Limiting Behavior of Invariant Measures of Stochastic Functional-Differential Neutral Equations in Hilbert Space

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We consider the following stochastic functional-differential neutral equation on Hilbert space with delay parameter $h \in (0,1]$:

$$d(u(t)+g(u(t-h),u(t))) = (f(u(t-h),u(t))+Au(t))\,dt + \sigma(u(t-h),u(t))\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t) = \phi(t), \quad t \in [-h,0]. \tag{0.2}$$

Here A is an inifinitesimal generator of a strong continuous semigroup $\{S(t), t \geq 0\}$ of bounded linear operators in real separable Hilbert space H. The noise $W(t)$ is a Q-Wiener process on separable Hilbert space K. For any $h \in (0,1)$ denote $C_h := C([-h,0],H)$, a space of continuous H-valued functions with a norm.

$$\|\phi\|_{C_h} := \sup_{t\in[-h,0]} \|\phi(t)\|_H.$$

Below we denote $\|\cdot\|_H$ as $\|\cdot\|$. The functions f and g map $H \times H$ into H and $\sigma : H \times H \to L_2^0$, where $L_2^0 = L(Q^{1/2}K, H)$ is the space of Hilbert–Schmidt operators from $Q^{1/2}K$ to a H. Finally, $\phi : [-h,0] \times \Omega \to H$ is the initial condition on probability space $(\Omega, \mathcal{F}, P)$.

We consider the limiting behavior of invariant measures of equation (0.1), (0.2) when delay parameter h converges to zero.

1 Preliminaries

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $\{F_t; t \geq 0\}$ generated by the Q-Wiener process W on (Ω, F, P) with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete orthonormal system e_k in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \dots$, and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion $W(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) e_k$, where $\beta_k(t)$ are real valued Brownian motions mutually independent on (Ω, F, P).

Let $U_0 = Q^{\frac{1}{2}}(U)$ and $L_0^2 = L_2(U_0, H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi) L_0^2 = \operatorname{tr}[\Phi Q \Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

Lemma 1.1 (Stochastic Gronwall Lemma [5, 10])**.** *Let Z, H be nonnegative stochastic processes adapted to filtration, M be a continuous local martingale. Then:*

1. *If $M(0) = 0$ and there exist $K, C \geq 0$ such that*
$$Z(t) \leq K \int_0^t \sup_{u \in [0,s]} (Z(u))\, ds + M(t) + C,$$
then for all $0 < \alpha < 1$ there exist $C_1, C_2 > 0$ such that
$$\mathbf{E}\Big(\sup_{t \in [0,T]} (Z(t))^{\alpha}\Big) \leq C^{\alpha} C_1 e^{C_2 K T}.$$

2. *If $M(0) = 0$, $H(0) = 0$ and there exists $K \geq 0$ such that*
$$Z(t) \leq K \int_0^t \sup_{u \in [0,s]} (Z(u)) ds + M(t) + H(t),$$
then for all $0 < \alpha < 1$ and $\beta > \frac{1+\alpha}{1-\alpha}$ there exist $C_3, C_4 > 0$ such that
$$\mathbf{E}\Big(\sup_{t \in [0,T]} (Z(t))^{\alpha}\Big) \leq C_3 e^{C_4 K T} \Big(\mathbf{E}\Big(\sup_{t \in [0,T]} H(t)\Big)^{\beta}\Big)^{\alpha/\beta}.$$

3. *If $H(t)$ is non-negative, then for all $0 < \alpha < 1$ there exists $C_{\alpha} \geq 0$ such that*
$$\mathbf{E}\Big(\sup_{t \in [0,T]} (Z(t))^{\alpha}\Big) \leq (C_{\alpha} + 1) e^{\alpha K T} \Big(\mathbf{E}\Big(\sup_{t \in [0,T]} H(t)\Big)^{\alpha}\Big).$$

