Initial Value Problem on an Infinite Interval for First Order Advanced Differential Equations

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In the present report, we give conditions guaranteeing, respectively, the existence and uniqueness of a solution to the Cauchy initial value problem

$$u'(t)=f(t,u(\tau(t))), \tag{1}$$

$$u(0)=c_0, \tag{2}$$

defined on the interval $\mathbb{R}_+=[0,+\infty[$.

Everywhere below it is assumed that c_0 is a positive number, $\tau:\mathbb{R}_+\to\mathbb{R}_+$ is a measurable and bounded on every finite interval contained in $\mathbb{R}_+$ function, satisfying the inequality

$$\tau(t)\geq t \ \text{ for } \ t\in\mathbb{R}_+, \tag{3}$$

while $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is a function from the Carathéodory space.

We use the following notation and definitions.

$L_{loc}(\mathbb{R}_+)$ is the space of real functions, defined on $\mathbb{R}_+$, which are Lebesgue integrable on every finite interval contained in $\mathbb{R}_+$;

$$f^*(t,y)=\max\big\{|f(t,x)|:\ |x|\leq y\big\} \ \text{ for } \ t\in\mathbb{R}_+, \ \ y>0;$$

$$f_*(t,y)=\min\big\{|f(t,x)|:\ y\leq x\leq c_0\big\} \ \text{ for } \ t\in\mathbb{R}_+, \ \ 0<y\leq c_0.$$

We say that a function $f:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ belongs to the Carathéodory space if $f(t,\cdot\,):\mathbb{R}\to\mathbb{R}$ is continuous for almost all $t\in\mathbb{R}_+$,

$$f(\,\cdot\,,x)\in L_{loc}(\mathbb{R}_+) \ \text{ for } \ x\in\mathbb{R},$$

and

$$f^*(\,\cdot\,,y)\in L_{loc}(\mathbb{R}_+) \ \text{ for } \ y\in\mathbb{R}_+.$$

A solution to problem (1), (2) is sought in the space of functions $u:\mathbb{R}_+\to\mathbb{R}$ which are absolutely continuous on every finite interval contained in $\mathbb{R}_+$.

The solution u to problem (1), (2) is said to be vanishing at infinity if

$$\lim_{t\to+\infty}u(t)=0.$$

If $\tau(t)\equiv t$ and on the set $\mathbb{R}_+\times\mathbb{R}$ the inequality

$$|f(t,x)|\leq g(t)|x|+h(t)$$

is fulfilled, where $g,\, h \in L_{loc}(\mathbb{R}_+)$, then, according to the Wintner theorem (see [5]), problem (1), (2) has at least one solution in $\mathbb{R}_+$ and each maximally extended to the right solution to this problem is defined on $\mathbb{R}_+$.

In the general case, when inequality (3) holds and $\tau(t) \not\equiv t$, Wintner's condition does not guarantee the solvability of problem (1), (2).

Moreover, the following proposition is valid.

Proposition 1. *Let the function f admit the estimate*

$$f(t,x) \le -g(t)|x| - h(t) \ \ \textit{for} \ \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R},$$

where $g,\, h \in L_{loc}(\mathbb{R}_+)$ are nonnegative functions. If, moreover, the function τ is nondecreasing and the inequalities

$$\limsup_{t\to+\infty} \int\limits_t^{\tau(t)} g(s)\, ds > 1, \tag{4}$$

$$\int\limits_0^{+\infty} h(s)\, ds > c_0 \tag{5}$$

hold, then problem (1), (2) *has no solution.*

Proposition 2. *Let the function f admit the estimate*

$$f(t,x) \ge g(t)|x| \ \ \textit{for} \ \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R},$$

where $g \in L_{loc}(\mathbb{R}_+)$ is a nonnegative function. If, moreover, the function τ is nondecreasing and inequality (4) *holds, then problem* (1), (2) *has no solution.*

As examples, we consider the differential equations

$$u'(t) = -g(t)|u(\tau(t))| - h(t), \tag{6}$$

$$u'(t) = g(t)|u(\tau(t))| + h(t), \tag{7}$$

where $g,\, h \in L_{loc}(\mathbb{R}_+)$ are nonnegative functions.

Propositions 1 and 2 yield the following corollary.

