Approximation of Stochastic Systems with Integral-Type Delay by Stochastic Systems without Delay

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Abstract

This work generalizes an approach for approximating stochastic delay systems of integral type by stochastic systems without delay. The proposed scheme is based on expanding the solution using Taylor's formula with respect to the delay parameter h, where $[-h, 0]$ is the delay interval, and provides convergence results in the mean square metric.

Introduction

Stochastic delay differential equations are mathematical models of real-world processes in the natural sciences that evolve under the influence of random factors and whose future behavior depends on past states. It is well known [2] that continuous (or integrable) functions serve as initial data here, making the phase space of such equations infinite-dimensional, which significantly complicates their study. One possible approach to investigating these equations is the scheme proposed in [1], which approximates the initial problem for systems with delay by a Cauchy problem for systems of ordinary differential equations (ODEs). As the dimension of such systems increases, their solutions approach the solutions of the original initial problem for the delayed system in the uniform metric. This scheme is based on an old idea by M . M. Krasovskii, related to expanding the solution of the delayed system using Taylor's formula with respect to h, where $[-h, 0]$ is the delay interval.

This work generalizes such an approach to stochastic systems.

1 Problem statement and the main result

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$, $t \geq 0$, relative to which a scalar Wiener process $W(t)$, $t \geq 0$, is adapted. Without loss of generality, and to simplify the exposition, we will assume it is one-dimensional.

Let $h > 0$ represent the delay interval, on which a continuous deterministic initial function $\phi(t)$ is defined. Denote by $C = C([-h, 0],\ \mathbb{R}^d)$ the class of continuous d-dimensional vector functions $\phi : [-h, 0] \to \mathbb{R}^d$ with the supremum norm

$$\|\phi\| = \sup_{t\in[-h,0]} |\phi(t)|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

We consider the following initial value problem for a system of stochastic functional-differential equations:

$$\begin{cases} dx(t) = f\left(t, x(t), \int\limits_{-h}^{0} x(t+\theta)\, d\theta\right) dt + \sigma\left(t, x(t), \int\limits_{-h}^{0} x(t+\theta)\, d\theta\right) dW(t), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1.1}$$

where the functions $f, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are defined, continuous in all variables, and satisfy the following conditions: there exists a constant $L > 0$ such that:

(1) Linear Growth Condition:

$$|f(t, x, y)|^2 + |\sigma(t, x, y)|^2 \le L\big(1 + |x|^2 + |y|^2\big),$$

for any $t \in [0, T]$, $x, y \in \mathbb{R}^d$.

(2) Lipschitz Condition:

$$\big|f(t, x_1, y_1) - f(t, x_2, y_2)\big|^2 + \big|\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)\big|^2 \le L\big(|x_1 - x_2|^2 + |y_1 - y_2|^2\big).$$

We will understand the solution to the initial value problem (1.1) in the standard sense [3, p. 61].

Definition 1.1. An $\mathcal{F}_t$-adapted stochastic process with continuous trajectories is called a strong solution to the initial value problem (1.1) on $[0, T]$ if:

1. $x(t) = \phi(t)$, $t \in [-h, 0]$;
2. $x(t) = \phi(0) + \int\limits_{0}^{t} f\left(s, x(s), \int\limits_{-h}^{0} x(s+\theta)\, d\theta\right) ds + \int\limits_{0}^{t} \sigma\left(s, x(s), \int\limits_{-h}^{0} x(s+\theta)\, d\theta\right) dW(s),$

with probability 1.

Note that equation (1.1) induces abstract mappings from the space C to $\mathbb{R}^d$ of the following form:

$$f_1(t, \phi) = f\left(t, \phi(0), \int\limits_{-h}^{0} \phi(\theta)\, d\theta\right), \quad \sigma_1(t, \phi) = \sigma\left(t, \phi(0), \int\limits_{-h}^{0} \phi(\theta)\, d\theta\right).$$

From conditions (1) and (2), we have:

$$|f_1(t, \phi)|^2 + |\sigma_1(t, \phi)|^2 \le L\big(1 + (1 + h^2)\|\phi\|^2\big),$$
$$|f_1(t, \phi) - f_1(t, \psi)|^2 + |\sigma_1(t, \phi) - \sigma_1(t, \psi)|^2 \le L\big((1 + h^2)\|\phi - \psi\|^2\big).$$

Therefore, the conditions of the existence and uniqueness theorem for a strong continuous solution of problem (1.1) on $[0,T]$ are satisfied, and $\sup\limits_{t\in[0,T]} \mathbb{E}\,|x(t)|^2<\infty$.