Definition 1.1 (Mild solution)**.** A continuous $\mathcal{F}_t$ adapted stochastic process $u : [-h, T] \times \Omega \to H$ is a mild solution for (0.1), (0.2) for $t \in [0, T]$ if it satisfies the integral equation

$$u(t) = S(t)(\phi(0) + g(\phi(-h), \phi(0))) - g(u(t-h), u(t)) - \int_0^t AS(t-s) g(u(s-h), u(s))\, ds$$
$$+ \int_0^t S(t-s) f(u(s-h), u(s))\, ds + \int_0^t S(t-s) \sigma(u(s-h), u(h))\, dW(s),$$

and $u(t) = \phi(t)$ a.s. for $t \in [-h, 0]$.

A non-delay equation will look as follows

$$d\big(u(t) + g(u(t), u(t))\big) = \big(f(u(t), u(t)) + Au(t)\big)\, dt + \sigma(u(t), u(t))\, dW(t), \quad t \geq 0, \tag{1.1}$$
$$u(0) = \phi(0). \tag{1.2}$$

For starting function ϕ and delay parameter $h \in (0,1)$ we denote mild solution of equation (0.1), (0.2) as $u^h(t,\phi)$. For starting point v we denote mild solution of equation (1.1), (1.2) as $u^0(t,v)$.

Let (Z,d) be a polish space with metric d. Suppose for every $\rho \in (0,1]$ and $\phi \in C([-\rho,0],Z)$, $\{X^\rho(t,0,\phi), t \geq 0\}$ is a stochastic process in the state space $C([-\rho,0],Z)$ with initial value ϕ at initial time 0. Similarly, assume for every $x \in Z$, $\{X^0(t,0,x), t \geq 0\}$ is a stochastic process in the state space Z with initial value x at initial time 0. We also assume that the probability transition operators of X^0 are Feller.

$UC_b(Z)$ is the Banach space of all bounded uniformly continuous functions defined on Z with uniform norm.

Given $\rho \in (0,1]$, define an operator $T_\rho : C([-\rho,0],Z) \to Z$ by $T_\rho\phi = \phi(0)$, and $\mathcal{T}_\rho : C([-1,0],Z) \to C([-\rho,0],Z)$ by $\mathcal{T}_\rho\phi(s) = \phi(s)$.

Condition (C1). For every compact set $K \subset C([-1,0],Z)$, $t \geq 0$, and $\eta > 0$,

$$\lim_{\rho\to 0} \sup_{\phi\in\mathcal{T}_\rho K} P\Big(d\big(X^\rho(t,0,\phi)(0), X^0(t,0,T_\rho\phi)\big) \geq \eta\Big) = 0.$$

Theorem 1.1 ([4])**.** *Assume* (C1) *holds true and* $\rho_n \in (0,1]$. *Let* μ^{ρ_n} *be an invariant measure of* X^{ρ_n} *in* $C([-\rho_n,0],Z)$ *for all* $n \in \mathbb{N}$. *Suppose* $\{\mu^{\rho_n}\}_{n=1}^\infty$ *is tight in a sense that for every* $\varepsilon > 0$ *there exists compact set* $K_1 \subset C([-1,0],Z)$ *such that*

$$\mu^{\rho_n}(\mathcal{T}_{\rho_n}K_1) > 1-\varepsilon, \tag{1.3}$$

for all $n \in \mathbb{N}$. *Then we have:*

1. *The sequence* $\{\mu^{\rho_n} \circ T_{\rho_n}^{-1}\}_{n=1}^\infty$ *is tight;*
2. *If* $\rho_n \to 0$ *and* μ *is a probability measure in* Z *such that* $\mu^{\rho_n} \circ T_{\rho_n}^{-1} \to \mu$ *weakly, then* μ *must be an invariant measure of* X^0.