Corollary 1. *If inequalities* (4) *and* (5) *hold* (*inequality* (4) *holds*), *then problem* (6), (2) (*problem* (7), (2)) *has no solution.*

It is easy to see that if for some $r > 0$ the function $f^*(\,\cdot\,, r)$ is integrable on $\mathbb{R}_+$ and satisfies the inequality

$$c_0 + \int\limits_0^{+\infty} f^*(t,r)\, dt \le r,$$

then problem (1), (2) has at least one solution.

The above Propositions 1 and 2, containing the sufficient conditions for the unsolvability of problem (1), (2), concern the case, where

$$\int\limits_0^{+\infty} f^*(t,y)\, dt = +\infty \ \ \text{for} \ \ y > 0.$$

In this case the questions on the solvability and unique solvability of the above mentioned problem still remain unstudied (see, for example, [1–4,6] and the references therein). The results we obtained fill this gap to some extent.

The following theorem is valid.

Theorem 1. *If*

$$f(t,0)=0,\ \ f(t,x)\le 0 \ \text{ for } \ t>0,\ \ x>0, \tag{8}$$

then problem (1), (2) *has at least one nonnegative solution. And if along with* (8) *the condition*

$$\int\limits_0^{+\infty} f_*(t,y)\,dt=+\infty \ \text{ for } \ 0<y\le c_0 \tag{9}$$

holds, then that solution is vanishing at infinity.

Sketch of the Proof of Theorem 1. Since the function τ is bounded on every finite interval, there exists a sequence of positive numbers $(a_k)_{k=1}^{+\infty}$ such that for every natural k in the interval $[0,a_k]$ the inequality

$$1+\tau(t)<a_{k+1}$$

holds.

Denote

$$\tau_k(t)=\begin{cases}\tau(t)+\frac{1}{k} & \text{for } \ 0\le t\le a_k,\\ a_{k+1} & \text{for } \ a_k<t\le a_{k+1},\end{cases}$$

and for each k in the interval $[0,a_{k+1}]$ consider the Cauchy problem

$$u'(t)=f(t,u(\tau_k(t))), \tag{10}$$

$$u(a_{k+1})=x, \tag{11}$$

where $x\in\mathbb{R}_+$.

Based on condition (8), it can be proved that for every $x\in\mathbb{R}_+$ problem (10), (11) in the interval $[0,a_{k+1}]$ has a unique solution $u(\,\cdot\,;x)$ which continuously depends on the parameter x. Also,

$$u(t;0)\equiv 0,$$

and

$$u(t,x)\ge x \ \text{ for } \ 0\le t\le a_{k+1},\ \ x>0.$$

Since

$$u(0;0)=0,\ \ \lim_{x\to+\infty}u(0;x)=+\infty,$$

there exists a positive number x_k such that

$$u(0;x_k)=c_0.$$

Therefore, for every natural k problem (10), (2) has a solution u_k such that

$$0<u_k(t)\le c_0 \ \text{ for } \ 0\le t\le a_{k+1},$$

$$|u_k'(t)|\le f^*(t,c_0) \ \text{ for almost all } \ t\in(0,a_{k+1}).$$

According to these last two inequalities and the Arzelà-Ascoli lemma, the sequence $(u_k)_{k=1}^{+\infty}$ contains a subsequence $(u_{k_m})_{m=1}^{+\infty}$ which is uniformly converging on every finite interval contained in $\mathbb{R}_+$. Evidently, the function

$$u(t) = \lim_{m\to+\infty} u_{k_m}(t) \ \text{ for } \ t \in \mathbb{R}_+$$

is a nonnegagive, nonincreasing solution to problem (1), (2). In addition,

$$0 \le u(t) \le c_0 - \int_0^t f_*(s,\delta)\,ds \ \text{ for } \ t \in \mathbb{R}_+,$$

where

$$\delta = \lim_{t\to+\infty} u(t).$$

From the last inequality it follows that if condition (9) is satisfied, then $\delta = 0$, i.e. the solution u is vanishing at infinity. □

