ODE-Systems with Boundary Inhomogeneous Conditions Containing Higher-Order Derivatives

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The study of systems of ODE is one of the part of investigations in modern analysis and its applications. Unlike Cauchy problems, the solutions to inhomogeneous boundary-value problems for differential systems may not exist and/or may not be unique. Therefore, the question about the solvability character of such problems is fundamental for the theory of differential equations.

The topic is most fully studied for linear ODE. Thus, Kiguradze [2] investigated the solutions of first order differential systems with general inhomogeneous boundary conditions of the form

$$y'(t) + A(t)y(t) = f(t), \;\; t \in (a,b), \quad By = c. \tag{1}$$

Here, the matrix-valued function $A(\,\cdot\,)$ is Lebesgue integrable over the finite interval (a,b); the vector-valued function $f(\,\cdot\,)$ belongs to $L((a,b);\mathbb{R}^m)$; the vector c pertains to $\mathbb{R}^m$, and B is an arbitrary linear continuous operator from the Banach space $C([a,b];\mathbb{R}^m)$ to $\mathbb{R}^m$ with $m \in \mathbb{N}$.

The boundary condition in (1) covers the main types of classical boundary conditions; namely: Cauchy problems, two-point and multipoint problems, integral and mixed problems. The Fredholm property with zero index was established for problems of the form (1). Moreover, the conditions for the problems to be well posed were obtained, and the limit theorem for their solutions was proved.

These results were further developed in a series of articles by Mikhailets and his colleagues. Specifically, they allow the differential system to have an arbitrary order $r \in \mathbb{N}$ and the boundary operator B to be any linear continuous operator from the space $C^{r-1}([a,b];\mathbb{C}^m)$ to $\mathbb{C}^{rm}$. They obtained conditions for the boundary-value problems to be well posed and proved limit theorems for solutions to these problems.

We arbitrarily choose a finite interval $(a,b) \subset \mathbb{R}$ and the following parameters:

$$n \in \mathbb{N} \cup \{0\}, \;\; \{m, r, l\} \subset \mathbb{N}, \;\; \text{and} \;\; 1 \leq p \leq \infty.$$

As usual,

$$W_p^{n+r}([a,b];\mathbb{C}) := \Big\{ y \in C^{n+r-1}([a,b];\mathbb{C}) : \;\; y^{(n+r-1)} \in AC[a,b], \;\; y^{(n+r)} \in L_p[a,b] \Big\}$$

is a complex Sobolev space; set $W_p^0 := L_p$. This space is Banach with respect to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r} \|y^{(k)}\|_p,$$

with $\|\cdot\|_p$ standing for the norm in the Lebesgue space $L_p([a,b];\mathbb{C})$. We need the Sobolev spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \ \text{ and } \ (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They respectively consist of vector-valued functions and matrix-valued functions whose elements belong to W_p^{n+r}. The norms in these spaces are defined to be the sums of the relevant norms in W_p^{n+r} of all elements of a vector-valued or matrix-valued function.

We preserve the same notation $\|\cdot\|_{n+r,p}$ for these norms. It will be clear from the context to which space (scalar or vector-valued or matrix-valued functions) relates the designation of the norm. The same concerns all other Banach spaces used in the sequel. Certainly, the above Sobolev spaces coincide in the $m=1$ case. If $p<\infty$, they are separable and have a Schauder basis.

Consider the linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \ \ t\in(a,b), \tag{2}$$

$$By = c. \tag{3}$$

Here, the matrix-valued functions $A_{r-j}(\cdot)\in(W_p^n)^{m\times m}$, vector-valued function $f(\cdot)\in(W_p^n)^m$, vector $c\in\mathbb{C}^l$, linear continuous operator

$$B : (W_p^{n+r})^m \to \mathbb{C}^l \tag{4}$$

are arbitrarily chosen; $y(\cdot)\in(W_p^{n+r})^m$ is unknown.

The boundary condition (3) consists of l scalar condition for system of m differential equations of r-th order, we representing vectors and vector-valued functions as columns.

A solution to the boundary-value problem (2), (3) is understood as a vector-valued function $y(\cdot)\in(W_p^{n+r})^m$ that satisfies both equation (2) (everywhere if $n\geq 1$, and almost everywhere if $n=0$) on (a,b) and equality (3).

If the parameter n increases, so does the class of linear operators (4). When $n=0$, this class contains all operators that set the general boundary conditions described above. Hence, the condition (3) with operator (4) is generic condition for this equation. It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order $\geq rm$. Thus, boundary conditions can contain derivatives whose order is greater than the order of the equation. If $l<rm$, then the boundary conditions are underdetermined. If $l>rm$, then the boundary conditions are overdetermined.