Proof.
1. Given $\varepsilon > 0$, let $K_1 \subset C([-1,0],Z)$ be the compact set satisfying (1.3). Denote by $K_0 = \{\phi(0) : \phi \in K_1\}$. Then K_0 is a compact subset of Z and for all $n \in \mathbb{N}$,

$$\mu^{\rho_n} \circ T_{\rho_n}^{-1}(K_0) \geq \mu^{\rho_n}(\mathcal{T}_{\rho_n}K_1) > 1-\varepsilon, \tag{1.4}$$

which shows that $\{\mu^{\rho_n} \circ T_{\rho_n}^{-1}\}$ is tight.

2. We need to prove that for all $\psi \in UC_b(Z)$ and $t > 0$,

$$\int_Z \mathbf{E}\psi(X^0(t,0,x))\mu\,(dx) = \int_Z \psi(x)\mu\,(dx). \tag{1.5}$$

One can notice that

$$\begin{aligned}\int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}\,(dx) &= \int_{C([-\rho_n,0],Z)} \psi(T_{\rho_n}\xi)\mu^{\rho_n}\,d\xi \\ &= \int_{C([-\rho_n,0],Z)} \psi\big(T_{\rho_n}X^{\rho_n}(t,0,\xi)\big)\mu^{\rho_n}\,d\xi = \int_{C([-\rho_n,0],Z)} \psi\big(X^{\rho_n}(t,0,\xi)(0)\big)\mu^{\rho_n}\,d\xi,\end{aligned}$$

which with (1.4) yields that

$$
\begin{aligned}
&\left| \int_Z \mathbf{E}\psi(X^0(t,0,x))\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) - \int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) \right| \\
&\qquad \le \int_{C([-\rho_n,0],Z)} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad \le \int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad\qquad + \int_{C([-\rho_n,0],Z)\setminus \mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&\qquad \le \int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) + 2\varepsilon \sup_{x\in Z}|\psi(x)|. \qquad (1.6)
\end{aligned}
$$

Since $\psi \in UC_b(Z)$ and for $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\psi(u) - \psi(v)| < \varepsilon,$$

if $d(u,v) < \eta$. Then we get

$$
\begin{aligned}
&\int_{\mathcal{T}_{\rho_n}K_1} \mathbf{E}\big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n}(d\xi) \\
&= \int_{\mathcal{T}_{\rho_n}K_1} \left(\int_{\{d(\psi(X^0(t,0,T_{\rho_n}\xi)),\psi(X^{\rho_n}(t,0,\xi)(0)))\ge\eta\}} \big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n} P(d\omega) \right)(d\xi) \\
&+ \int_{\mathcal{T}_{\rho_n}K_1} \left(\int_{\{d(\psi(X^0(t,0,T_{\rho_n}\xi)),\psi(X^{\rho_n}(t,0,\xi)(0)))<\eta\}} \big|\psi(X^0(t,0,T_{\rho_n}\xi)) - \psi(X^{\rho_n}(t,0,\xi)(0))\big|\mu^{\rho_n} P(d\omega) \right)(d\xi) \\
&\qquad \le 2\sup_{x\in Z}|\psi(x)| \cdot \sup_{\xi\in\mathcal{T}_{\rho_n}K_1} P\Big(\Big\{d\Big(\psi(X^0(t,0,T_{\rho_n}\xi)), \psi(X^{\rho_n}(t,0,\xi)(0))\Big) \ge \eta\Big\}\Big) + \varepsilon. \qquad (1.7)
\end{aligned}
$$

Then, from (C1) and (1.6), (1.7) we can deduce that

$$\left| \int_Z \mathbf{E}\psi(X^0(t,0,x))\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) - \int_Z \psi(x)\mu^{\rho_n} \circ T_{\rho_n}^{-1}(dx) \right| \le \varepsilon + 2\varepsilon \sup_{x\in Z}|\psi(x)|,$$

and since $\varepsilon > 0$ is arbitrary and $\mu^{\rho_n} \circ T_{\rho_n}^{-1} \to \mu$ weakly, we get that μ is an invariant measure for X^0 by (1.5). □

2 Conditions on functions

Condition (H1). If $\sigma(-A)$ is the spectrum of $(-A)$, we have

$$\operatorname{Re}\sigma(-A) > \delta > 0,$$

and A^{-1} is compact in H.

