On Admissible Perturbations of 3D Quadratic ODE System with an Infinite Number of Limit Cycles

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1 Introduction

The qualitative study of parameter depending systems of autonomous ordinary differential equations requires the study of limit sets and their bifurcations. With respect to applications, equilibria, limit cycles, homoclinic orbits and invariant manifolds play a crucial role. In case of planar autonomous systems a lot of methods for qualitative investigations has been established (see [1, 8]), nevertheless there are unsolved basic questions [9]. According to the results of H. Dulac, Yu. Ilyasenko and J. Ecale a planar polynomial autonomous system has only a finite number of limit cycles (individual finiteness) [7, 12]. The question for the maximum number of limit cycles of polynomial systems in dependence of the the degree of the polynomials and their bifurcations (Hilberts sixteenth problem) is still open. It has been proved that the cyclicity of a focus and of a period annulus (continuum of periodic orbits) of quadratic systems is three [27].

It is well known that already in the case of of quadratic polynomial three-dimensional autonomous systems new limit sets exist and new bifurcation scenarios occur [2, 10, 11, 14, 29]. The motivation for our work is to due two papers [4, 5] of V. Bulgakov devoted to the bifurcation of limit cycles in polynomial three-dimensional systems. In the first paper the focus is on Hopf bifurcation using the approach of Y. Bibikov [3] which essentially coincides with the center manifold approach [28]. In the second paper Bulgakov and Grin [5] proved that the system

$$\begin{aligned} \dot{x} &= a_0x - a_1y + a_2xy + a_3y^2 + a_4xz + a_5yz,\\ \dot{y} &= a_1x + a_0y - a_2x^2 - a_3xy + a_4yz - a_5xz,\\ \dot{z} &= 2(a_0z + a_4z^2); \ \ (x, y, z) \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $a_i \in \mathbb{R}$ $(i = \overline{0, 5})$ are system parameters, has infinitely many (continuum) limit cycles, which represent intersection curves of the family of invariant surfaces $z = (x^2 + y^2)/k$, $k \in \mathbb{R} \setminus \{0\}$ and the plane $z = -a_0/a_4$. But the results presented in the mentioned paper [5] are local. For system (1.1) under consideration, the non-local existence of an infinite number of limit cycles is proved in [25]. The focus was also on Hopf bifurcation of the reduced system on the invariant manifolds, but since these manifolds are no center manifolds this type of bifurcation did not explain the existence of limit cycles in the three-dimensional system. The underlying mechanism to generate a continuum of limit cycles is related to the existence of a period annulus.

In planar systems it is usual to define a limit cycles as an isolated periodic solution [1, 8]. To be able to speak about a continuum of limit cycles we have to use another definition of a limit cycle. In the monograph of C. Chicone [6] we find the following definition: A limit cycle Γ is a periodic orbit that is either the ω-limit set or the α-limit set of some point in the phase space with the periodic orbit removed. This monograph further emphasizes that the above definitions are not equivalent to each other in general, but they are equivalent in the case of real analytical systems. We will use Chicones definition in what follows.

The goal of our work is to perturb system (1.1) and find such a perturbed non-autonomous system in which the continuum of limit cycles retained its existence.

2 Preliminaries

In paper [15] V. I. Mironenko introduced the concept of a reflecting function to study the qualitative behavior of solutions of ODE systems. This function is now known as the *Mironenko reflecting function* (MRF) and has been successfully used to solve many problems in qualitative theory of ODE [16–18, 20].

ODE systems with the same MRF have the same translation operator (see [13]) on any interval $(-\beta, \beta)$, and 2ω-periodic ODE systems with the same MRF have the same mapping on the period $[-\omega, \omega]$ (Poincare mapping). Therefore, some qualitative properties (such as the existence of periodic solutions and their stability) of solutions of ODE systems that have the same MRF are common.

So it is advisable to look for perturbations that do not change the MRF (the so-called *admissible perturbations*) of known (well-studied) systems. If we manage to find admissible perturbations, then we thereby know which perturbations not change the qualitative properties of the solutions inherent in the solutions of the original unperturbed system.