In case $1\leq p<\infty$, the linear continuous operator $B:(W_p^{n+r})^m\to\mathbb{C}^l$ admits the unique analytic representation

$$By = \sum_{i=0}^{n+r-1} \alpha_i\, y^{(i)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\,\mathrm{d}t, \ \ y(\cdot)\in(W_p^{n+r})^m,$$

for certain number matrices $\alpha_s\in\mathbb{C}^{rl\times m}$ and matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{rl\times m})$; as usual, $1/p+1/p'=1$. If $p=\infty$, this formula also defines a bounded operator $B:(W_\infty^{n+r})^m\to\mathbb{C}^{rl}$. However, there exist other operators of this class generated by integrals over finitely additive measures. Hence, unlike $p<\infty$, the case of $p=\infty$ contains additional analytical difficulties.

We rewrite the inhomogeneous boundary-value problem (2), (3) in the form of a linear operator equation

$$(L,B)y = (f,c).$$

Here, (L, B) is a bounded linear operator on the pair of Banach spaces

$$(L, B) : (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l, \tag{5}$$

which follows from the definition of the Sobolev spaces involved and from the fact that W_p^n is a Banach algebra.

Let E_1 and E_2 be Banach spaces. A linear bounded operator $T : E_1 \to E_2$ is called a Fredholm one if its kernel and co-kernel are finite-dimensional. If T is a Fredholm operator, then its range $T(E_1)$ is closed in E_2, and its index is finite

$$\operatorname{ind} T := \dim\ker T - \dim \frac{E_2}{T(E_1)} \in \mathbb{Z}.$$

Theorem 1. *The bounded linear operator* (5) *is a Fredholm one with index* $rm - l$.

The proof of Theorem 1 uses the well-known theorem on the stability of the index of a linear operator with respect to compact additive perturbations.

Theorem 1 naturally raises the question of finding d-characteristics of the operator (L, B), i.e. $\dim\ker(L, B)$ and $\dim\operatorname{coker}(L, B)$. This is a quite difficult task because the Fredholm numbers may vary even with arbitrarily small one-dimensional additive perturbations.

To formulate the following result, let us introduce some notation and definitions.

For each number $i \in \{1, \dots, r\}$, we consider the family of matrix Cauchy problems:

$$Y_i^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_i^{(r-j)}(t) = O_m, \quad t \in (a, b),$$
$$Y_i^{(j-1)}(a) = \delta_{i,j} I_m, \quad j \in \{1, \dots, r\},$$

where $Y_i(\,\cdot\,)$ is an unknown $m \times m$ matrix-valued function.

Let $[BY_i]$ denote the number $l \times m$ matrix whose j-th column is the result of the action of B on the j-th column of the matrix-valued function Y_i.

Definition 1. A block rectangular number matrix

$$M(L, B) := \big([BY_1], \dots, [BY_r]\big) \in \mathbb{C}^{l \times rm} \tag{6}$$

is called the characteristic matrix to problem (2), (3).

Note that this matrix consists of r rectangular block columns $[BY_k] \in \mathbb{C}^{m \times l}$.

Theorem 2. *The dimensions of the kernel and co-kernel of the operator* (5) *are equal to the dimensions of the kernel and co-kernel of the characteristic matrix* (6)*, resp; i.e.,*

$$\dim\ker(L, B) = \dim\ker(M(L, B)),$$
$$\dim\operatorname{coker}(L, B) = \dim\operatorname{coker}(M(L, B)).$$

Theorem 2 implies the following necessary and sufficient conditions for the invertibility of (5).

Corollary. *The operator* (5) *is invertible if and only if* $l = rm$ *and the square matrix* $M(L, B)$ *is nonsingular.*

If all the coefficients of the differential expression L are constant, then the characteristic matrix can be explicitly found. In this case, the characteristic matrix is an analytic function of a certain square number matrix and coincides hence with some polynomial of this matrix.

Example 1 (One-point problem)**.** Consider a linear one-point boundary-value problem

$$(Ly)(t) := y'(t) + Ay(t) = f(t), \ \ t \in (a,b), \tag{7}$$

$$By := \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) = c. \tag{8}$$

Here, A is a constant $(m \times m)$-matrix, $f(\,\cdot\,) \in (W_p^n)^m$, $\alpha_k \in \mathbb{C}^{l\times m}$, $c \in \mathbb{C}^l$, $y(\,\cdot\,) \in (W_p^{n+1})^m$,

$$B : (W_p^{n+1})^m \to \mathbb{C}^l, \quad (L,B) : (W_p^{n+1})^m \to (W_p^n)^m \times \mathbb{C}^l.$$