It follows from [6] that for $0 \le \alpha \le 1$ one can define fractional power $(-A)^\alpha$, which is closed linear operator with domain $D(-A)^\alpha$. We denote H_α to be a Banach space $D(-A)^\alpha$ with the norm

$$\|u\|_\alpha := \|(-A)^\alpha u\|,$$

which is equivalent to the graph norm of $(-A)^\alpha$. This way $H_0 = H$. It follows from [3, Section 1.4] that if A^{-1} is compact, then $S(t)$ is compact for $t > 0$. Next, it follows from [6, Theorem 3.2, p. 48] that under assumption (H1) semigroup $S(t)$ is continuous with respect to uniform operator topology for $t > 0$. Thus, using [6, Theorem 3.3, p. 48], we may conclude that the operator A has a compact resolvent. Consequently, from [3, Theorem 1.4.8], we have the following result.

Proposition 2.1. *Under condition* (H1) *the embedding $H_\alpha \subset H_\beta$ is compact if $0 \le \beta < \alpha \le 1$.*

Proposition 2.2 ([3, Theorem 1.4.3])**.** *Under condition* (H1)*, for every $\alpha \ge 0$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \le C_\alpha t^{-\alpha} e^{-\delta t},$$

for $t > 0$. In particular,

$$\|S(t)\| \le C_0 e^{-\delta t},$$

for $t > 0$.

Proposition 2.3 ([1])**.** *Let $p > 2$, $T > 0$ and let Φ be an L_2^0 valued, predictable process such that*

$$\mathbf{E} \int_0^T \|\Phi(t)\|_{L_2^0}^p \, dt < \infty.$$

Then there is a constant $M_T > 0$ such that

$$\mathbf{E} \sup_{t\in[0,T]} \bigg\| \int_0^t S(t-s)\Phi(s)\, dW(s) \bigg\| \le M_T \mathbf{E} \int_0^T \|\Phi(s)\|_{L_2^0}^p \, ds.$$

Condition (H2). The mappings $f : H \times H \to H$ and $\sigma : H \times H \to L_2^0$ are continuous and satisfy:

1. There exist a positive constant $K > 0$ such that

   $$\|f(u,v)\| + \|\sigma(u,v)\|_{L_2^0} \le K\big(1 + \|u\| + \|v\|\big)$$

   for all $u, v \in H$.

2. There exist a positive constant $L > 0$ such that

   $$\|f(u,v) - f(u_1,v_1)\|^2 + \|\sigma(u,v) - \sigma(u_1,v_1)\|_{L_2^0}^2 \le L\big(1 + \|u - u_1\|^2 + \|v - v_1\|^2\big)$$

   for all $u, v, u_1, v_1 \in H$.

Condition (H3). There exist positive constants $\alpha \in (0,1)$ and $M_g \in (0,1)$ such that for all $u, v, u_1, v_1 \in H$ the function $g : H \times H \to H_\alpha$ satisfies

$$\|g(u,v) - g(u_1,v_1)\|_{H_\alpha}^2 \le M_g\big(\|u - u_1\|^2 + \|v - v_1\|^2\big).$$

Condition (H4). The initial condition $\phi : [-h,0] \times \Omega \to H$ is an $\mathcal{F}_0$-measurable random variable, independent of W, which has continuous trajectories.

Remark. It is easy to see from [9], that under conditions above equation (0.1), (0.2) have unique mild solution, and this solution have an invariant measure.

3 Main results

Lemma 3.1. *Suppose that* (H1)–(H4) *hold. Then for every compact set K in C_h, $t > 0$ and $\eta > 0$ the following holds*

$$\lim_{h\to 0}\sup_{\xi\in\mathcal{T}_h K} P\Big(\|u^h(t,\xi) - u^0(t,T_h\xi)\| \geq \eta\Big) = 0.$$

Sketch of the proof.