Based on the system of stochastic functional differential equations (1.1), we construct a system of stochastic differential equations without delay, which we call the approximating system, as follows. Fix $m\in\mathbb{N}$ and partition the interval $[-h,0]$ with points $-\frac{hj}{m}$, $j=\overline{0,m}$, into m parts.

Define functions $z_j(t)\in\mathbb{R}^d$ on $[0,T]$ as solutions to the following Cauchy problems:

$$\begin{cases} dz_0(t)=f\Big(t,z_0(t),\dfrac{h}{m}\displaystyle\sum_{j=1}^m z_j(t)\Big)\,dt+\sigma\Big(t,z_0(t),\dfrac{h}{m}\displaystyle\sum_{j=1}^m z_j(t)\Big)\,dW(t),\\ dz_j(t)=\dfrac{m}{h}\,\big[z_{j-1}(t)-z_j(t)\big],\ \ j=\overline{1,m},\\ z_j(0)=\phi\Big(-\dfrac{hj}{m}\Big),\ \ j=\overline{0,m}. \end{cases} \tag{1.2}$$

Definition 1.2. System (1.2) is called an approximating system for system (1.1) in the mean square sense on $[0,T]$ if

$$\sup_{t\in[0,T]}\mathbb{E}\,\Big|x\Big(t-\frac{hj}{m}\Big)-z_j(t)\Big|^2\longrightarrow 0,\ \ m\to\infty,\ \ j=\overline{0,m}.$$

The main result of this work is the following theorem.

Theorem 1.1. *Under conditions* (1) *and* (2) *system* (1.2) *is an approximating system in the mean square sense for the initial problem* (1.1)*, uniformly over $j=\overline{0,m}$, i.e.,*

$$\sup_{j=\overline{0,m}}\ \sup_{t\in[0,T]}\mathbb{E}\,\Big|x\Big(t-\frac{hj}{m}\Big)-z_j(t)\Big|^2\longrightarrow 0,\ \ m\to\infty.$$

2 Proof of the main result

To prove the theorem, we need a lemma about estimating the mean square modulus of continuity of the solution to problem (1.1).

Lemma 2.1 (On the Modulus of Continuity)**.** *Under conditions* (1) *and* (2)*, for the solution of the initial problem* (1.1)*, the following relation holds:*

$$\sup_{t_1\in[-h,T]}\mathbb{E}\sup_{t_2\in[t_1,t_1+l]}|x(t_2)-x(t_1)|^2\le C\big(T,\|\phi\|,h,\big)\longrightarrow 0,\ \ l\to 0.$$

Proof. Since the solution to the initial problem (1.1) exists on $[0,T]$ and has a bounded second moment, by the linear growth condition, we have

$$|x(t)|^2\le 3\bigg(|\phi(0)|^2+\bigg|\int_0^t f\bigg(s,x(s),\int_{-h}^0 x(s+\theta)\,d\theta\bigg)\,ds\bigg|^2 \\ +\bigg|\int_0^t \sigma\bigg(s,x(s),\int_{-h}^0 x(s+\theta)\,d\theta\bigg)\,dW(s)\bigg|^2\bigg). \tag{2.1}$$

Next, note the inequality

$$\sup_{t\in[0,T]}\ \sup_{\theta\in[-h,0]}|x(t+\theta)|^2\le\|\phi\|^2+\sup_{t\in[0,T]}|x(t)|^2. \tag{2.2}$$

Considering (2.2), the Cauchy–Bunyakovsky inequality and using maximal inequality for stochastic integrals, from (2.1) we get

$$\mathbb{E}\sup_{s\in[0,t]}|x(s)|^2 \le 3|\phi(0)|^2 + 3T^2L\|\phi\|^2h^2 + 3T^2L + 6TL\int_0^t \mathbb{E}\sup_{\tau\in[0,s]}|x(\tau)|^2\,d\tau + 12L\int_0^t \Big(1 + h^2\|\phi\|^2 + h^2\mathbb{E}\sup_{\tau\in[0,s]}|x(\tau)|^2\Big)\,d\tau.$$

Applying Gronwall's inequality, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}|x(s)|^2 \le C_3\big(T,\|\phi\|,h\big). \tag{2.3}$$