For example, in papers [21–24, 26], admissible perturbations of various systems, such as the Lorenz-84 system, Langford system, generalized Langford system and Hindmarsh-Rose system, were obtained, and the qualitative properties of solutions of perturbed systems were also studied.

To search for admissible perturbations, we can use theorem from [19], which we formulate here in the form of the following lemma.

Lemma 2.1. *Let the vector functions* $\Delta_i(t,x)$ $(i=\overline{1,m}$*, where* $m \in \mathbb{N}$ *or* $m=\infty)$ *be solutions of the equation*

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X - \frac{\partial X}{\partial x} \Delta = 0 \tag{2.1}$$

and $\alpha_i(t)$ *be any scalar continuous odd functions. Then the MRF of any perturbed system of the form*

$$\dot{x} = X(t,x) + \sum_{i=1}^{m} \alpha_i(t) \Delta_i(t,x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n$$

is equal to the MRF of the system

$$\dot{x} = X(t,x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n. \tag{2.2}$$

3 Main results

For system (1.1), we look for admissible perturbations of the form $\Delta \cdot \alpha(t)$, where $\alpha(t)$ is an arbitrary continuous scalar odd function and

$$\Delta = \Bigg(\sum_{i+j+k=0}^{l} q_{ijk}\, x^i y^j z^k, \sum_{i+j+k=0}^{l} r_{ijk}\, x^i y^j z^k, \sum_{i+j+k=0}^{l} s_{ijk}\, x^i y^j z^k \Bigg)^{\mathrm{T}},$$

where $q_{ijk}, r_{ijk}, s_{ijk} \in \mathbb{R}$, $i,j,k,l \in \mathbb{N} \cup \{0\}$. For the polynomial Δ under consideration, relation (2.1) takes the form

$$\frac{\partial \Delta(x,y,z)}{\partial(x,y,z)}\, X(x,y,z) \equiv \frac{\partial X(x,y,z)}{\partial(x,y,z)}\, \Delta(x,y,z). \tag{3.1}$$

Substituting Δ into relation (3.1) and using the method of indefinite coefficients we obtain a system of equations for q_{ijk}, r_{ijk}, s_{ijk}. As a result, we obtained the following theorem.

Theorem 3.1. *Let $\alpha_i(t)$ $(i=\overline{1,3})$ be arbitrary scalar continuous odd functions. Then for $a_2=a_3=0$, the MRF of system* (1.1) *coincides with the MRF of the system*

$$\begin{aligned}
\dot{x} &= (a_0x - a_1y + a_4xz + a_5yz)(1+\alpha_1(t)) + x\alpha_2(t) + y\alpha_3(t),\\
\dot{y} &= (a_1x + a_0y + a_4yz - a_5xz)(1+\alpha_1(t)) + y\alpha_2(t) - x\alpha_3(t),\\
\dot{z} &= 2z(a_0 + a_4z)(1+\alpha_1(t)).
\end{aligned} \tag{3.2}$$

Proof. For $a_2=a_3=0$, the right-hand side of system (1.1) is

$$X = \Big(a_0x - a_1y + a_4xz + a_5yz, a_0y + a_1x + a_4yz - a_5xz, 2(a_0+a_4z)z\Big)^{\mathrm{T}}$$

and its Jacobi matrix is

$$\frac{\partial X}{\partial(x,y,z)} = \begin{pmatrix} a_0+a_4z & -a_1+a_5z & a_4x+a_5y \\ a_1-a_5z & a_0+a_4z & a_4y-a_5x \\ 0 & 0 & 2(a_0+2a_4z) \end{pmatrix}.$$

Let us write out the vector factors for $\alpha_i(t)$ from the right-hand side of system (3.2):

$$\Delta_1 = \begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz \\ a_0y + a_1x + a_4yz - a_5xz \\ 2z(a_0+a_4z) \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \quad \Delta_3 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}.$$