Remark 1. If condition (8) holds and the function τ satisfies a more stringent condition than (3)

$$\operatorname{ess\,inf}\left\{\tau(s) - s : \ 0 \le s \le t\right\} > 0 \ \text{ for } \ t > 0, \tag{12}$$

then every nonnegative solution to problem (1), (2) is positive. It should be noted that condition (12) cannot be replaced by the condition

$$\operatorname{ess\,inf}\left\{\tau(s) - s : \ t_0 \le s \le t\right\} > 0 \ \text{ for } \ t \ge t_0,$$

no matter how small the positive number t_0 is. Indeed, if

$$0 < \lambda < 1, \ \ \alpha = (1-\lambda)^{-1}, \ \ p = \alpha\, c_0^{1-\lambda}/t_0,$$
$$\tau(t) = \begin{cases} t & \text{for } \ 0 \le t < t_0, \\ t+1 & \text{for } \ t \ge t_0, \end{cases}$$

then the function

$$u(t) = \begin{cases} c_0(1 - t/t_0)^\alpha & \text{for } \ 0 \le t < t_0, \\ 0 & \text{for } \ t \ge t_0 \end{cases}$$

is a nonnegative but not positive solution to the differential equation

$$u'(t) = -p|u(\tau(t))|^\lambda \operatorname{sgn}(u(\tau(t)))$$

under the initial condition (2).

Remark 2. According to Proposition 1, condition (8) in Theorem 1 cannot be replaced by the condition

$$f(t,x) \le 0 \ \text{ for } \ t \in \mathbb{R}_+, \ \ x \in \mathbb{R}_+,$$

i.e. the requirement

$$f(t,0) \equiv 0$$

cannot be removed from (8).

As an example, we consider the differential equation

$$u'(t) = -\sum_{i=1}^{n} p_i(t) f_i(u(\tau(t))), \tag{13}$$

where $p_i \in L_{loc}(\mathbb{R}_+)$ $(i = 1, \dots, n)$, and $f_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous functions.

Theorem 1 implies the following corollary.

Corollary 2. *Let the functions p_i and f_i $(i = 1, \dots, n)$ be nonnegative in $\mathbb{R}_+$, and*

$$f_i(0) = 0 \ \ (i = 1, \dots, n). \tag{14}$$

Then problem (13), (2) has at least one nonnegative solution. And if along with the above conditions the following conditions

$$\int_0^{+\infty} p_m(t)\,dt = +\infty, \quad f_m(x) > 0 \ \ \text{for} \ \ x > 0 \tag{15}$$

are satisfied for some $m \in \{1, \dots, n\}$, then that solution is vanishing at infinity.

So far we have been able to prove the unique solvability of problem (1), (2) only in the case where τ is a step function of the type

$$\tau(t) = t_k \ \ \text{for} \ \ t_{k-1} < t \le t_k \ \ (k = 1, 2, \dots), \tag{16}$$

where $t_0 = 0$, and $(t_k)_{k=1}^{+\infty}$ is some increasing and unbounded sequence of positive numbers.

In particular, the following theorem is proved.

Theorem 2. *Let the function τ have the form (16), and let the function f be nonincreasing in the second argument and satisfy the equality*

$$f(t, 0) = 0 \ \ \text{for} \ \ t \in \mathbb{R}_+.$$

Then problem (1), (2) has a unique solution, admitting the representation

$$u(t) = c_k - \int_t^{t_k} f(s, c_k)\,ds \ \ \text{for} \ \ t_{k-1} \le t \le t_k \ \ (k = 1, 2, \dots),$$

where $(c_k)_{k=1}^{+\infty}$ is a sequence of positive numbers such that

$$c_k - \int_{t_{k-1}}^{t_k} f(s, c_k)\,ds = c_{k-1} \ \ (k = 1, 2, \dots).$$

Corollary 3. *Let the function τ have the form (16), let the functions p_i $(i = 1, \dots, n)$ be nonnegative, and let the functions f_i $(i = 1, \dots, n)$ be nonnegative and satisfy equalities (14). Then problem (13), (2) has a unique solution, admitting the representation*

$$u(t) = c_k + \sum_{i=1}^{n} f_i(c_k) \int_t^{t_k} p_i(s)\,ds \ \ \text{for} \ \ t_{k-1} \le t \le t_k \ \ (k = 1, 2, \dots),$$

where $(c_k)_{k=1}^{+\infty}$ is a sequence of positive numbers such that

$$c_k + \sum_{i=1}^{n} f_i(c_k) \int_{t_{k-1}}^{t_k} p_i(s)\,ds = c_{k-1} \ \ (k = 1, 2, \dots).$$

Remark 3. It is evident that if the conditions of Theorem 2 (of Corollary 3) are satisfied, then a solution to problem $(1),(2)$ (to problem $(13),(2)$) is positive and vanishes at infinity if

$$\int\limits_0^{+\infty} f(t,y)\,dt = -\infty \ \text{ for } \ y > 0$$

(if for some $m \in \{1,\dots,n\}$ conditions (15) are satisfied).

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