Next, if $t_1 \ge 0$, we have

$$\mathbb{E}\sup_{t\in[t_1,t_1+l]}|x(t)-x(t_1)|^2 \le 2\Bigg(l\int_{t_1}^{t_1+l} L\Bigg(1+\mathbb{E}\,|x(t)|^2+\mathbb{E}\bigg|\int_{-h}^{0} x(t+\theta)\,d\theta\bigg|^2\Bigg)\,dt + \mathbb{E}\sup_{t\in[t_1,t_1+l]}\bigg|\int_{t_1}^{t}\sigma\bigg(s,x(s),\int_{-h}^{0}x(s+\theta)\,d\theta\bigg)\,dW(s)\bigg|^2\Bigg).$$

Using (2.3) and the previous inequality, and considering (2.2), we obtain

$$\mathbb{E}\sup_{t\in[t_1,t_1+l]}|x(t_2)-x(t_1)| \le C\big(T,\|\phi\|,h,l\big) \longrightarrow 0, \quad l\to 0.$$

If $t_1, t_1+l \in [-h,0]$, then, by the definition of the solution, we have

$$\mathbb{E}\sup_{t_2\in[t_1,t_1+l]}|x(t_2)-x(t_1)|^2 = \sup_{t_2\in[t_1,t_1+l]}|\phi(t_2)-\phi(t_1)|^2 \longrightarrow 0, \quad l\to 0,$$

due to the uniform continuity of the function $\phi(t)$, which completes the proof of the lemma. □

Continuation of the Proof of Theorem 1.1: Let us proceed with the proof of the main theorem. It is well known that the trajectories of the solution to (1.1) are continuous but nowhere differentiable functions, so we smooth the solution as follows. For any sufficiently small $\mu > 0$, we set

$$x_\mu(t) = \frac{1}{\mu}\int_t^{t+h} x(s)\,ds, \quad t\in[-h,T],$$

where, for $t \ge T$, we extend the process $x(s)$ by a constant random variable due to continuity. It is obvious that the process $x_\mu(t)$ has smooth trajectories with probability 1, and

$$\dot{x}_\mu(t) = \frac{1}{\mu}\big[x(t+h)-x(t)\big].$$

Using the mean value theorem, we have

$$\sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x_\mu(t)|^2 = \sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x(\theta)|^2,$$

where $\theta=\theta(\omega)$ is a random variable with $\theta\in[t,t+\mu]$. Therefore,

$$\sup_{t\in[-h,T]}\mathbb{E}\,|x(t)-x(\theta)|^2\le\sup_{t\in[-h,T]}\mathbb{E}\sup_{s\in[t,t+\mu]}|x(t)-x(s)|^2\le C\big(T,\|\phi\|,h,\mu\big)\longrightarrow 0,\ \ \mu\to 0,$$

by the lemma on the modulus of continuity. Let $y_j(t)=x(t-\frac{hj}{m})$, and introduce the differences

$$N_j(t)=\mathbb{E}\,|y_j(t)-z_j(t)|^2,\ \ j=\overline{0,m},$$

where $z_j(t)$ are solutions of system (1.2). Note that by the classical existence and uniqueness theorems for the Cauchy problem for systems of stochastic equations without delay, considering conditions (1) and (2), we obtain that system (1.2) for each natural m has a unique strong solution defined on $[0,T]$. The proof now proceeds through several steps.

Step 1. We decompose (1.2) into two systems and represent its solution as a sum:

$$z_j(t)=z_j^{(1)}(t)+z_j^{(2)}(t),$$

where $z_j^{(1)}$ is the solution of the system

$$\begin{cases}\dfrac{h}{m}\,\dot z_0^{(1)}=x(t)-z_1^{(1)}(t),\\ \dfrac{h}{m}\,\dot z_j^{(1)}=z_{j-1}^{(1)}(t)-z_j^{(1)}(t),\ \ j=\overline{1,m},\\ z_j^{(1)}(0)=x\Big(-\dfrac{hj}{m}\Big),\end{cases}$$

and $z_j^{(2)}$ is the corresponding solution of

$$\begin{cases}\dfrac{h}{m}\,\dot z_1^{(2)}=-z_1^{(2)}(t)+z_0(t)-x(t),\\ \dfrac{h}{m}\,\dot z_j^{(2)}=z_{j-1}^{(2)}(t)-z_j^{(2)}(t),\ \ j=\overline{1,m},\\ z_j^{(2)}(0)=0.\end{cases}$$