Step 1. Rewrite solutions $u^h(t,\xi)$ and $u^0(t,T_h\xi)$ using Definition 1 as follows

$$\begin{aligned} u^h(t,\xi) &= S(t)\big(\xi(0) - g(\xi(-h),\xi(0))\big) \\ &\quad + g\big(u^h(t-h,\xi),u^h(t,\xi)\big) - \int_0^t AS(t-s)g\big(u^h(s-h,\xi),u^h(s,\xi)\big)\,ds \\ &\quad + \int_0^t S(t-s)f\big(u^h(s-h,\xi),u^h(s,\xi)\big)\,ds + \int_0^t S(t-s)\sigma\big(u^h(s-h,\xi),u^h(s,\xi)\big)\,dW(s), \end{aligned}$$

and

$$\begin{aligned} u^0(t,T_h\xi) &= S(t)(T_h\xi - g(T_h\xi,T_h\xi)) \\ &\quad + g\big(u^0(t,T_h\xi),u^0(t,T_h\xi)\big) - \int_0^t AS(t-s)g\big(u^0(s,T_h\xi),u^0(s,T_h\xi)\big)\,ds \\ &\quad + \int_0^t S(t-s)f\big(u^0(s,T_h\xi),u^0(s,T_h\xi)\big)\,ds + \int_0^t S(t-s)\sigma\big(u^0(s,T_h\xi),u^0(s,T_h\xi)\big)\,dW(s). \end{aligned}$$

Step 2. Estimate $\mathbf{E}\|u^h(t,\xi) - u^0(t,T_h\xi)\|^2$ from conditions (H1)–(H4) and Propositions 2.1–2.3 and using Lemma 1.1 (Stochastic Gronwall Lemma).

Step 3. Proposition of the lemma is a direct consequence of Chebyshev inequality. □

Given $h \in [0,1]$, let $p^h(r,\xi;t,\cdot)$ be the transition probability function of $u^h(t,\xi)$ with $0 \leq r \leq t$ and $\xi \in C_h$. Denote by $\mathcal{M}^h$ –collection of all limit points of probability measure

$$\frac{1}{n}\int_0^n p^h(0,0;t,\cdot)\,dt.$$

Then we have the following result.

Theorem 3.1. *Suppose that* (H1)–(H4) *hold. Then:*

1. *The union* $\bigcup_{h\in[0,1]} \mathcal{M}^h$ *is tight;*

2. *If $h_n \to 0$ and $\mu^{h_n} \in \mathcal{M}^{h_n}$, then there exist a subsequence $h_{k(n)}$ and an invariant measure $\mu^0 \in \mathcal{M}^0$ such that $\mu^{h_{k(n)}} \circ T^{-1}_{h_{k(n)}} \to \mu^0$ weakly.*

Proof.

1. Direct consequence of [9].

2. By first item we know that $\{\mu^{h_n}\}$ is tight and hence by Theorem 1.1 and Lemma 3.1 we infer that sequence $\{\mu^{h_n} \circ T_{h_n}^{-1}\}_{n=1}^{\infty}$ is also tight. Consequently, there exists a subsequence h_{n_k} and a probability measure μ^* such that $\mu^{h_{n_k}} \circ T_{h_{n_k}}^{-1} \to \mu^*$ weakly. By Theorem 1.1 and Lemma 3.1 we find that μ^* is invariant and $\mu^* \in \mathcal{M}^0$. □

As an immediate corollary of Theorem 3.1, we have the following result.

Theorem 3.2. *Suppose that* (H1)–(H4) *hold and* $h_n \to 0$. *Then, if* μ^{h_n} *and* μ^0 *are the unique invariant measures of equations* (0.1), (0.2) *and* (1.1), (1.2) *correspondingly, then* $\mu^{h_n} \to \mu^0$ *weakly.*

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