Positive Solutions of Superlinear Mean Curvature Problems in Planar Domains: Topological Versus Variational Approach

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It is a rather common belief that, for analysing problems having a variational structure, variational methods are more powerful and give better results than topological ones. In this note we exhibit a class of variational problems for which the topological approach reveals instead much more effective. Namely, we look for positive regular solutions of the prescribed mean curvature problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded planar domain, the function f grows superlinearly, and λ is a real parameter.

In a recent paper Figueiredo and Rădulescu proved the existence of positive solutions for (1) assuming that f is a superlinear function having critical exponential growth at infinity with respect to the Moser–Trudinger inequality in $\mathbb{R}^2$, also providing in their article detailed history, motivations, and references concerning this topic. Precisely, in [6] they proved the following result.

Theorem 1 ([6, Theorem 1.1])**.** *Assume that*

(h_1) *Ω is a bounded domain in $\mathbb{R}^2$ having a smooth boundary $\partial\Omega$,*

(h_2) *$f : \mathbb{R} \to \mathbb{R}$ is a continuous function,*

(h_3) *there exists $\alpha_0 > 0$ such that*

$$\lim_{s\to+\infty} \frac{f(s)}{\exp(\alpha s^2)} = 0 \ \text{ for } \alpha > \alpha_0 \ \text{ and } \ \lim_{s\to+\infty} \frac{f(s)}{\exp(\alpha s^2)} = +\infty \ \text{ for } \alpha < \alpha_0,$$

(h_4) $\lim\limits_{s\to 0} \frac{f(s)}{s} = 0,$

(h_5) *the function $s \mapsto \frac{f(s)}{s}$ is increasing in $(0, +\infty)$,*

(h_6) *there exists $p > \frac{32}{7}\sqrt{2} \approx 6.465$ and $\lambda > 0$ such that, for all $s > 0$,*

$$f(s) \ge s^{p-1},$$

(h_7) *for all $s > 0$,*

$$f(s)s \ge p \int_0^s f(t)\,dt,$$

where p is the same exponent as in (h_6).

Then, there exists a constant $\lambda^* > 0$ such that problem (1) *has at least one positive weak solution $u \in C^1(\overline{\Omega})$ provided that $\lambda > \lambda^*$.*

Example. A paradigmatic model for f is provided by the function

$$f(s) = (s^+)^{p-1} \exp(\alpha_0 s^2), \tag{2}$$

where $p > 2$ and $\alpha_0 > 0$ are given exponents and $s^+ = \max\{s, 0\}$.

The proof of Theorem 1 produced in [6] strongly exploits the variational structure of problem (1) and cleverly combines a truncation argument [8], along with the use of the Nehari manifold method [4], Moser iteration techniques [10], and Stampacchia estimates [9].

The aim of this note is to show that Theorem 1 can be significantly improved, as well as extended in several directions, through a few minor modifications of a result recently established by Omari and Sovrano in [11], namely Theorem 2.2 therein. Like there, we consider here a more general problem than (1), specifically

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(x, u, \nabla u; \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where λ plays the role of a parameter and

(k_1) *Ω is a bounded domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$ of class C^2,*

(k_2) *$f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ is a continuous function.*

The following notion of solution for problem (3) is adopted in the sequel.

Definition. By a solution of (3) we mean a function $u \in W^{2,q}(\Omega)$ for all finite $q \geq 1$, which satisfies the equation almost everywhere in Ω and the boundary condition everywhere on $\partial\Omega$. A solution u is said strictly positive if $u(x) > 0$ in Ω and $\partial_\nu u(x) < 0$ on $\partial\Omega$, $\nu = \nu(x)$ being the unit outer normal to Ω at $x \in \partial\Omega$.

We also introduce the set

$$\mathscr{S} = \Big\{(u, \lambda) \in C^1(\overline{\Omega}) \times (0, +\infty) : \ \ u \text{ is a strictly positive solution of (3) for some } \lambda > 0\Big\}$$

and we endow $\mathscr{S}$ with the product topology of $C^1(\overline{\Omega}) \times \mathbb{R}$.