By successively checking identity (3.1) for each vector-multiplier Δ_i we will make sure that it is true. Let us show this, for example, for Δ_1. The Jacobi matrix is

$$\frac{\partial \Delta_1}{\partial(x,y,z)} = \begin{pmatrix} a_0+a_4z & -a_1+a_5z & a_4x+a_5y \\ a_1-a_5z & a_0+a_4z & a_4y-a_5x \\ 0 & 0 & 2(a_0+2a_4z) \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned}
\frac{\partial \Delta_1}{\partial(x,y,z)}\,X &= \begin{pmatrix} a_0+a_4z & -a_1+a_5z & a_4x+a_5y \\ a_1-a_5z & a_0+a_4z & a_4y-a_5x \\ 0 & 0 & 2(a_0+2a_4z) \end{pmatrix} \begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz \\ a_0y + a_1x + a_4yz - a_5xz \\ 2(a_0+a_4z)z \end{pmatrix} \\
&\equiv \begin{pmatrix} a_0+a_4z & -a_1+a_5z & a_4x+a_5y \\ a_1-a_5z & a_0+a_4z & a_4y-a_5x \\ 0 & 0 & 2(a_0+2a_4z) \end{pmatrix} \begin{pmatrix} a_0x - a_1y + a_4xz + a_5yz \\ a_0y + a_1x + a_4yz - a_5xz \\ 2z(a_0+a_4z) \end{pmatrix} = \frac{\partial X}{\partial(x,y,z)}\,\Delta_1.
\end{aligned}$$

Then the assertion of the theorem follows from Lemma 2.1. □

If, as usual, we consider non-negative time, then the requirement that the functions $\alpha_i(t)$ be odd is not essential, since they can be continued in an odd way continuously to the negative semi-axis of time (assuming that $\alpha_i(0)=0$).

In some cases, it is possible to find solutions of system (1.1) corresponding to limit cycles.

Lemma 3.1. *Suppose $a_2 = a_3 = 0$ and $a_4 \neq 0$. Then $\forall\, k \in \mathbb{R} \setminus \{0\}$ such that $a_0 k/a_4 < 0$, system* (1.1) *has a solution*

$$x(t) = \sqrt{\frac{-a_0 k}{a_4}}\, \cos\Big(\Big(\frac{a_0 a_5}{a_4} + a_1\Big)t\Big),$$
$$y(t) = \sqrt{\frac{-a_0 k}{a_4}}\, \sin\Big(\Big(\frac{a_0 a_5}{a_4} + a_1\Big)t\Big), \tag{3.3}$$
$$z(t) = -\frac{a_0}{a_4}$$

corresponding to the cycle $x^2 + y^2 = -a_0 k/a_4$, $z = -a_0/a_4$. Moreover, for $a_1 \neq -a_0 a_5/a_4$ this solution is $\frac{2\pi|a_4|}{|a_0 a_5 + a_1 a_4|}$-periodic.

The assertions of the lemma are proved by direct substitution of (3.3) into system (1.1).

The following theorem tells us about the cases when system (3.2) has infinitely many periodic solutions, and what is the character of the stability of these solutions.

Theorem 3.2. *Let $\alpha_i(t)$ $(i = \overline{1,3})$ be scalar twice continuously differentiable odd functions, $a_4 \neq 0$, $a_1 \neq -a_0 a_5/a_4$ and the right-hand side of system* (3.2) *be $\frac{2\pi|a_4|}{|a_0 a_5 + a_1 a_4|}$-periodic with respect to time t. Then $\forall\, k \in \mathbb{R} \setminus \{0\}$ such that $a_0 k/a_4 < 0$, a solution of system* (3.2)*, satisfying the initial conditions*

$$x\Big(\frac{-\pi|a_4|}{|a_0 a_5 + a_1 a_4|}\Big) = \sqrt{\frac{-a_0 k}{a_4}}, \quad y\Big(\frac{-\pi|a_4|}{|a_0 a_5 + a_1 a_4|}\Big) = 0, \quad z\Big(\frac{-\pi|a_4|}{|a_0 a_5 + a_1 a_4|}\Big) = -\frac{a_0}{a_4}, \tag{3.4}$$

is $\frac{2\pi|a_4|}{|a_0 a_5 + a_1 a_4|}$-periodic. Moreover, the character of stability of this solution and solution (3.3) *of system* (1.1) *with the same initial conditions* (3.4) *coincides.*

