln^\gamma t)|y|^{\sigma_0}|y'|^{\sigma_1}\exp\big(\big(|\ln|yy'|\big)^\mu\big) \tag{6}$$

on the interval $[t_0;+\infty[$ $(t_0>0)$, where $m\in\,]-\infty,0[\,$, $k\in\,]0,+\infty[\,$, $\gamma,\mu\in\,]0;1[\,$, $\sigma_0,\sigma_1\in\mathbb{R}$, $\sigma_0+\sigma_1\neq 1$, $\sigma_1\neq1$ is the equation of the form (1), where $\alpha_0=\operatorname{sign} m=-1$, $p(t)=mt^{\sigma_1-2}\exp(k\ln^\gamma t)$,

$\varphi_0 = |y|^{\sigma_0}$, $\varphi_1 = |y|^{\sigma_1}$, $R(z) = z^{\mu}$. This function φ_1 satisfies the condition S. Let us consider the case, when $\omega = Y_0 = Y_1 = +\infty$.

If $\mu - \gamma < 0$, then for the existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions of the equation (6) the following condition

$$1 - \sigma_0 - \sigma_1 > 0 \tag{7}$$

is necessary and sufficient.

Moreover, for each such solution the following asymptotic representations take place as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\gamma k} \exp\Big(\frac{k\ln^{\gamma} t}{1-\sigma_1}\Big) \ln^{1-\gamma} t[1+o(1)],$$

$$\frac{y(t)}{y'(t)} = \frac{(1-\sigma_0-\sigma_1)\gamma k}{(1-\sigma_1)^2} \frac{\ln^{\gamma-1} t}{t} [1+o(1)].$$

Let us now consider the case $\mu - \gamma > 0$. In this case for $\mu - \gamma > 0$ for existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions to the equation (6) the condition (7) is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1)} \exp\Big(\frac{k\ln^{\gamma} t}{1-\sigma_1}\Big) \ln^{1-\mu} t[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\mu}{\sigma_0+\sigma_1-1} t^{\sigma_1-2} \ln^{\gamma-1} t[1+o(1)].$$

In case $\mu = \gamma$ we obtain that for the existence of $P_{+\infty}(+\infty, +\infty, 0)$-solutions to the equation (6) the condition (7) together with the condition

$$(1-\sigma_1-k)(1-\sigma_1) > 0$$

is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t \to +\infty$

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}} \exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1 - 1}\Big) = \frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1-k)} \exp\Big(\frac{k\ln^{\gamma} t}{1-\sigma_1}\Big) \ln^{1-\mu} t[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\mu(1-\sigma_1-k)}{(\sigma_0+\sigma_1-1)(1-\sigma_1)} t^{\sigma_1-2} \ln^{\gamma-1} t[1+o(1)].$$

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C o n t e n t s