Since the right-hand side of the equation in (3) also depends on the gradient of the solution, the variational structure of this problem may be lost, thus ruling out the use of critical point theory in any existence proof. As a consequence, Theorem 2.2 in [11] is proven via topological methods and perturbative techniques. Specifically, assuming a structure condition on f expressed by (k_3) below, the quasilinear problem (3) is first interpreted, when λ is large, as a small perturbation of a limiting semilinear problem for which the existence of a priori bounds for the possible positive solutions is known from [7] or [1, 2, 5]. Then, the existence of a connected branch $\mathscr{C}$ of positive solutions $(u, \lambda) \in \mathscr{S}$ of (3), bifurcating from 0 as $\lambda \to +\infty$, is eventually established by relying on a fixed point index calculation inspired to [1] and by using a general Leray-Schauder continuation theorem on metric ANRs stated in [3].

Hence, the following two results can be obtained. The first one, Theorem 2 below, improves and generalises the result in [6]. Indeed, with respect to Theorem 1, Theorem 2 allows to remove, as far as problem (1) is concerned, assumptions (h_3), (h_5), (h_7), which therefore reveal to be of a merely technical nature related to the method of proof, as well as to extend the range of the admissible exponents p from the interval $(\frac{32}{7}\sqrt{2}, +\infty)$, considered in Theorem 1, to the natural one $(2, +\infty)$. Accordingly, eliminating (h_3) permits f to exhibit a totally arbitrary behaviour at infinity. In particular, when considering the model given by (2), no additional restrictions on $p \in (2, +\infty)$ are needed.

Theorem 2. *Assume* (k_1), (k_2),

(k_3) $f(x, s, \xi; \lambda) = \lambda g(x, s, \xi) + h(x, s, \xi)$, *where*

$(k_{3,1})$ $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function for which there exist a finite exponent* $p > 2$ *and a function* $w \in C^0(\overline{\Omega})$ *such that*

$$\lim_{(s,\xi)\to(0,0)} \frac{g(x, s, \xi)}{|s|^{p-2}s} = w(x) \text{ uniformly in } \overline{\Omega},$$

$(k_{3,2})$ $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *is a continuous function such that*

$$\lim_{(s,\xi)\to(0,0)} \frac{h(x, s, \xi)}{s} = 0 \text{ uniformly in } \overline{\Omega},$$

(k_4) $w(x) > 0$ *for all* $x \in \overline{\Omega}$.

Then, there exist a constant $\lambda^* \geq 0$ *and a connected component* $\mathscr{C}$ *of* $\mathscr{S}$ *such that* $\mathrm{proj}_{\mathbb{R}}\mathscr{C} = (\lambda^*, +\infty)$ *and*

$$\lim_{\lambda\to+\infty} \max\left\{\|u\|_{C^1} : (u, \lambda) \in \mathscr{C}\right\} = 0.$$

Proof. The proof of Theorem 2 essentially exploits the same argument we developed for establishing Theorem 2.2 in [11], under the choice $\mu = 0$ in assumption (H_3) therein. While Steps 3.1, 3.3, and 3.4 in [11] remain unchanged, a few differences occur in Step 3.2 in order to get the conclusions of Lemmas 3.5 and 3.7 in [11] for the semilinear problem

$$\begin{cases} -\Delta v = \sigma v + w(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where w and p come from $(k_{3.1})$ and $\sigma \in \mathbb{R}$ is a given constant. Indeed, if (k_4) holds, the non-existence result of Lemma 3.5 in [11] can be obtained just by testing (4) against a positive principal eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, whereas the a priori estimates of Lemma 3.7 in [11] now follow directly from Theorem 1.1 in [7] and the linear elliptic regularity theory. □