For brevity, denote the norm

$$\|\xi\|_2=\sqrt{\mathbb{E}\,\xi^2}\,.$$

Then,

$$\sup_{t\in[0,T]}\Big\|x\Big(t-\frac{hj}{m}\Big)-z_j(t)\Big\|_2\le\sup_{t\in[0,T]}\|y_j(t)-z_j^{(1)}(t)\|_2+\sup_{t\in[0,T]}\|z_j^{(2)}(t)\|_2. \tag{2.4}$$

Step 2. At this step, we estimate the first term in (2.4). We show that the following inequality holds:

$$\sup_{t\in[0,T]}\|y_j(t)-z_j^{(1)}(t)\|_2\le\alpha\Big(T,\|\phi\|,h,\frac{h}{m}\Big)\longrightarrow 0,\ \ m\to\infty. \tag{2.5}$$

Step 3. To estimate the second term in (2.4), using the method of variation of constants, we obtain the inequality

$$\mathbb{E}\,z_1^{(2)}(t)\le\sup_{t\in[0,T]}\mathbb{E}\,|z_0(t)-x(t)|^2=\mathbb{E}\,N_0(t).$$

Step 4. We estimate $z_0(t) - x(t)$. From the Lipschitz condition (2), we have

$$\mathbb{E}\, N_0(t) \le 2(T+1) \int_0^t \left[\mathbb{E}\,|x(s) - z_0(s)|^2\right] + \mathbb{E}\left|\int_{-h}^{0} x(s+\theta)\, d\theta - \frac{h}{m}\sum_{j=0}^{m-1} z_j(s)\right|^2 dt.$$

However,

$$\int_{-h}^{0} x(s+\theta)\, d\theta - \frac{h}{m}\sum_{j=1}^{m} z_j(s) = \sum_{j=0}^{m-1} \frac{h}{m}\, x(s+\rho_j) - \frac{h}{m}\sum_{j=1}^{m} z_j(s),$$

where $\rho_j(\omega) \in (-\frac{h}{m}\,(j+1), -\frac{h}{m}\, j)$ by the mean value theorem. Then

$$\frac{h}{m}\sum_{j=0}^{m-1}\Big(s(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big) + \frac{h}{m}\sum_{j=1}^{m}\Big(x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big),$$

so

$$\begin{aligned}\mathbb{E}\, N_0(t) &\le 2(T+1)\\ &\times \int_0^t \left[\mathbb{E}\, N_0(s) + \frac{2h^2}{m^2}\left(\mathbb{E}\left(\sum_{j=1}^{m}\Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|\right)^2 + \mathbb{E}\left(\sum_{j=1}^{m}\Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|\right)^2\right)\right] ds. \qquad (2.6)\end{aligned}$$

Let us estimate the sums in inequality (2.6). For the first of them, by the lemma on the modulus of continuity, we have

$$\mathbb{E}\left(\sum_{j=1}^{m}\Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|\right)^2 \le m\sum_{j=1}^{m}\mathbb{E}\,\Big|x(s+\rho) - x\Big(s - \frac{hj}{m}\Big)\Big|^2 \le m^2 C\Big(T, \|\phi\|, l, \frac{h}{m}\Big). \qquad (2.7)$$

For the second sum, we have the estimate

$$\mathbb{E}\left(\sum_{j=1}^{m}\Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|\right)^2 \le m\sum_{j=1}^{m}\mathbb{E}\,\Big|x\Big(s - \frac{hj}{m}\Big) - z_j(s)\Big|^2 \le m^2\alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big), \qquad (2.8)$$

under estimate (2.5). Then, from (2.6)–(2.8), we get

$$\mathbb{E}\, N_0(t) \le 2(T+1)L\int_0^t \mathbb{E}\, N_0(s)\, ds + 2(T+1)T2h^2\Big(C\Big(T, \|\phi\|, l, \frac{h}{m}\Big) + \alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big)\Big).$$

From this, using Gronwall's lemma, we obtain the estimate

$$\mathbb{E}\, N_0(t) \le 2(T+1)T2h^2\Big(C\Big(T, \|\phi\|, l, \frac{h}{m}\Big) + \alpha^2\Big(T, \|\phi\|, h, \frac{h}{m}\Big)\Big)e^{2(T+1)LT}.$$

This last estimate proves the theorem. □

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