Let $Y(\,\cdot\,) = (y_{i,j})_{i,j=1}^m \in (W_p^{n+1})^{m\times m}$ be the unique solution of the linear homogeneous matrix equation with the initial Cauchy condition

$$Y'(t) + AY(t) = O_m, \ \ t \in (a,b), \quad Y(a) = I_m.$$

Hence,

$$Y(t) = \exp(-A(t-a)), \quad Y(a) = I_m;$$
$$Y^{(k)}(t) = (-A)^k \exp(-A(t-a)), \quad Y^{(k)}(a) = (-A)^k, \ \ k \in \mathbb{N}.$$

Recall that

$$M(L,B) = \left(B \begin{pmatrix} y_{1,1} \\ \vdots \\ y_{m,1} \end{pmatrix}, \dots, B \begin{pmatrix} y_{1,m} \\ \vdots \\ y_{m,m} \end{pmatrix} \right) \in \mathbb{C}^{l\times m}.$$

Substituting these values into the equality (8), we have

$$M(L,B) = \sum_{k=0}^{n-1} \alpha_k (-A)^k.$$

It follows from Theorem 1 that $\operatorname{ind}(L,B) = \operatorname{ind}(M(L,B)) = m - l$. Therefore, owing to Theorem 2, we obtain

$$\dim\ker(L,B) = \dim\ker\Big(\sum_{k=0}^{n-1} \alpha_k (-A)^k\Big) = m - \operatorname{rank}\Big(\sum_{k=0}^{n-1} \alpha_k (-A)^k\Big),$$
$$\dim\operatorname{coker}(L,B) = -m + l + \dim\ker\Big(\sum_{k=0}^{n-1} \alpha_k (-A)^k\Big) = l - \operatorname{rank}\Big(\sum_{k=0}^{n-1} \alpha_k (-A)^k\Big).$$

It follows from these formulas that d-characteristics of the problem do not depend on the length of the interval (a,b).

Example 2 (Multipoint problem)**.** Let us consider a multipoint boundary-value problem for the differential system (7) with $A(t) \equiv O_m$. The boundary conditions contain derivatives of integer and/or fractional orders (in the sense of Caputo) at certain points $t_k \in [a,b]$, $k = \{0,\dots,N\}$. These conditions become

$$Ly(t) := y'(t) = f(t), \ \ t \in (a,b),$$
$$By := \sum_{k=0}^{N} \sum_{j=0}^{s} \alpha_{k,j} \big({}^C\mathrm{D}_{a+}^{\beta_{k,j}} y\big)(t_k) = c.$$

Here, all $\alpha_{k,j} \in \mathbb{C}^{l\times m}$, whereas the nonnegative numbers $\beta_{k,j}$ satisfy

$$\beta_{k,0} = 0 \ \text{ whenever } \ k \in \{1, 2, \dots, N\}.$$

Theorem 1 asserts that index of the operator (L, B) equals $m - l$. Let us find its Fredholm numbers. Since $Y(\,\cdot\,) = I_m$, we have

$$M(L,B) = \sum_{k=0}^{N}\sum_{j=0}^{s} \alpha_{k,j}(\,{}^{C}\mathrm{D}_{a+}^{\beta_{k,j}} I_m) = \sum_{k=0}^{N} \alpha_{k,0},$$

because the derivatives $(\,{}^{C}\mathrm{D}_{a+}^{\beta_{k,j}} I_m) = 0$ whenever $\beta_{k,j} > 0$. Hence, by Theorem 2, we conclude that

$$\dim\ker(L,B) = \dim\ker\Big(\sum_{k=0}^{N} \alpha_{k,0}\Big) = m - \operatorname{rank}\Big(\sum_{k=0}^{N} \alpha_{k,0}\Big),$$

$$\dim\operatorname{coker}(L,B) = -m + l + \dim\operatorname{coker}\Big(\sum_{k=0}^{N} \alpha_{k,0}\Big) = l - \operatorname{rank}\Big(\sum_{k=0}^{N} \alpha_{k,0}\Big).$$

These formulas show that d-characteristics of the problem do not depend on the length of the interval (a, b) and on the choice of the points $\{t_k\}_{k=0}^{N} \subset [a, b]$ and matrices $\alpha_{k,j}$ with $j \geq 1$.

Conclusions

We prove that the generic problem (2), (3) is a Fredholm one and find its Fredholm numbers, i.e. the dimensions of its kernel and cokernel. Along the way, we find the index of the problem. Note that, unlike the index, the Fredholm numbers are unstable with respect to one-dimensional additive perturbations with an arbitrarily small norm. To find these numbers, we introduce a rectangular number characteristic matrix $M(L, B)$ of the problem and prove that the Fredholm numbers of this matrix coincide with the Fredholm numbers of the problem. We give examples in which the characteristic matrix can be explicitly found [1, 3].

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