The second result is a variant of Theorem 3 where the function w, considered in $(k_{3.1})$ and (k_4), is allowed to change sign, provided that its nodal domains satisfy certain conditions, introduced in [1, 5] and exploited in [11] in a context similar to the present one. Namely, we assume that

(k_5) $w \in C^2(\overline{\Omega})$,

(k_6) $\Omega^+ = \{x \in \Omega : w(x) > 0\} \neq \varnothing$, $\Omega^- = \{x \in \Omega : w(x) < 0\} \neq \varnothing$, *and* $\Omega^0 = \{x \in \Omega : w(x) = 0\}$ *is such that* $\partial\Omega^0 \subset \Omega$; *the boundaries* $\partial(\mathrm{int}\,\Omega^0)$, $\partial\Omega^+$, *and* $\partial\Omega^-$ *are of class* C^2; Ω^0 *has a finite number of connected components, that we denote by* D_i^+, D_j^-, *and* $D_k^\pm$.

Hence, we can represent Ω^0 in the form

$$\Omega^0 = \bigcup_i D_i^+ \cup \bigcup_j D_j^- \cup \bigcup_k D_k^\pm,$$

where the components D_i^+, D_j^-, and $D_k^\pm$ are supposed to satisfy:

(k_7) *for each* i, $\partial D_i^+ \subset \overline{\Omega}^+$ *and there exist* $\gamma_{1,i} > 0$, *a neighbourhood* U_i^+ *of* ∂D_i^+, *and* $\omega_i^+ : \overline{U}_i^+ \to]0,+\infty[$ *such that*

$$w(x) = \omega_i^+(x)\operatorname{dist}(x, \partial D_i^+)^{\gamma_{1,i}} \ \text{ for all } \ x \in \Omega^+ \cap U_i^+,$$

(k_8) *for each* j, $\partial D_j^- \subset \overline{\Omega}^-$ *and there exist* $\gamma_{2,j} > 0$, *a neighbourhood* U_j^- *of* ∂D_j^-, *and* $\omega_j^- : \overline{U}_j^- \to]-\infty,0[$ *such that*

$$w(x) = \omega_j^-(x)\operatorname{dist}(x, \partial D_j^-)^{\gamma_{2,j}} \ \text{ for all } \ x \in \Omega^- \cap U_j^-,$$

(k_9) *for each* k, *the following alternative holds*

$(k_{9,1})$ *if* $\operatorname{int}(D_k^\pm) = \varnothing$, *then*

- $\partial D_k^\pm = \Gamma_k$ *are of class* C^2,
- *there exist* $\gamma_{3,k} > 0$, *a neighbourhood* U_k^+ *of* Γ_k, *and* $\omega_k^+ : \overline{U}_k^+ \to]0,+\infty[$ *such that*
$$w(x) = \omega_k^+(x)\operatorname{dist}(x, \Gamma_k)^{\gamma_{3,k}} \ \text{ for all } \ x \in \Omega^+ \cap U_k^+, \tag{5}$$
- *there exist* $\gamma_{4,k} > 0$, *a neighbourhood* U_k^- *of* Γ_k, *and* $\alpha_k^- : \overline{U}_k^- \to]-\infty,0[$ *such that*
$$w(x) = \omega_k^-(x)\operatorname{dist}(x, \Gamma_k)^{\gamma_{4,k}} \ \text{ for all } \ x \in \Omega^- \cap U_k^-, \tag{6}$$

$(k_{9,2})$ *if* $\operatorname{int}(D_k^\pm) \neq \varnothing$, *then*

- $\partial D_k^\pm = \Gamma_k^+ \cup \Gamma_k^-$, *with* $\Gamma_k^+ \cap \Gamma_k^- = \varnothing$, $\Gamma_k^+ \subset \overline{\Omega}^+$, $\Gamma_k^- \subset \overline{\Omega}^-$ *of class* C^2,
- *there exist* $\gamma_{3,k} > 0$, *a neighbourhood* U_k^+ *of* Γ_k^+, *and* $\omega_k^+ : \overline{U}_k^+ \to]0,+\infty[$ *satisfying condition* (5),
- *there exist* $\gamma_{4,k} > 0$, *a neighbourhood* U_k^- *of* Γ_k^-, *and* $\omega_k^- : \overline{U}_k^- \to]-\infty,0[$ *satisfying condition* (6).