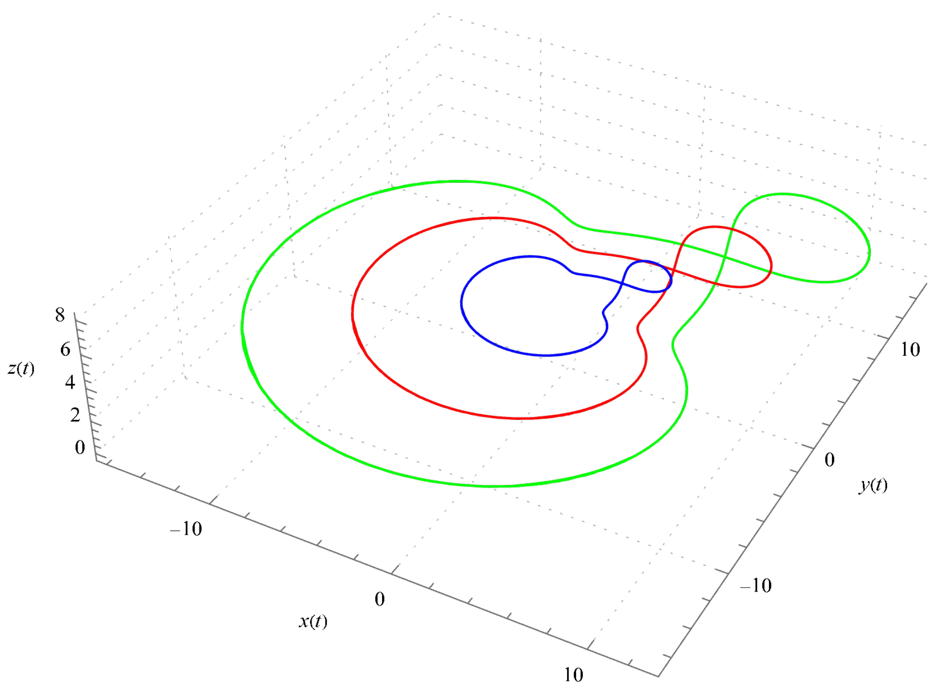

Figure 1. Phase portrait of periodic solutions of system (3.2) for $a_0 = 4$, $a_1 = 5$, $a_4 = -1$, $a_5 = 1$, $\alpha_i(t) = \sin(i \cdot t)$ $(i = \overline{1,3})$ and satisfying the initial conditions $x(-\pi) = 2\sqrt{k}$, $y(-\pi) = 0$, $z(-\pi) = 4$ (blue for $k = 1$, red for $k = 4$, and green for $k = 9$).

The proof of the theorem follows from the coincidence of the mappings over the period for systems (1.1) and (3.2).

Example. Let $a_0 = 4$, $a_1 = 5$, $a_2 = a_3 = 0$, $a_4 = -1$, $a_5 = 1$. Then, by Lemma 3.1, $\forall k \in (0, +\infty)$ system (1.1) has 2π-periodic solution (3.3). If $\alpha_i(t) = \sin(i \cdot t)$ $(i = \overline{1,3})$, then the right-hand side of system (3.2) is 2π-periodic. Therefore, by Theorem 3.2, $\forall\, k \in (0, +\infty)$ system (3.2) has 2π-periodic solution which satisfies the initial conditions $x(-\pi) = 2\sqrt{k}$, $y(-\pi) = 0$, $z(-\pi) = 4$ (see Figure 1).

4 Conclusion

Admissible perturbations were found for system (1.1) in the case when $a_2 = a_3 = 0$. The resulting perturbed non-autonomous systems have the same Mironenko reflecting function as the original unperturbed system. Solutions of different systems of ODEs with the same Mironenko reflecting function have many of the same qualitative properties. In particular, we proved that admissibly perturbed systems have infinitely many periodic solutions and that the character of their stability coincides with the character of stability of the corresponding solutions of unperturbed systems.

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