Let us define

$$D^+ = \bigcup_i D_i^+, \quad D^- = \bigcup_j D_j^-, \quad D^\pm = \bigcup_k D_k^\pm.$$

The set D^+ (respectively, D^-) is constituted by the connected components D_i^+ (respectively, D_j^-) of Ω^0, that are surrounded by regions of positivity (respectively, negativity) of w. Instead, $D^\pm$ is constituted by the connected components D_j^- of Ω^0, that are in between a region of positivity and one of negativity of w. $D^\pm$ can be either a "thin" nodal set, like when assuming condition (7) below, or a "thick" nodal set, that is, a set of positive measure. Figure 1, taken from [11], illustrates an admissible nodal configuration for the function w.

Remark. Suppose that the function $w \in C^2(\overline{\Omega})$ satisfies the following condition introduced in [2]:

$$\Omega^+ \neq \varnothing, \quad \Omega^- \neq \varnothing, \quad \Omega^0 = \overline{\Omega}^+ \cap \overline{\Omega}^- \subset \Omega, \ \text{ and } \ \nabla w(x) \neq 0 \ \text{ for all } \ x \in \Omega^0. \tag{7}$$

In this case, D^+, D^-, and $\operatorname{int}(D^\pm)$ are all empty sets and assumption $(H_{9.1})$ holds. Indeed, let Γ_k be a connected component of Ω^0. Then, condition (5) is satisfied taking $\gamma_{1,k} = 1$ and $\omega_k^+ : U_k^+ \to]0,+\infty[$ defined by

$$\omega_k^+(x) = \begin{cases} -|\nabla w(x)| & \text{if } x \in U_k^+ \setminus \Gamma_k, \\ \dfrac{w(x)}{\operatorname{dist}(x,\Gamma_k)} & \text{if } x \notin U_k^+ \setminus \Gamma_k, \end{cases}$$

where U_k^+ is a suitable tubular neighbourhood of Γ_k. Condition (6) can be verified similarly.

Theorem 3. *Assume* (k_1)–(k_3) *and* (k_5)–(k_9). *Then, the same conclusions of Theorem 2 hold.*

Proof. Theorem 3 is a direct consequence of Theorem 2.2 in [11], when the choice $\mu = 0$ in assumption (H_3) therein is made. □

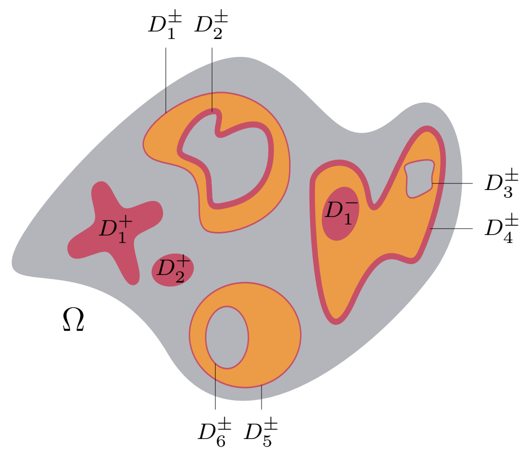

Figure 1. Example of an admissible nodal configuration for the weight function w. The sets Ω^+, Ω^0, and Ω^- are respectively the union of the grey, the red, and the yellow regions. Here, $D^+ = \bigcup_{i=1}^{2} D_i^+$, $D^{\pm} = \bigcup_{k=1}^{6} D_k^{\pm}$, and $D^- = D_1^-$.

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