Two Point Boundary Value Problems for the Fourth Order Ordinary Differential Equations

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1 Introduction

We study, on the interval $I := [a, b]$, the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t) \tag{1.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{1.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \ \ (j = 0, 1), \tag{1.3_1}$$

$$u^{(j)}(a) = 0 \ \ (j = 0, 1, 2), \quad u(b) = 0, \tag{1.3_2}$$

where $p, h \in L(I;\, R)$, $f \in K(I \times R; R)$.

By a solution of problem $(1.2), (1.3_i)$ $(i \in \{1, 2\})$ we understand a function $u \in \widetilde{C}^3(I; R)$ which satisfies equation (1.2) a.e. on I, and conditions (1.3_i).

We use the following notations here.

$\widetilde{C}^{(3)}(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their third derivatives;

$L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\, ds$;

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions.

For arbitrary $x, y \in L(I;\, R)$, the notation

$$x(t) \preccurlyeq y(t) \ (x(t) \succcurlyeq y(t)) \ \text{ for } \ t \in I$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$; We also use the notations $[x]_\pm = (|x| \pm x)/2$.

The aim of our work is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ depends only on the nonnegative (nonpositive) part of the coefficient p if this nonnegative (nonpositive) part is small enough. On the basis of these results for the nonlinear problems, we have proved sufficient conditions of solvability, which in some sense improve previously known results.

The results of the given work are based on our previous results from the papers [1] and [2].

Below we present some definitions and results from these papers.

Definition 1.1. Equation
$$u^{(4)}(t) = p(t)u(t) \ \text{ for } \ t \in I \tag{1.4}$$
is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less than four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 1.2. We will say that $p \in D_+(I)$ if $p \in L(I; R_0^+)$, and problem $(1.4), (1.3_1)$ has a solution u such that
$$u(t) > 0 \ \text{ for } \ t \in \,]a, b[\,. \tag{1.5}$$

Definition 1.3. We will say that $p \in D_-(I)$ if $p \in L(I; R_0^-)$, and problem $(1.4), (1.3_2)$ has a solution u such that inequality (1.5) holds.

Theorem 1.1 ([1])**.**

(a) *Let the equation*
$$u^{(4)}(t) = [p(t)]_+ u(t)$$
be disconjugate on I. Then problem $(1.1), (1.3_1)$ is uniquely solvable for arbitrary $[p]_-$ and q.

(b) *Let the equation*
$$u^{(4)}(t) = -[p(t)]_- u(t)$$
be disconjugate on I. Then problem $(1.1), (1.3_2)$ is uniquely solvable for arbitrary $[p]_+$ and q.

Theorem 1.2 ([2])**.** *Let $p \in L(I;\, R_0^+)$. Then for the discojugacy of equation (1.4) on I it is necessary and sufficient the existence of $p^* \in D_+(I)$, such that*
$$p(t) \preccurlyeq p^*(t) \ \textit{ for } \ t \in I.$$

Theorem 1.3 ([2])**.** *Let $p \in L(I;\, R_0^-)$. Then for the dsconjugacy of equation (1.4) on I it is necessary and sufficient the existence of $p_* \in D_-(I)$ such that*
$$p_*(t) \preccurlyeq p(t) \ \textit{ for } \ t \in I.$$

2 Linear problems

The proofs of the following results of the unique solvability of problems $(1.1), (1.3_1)$ and $(1.1), (1.3_2)$ are based on Theorems 1.1–1.3.

Theorem 2.1 ([3])**.** *Let $i \in \{1, 2\}$ and the function $p_0 \in L(I; R)$ be such that the equation*
$$u^{(4)}(t) = [p_0(t)]_+ u(t) \ \textit{ if } \ i = 1,$$
$$u^{(4)}(t) = -[p_0(t)]_- u(t) \ \textit{ if } \ i = 2,$$
is disconjugate on I. Then if the inequality
$$(-1)^{i-1}[p(t) - p_0(t)] \le 0 \ \textit{ for } \ t \in I$$
holds, problem $(1.1), (1.3_i)$ is uniquely solvable.

From the last theorem with $p_0 = [p]_+$ or $p_0 = -[p]_-$ it immediately follows

Corollary 2.1. *Let there exist $p^* \in D_+(I)$ $(p_* \in D_-(I))$ such that the inequality*

$$[p(t)]_+ \preccurlyeq p^*(t) \ \ \big(-[p(t)]_- \succcurlyeq p_*(t) \big) \ \ \text{for } \ t \in I$$

holds. Then problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ is uniquely solvable.

Corollary 2.2. *Let inequality*

$$p(t) \le \frac{\lambda_1^4}{(b-a)^4} \approx \frac{500}{(b-a)^4} \ \ \Big([p(t)]_- \le \frac{\lambda_2^4}{(b-a)^4} \approx \frac{949}{(b-a)^4} \Big)$$

holds, where λ_1 (λ_2) is the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \ \ u^{(j)}(0) = 0, \ \ u^{(j)}(1) = 0 \ \ (j = 0, 1) \ \ \ \big(u^{(j)}(0) = 0 \ \ (j = 0, 1, 2), \ \ u(1) = 0 \big).$$

Then problem $(1.1), (1.3_1)$ $((1.1), (1.3_2))$ is uniquely solvable.

3 Nonlinear problem

On the basis of our results for the linear problems, for the nonlinear problems we have proved the following solvability theorem.

Theorem 3.1 ([3])**.** *Let $i \in \{1, 2\}$ and there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a.e. on I the inequality*

$$-g(t)|x| \le (-1)^{i-1} f(t, x) \operatorname{sgn} x \le \delta(t, |x|) \ \ \text{for} \ \ |x| > r \tag{3.1}$$

holds, where the function $\delta \in K(I \times R_0^+; R_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \delta(s, \rho)\, ds = 0.$$

Then if the equation

$$u^{(4)}(t) = [p(t)]_+ u(t) \ \ \ \text{if} \ \ i = 1, \ \ \ \ u^{(4)}(t) = -[p(t)]_- u(t) \ \ \ \text{if} \ \ i = 2$$

is disconjugate, problem $(1.2), (1.3_i)$ has at least one solution.

From the last theorem and Corollary 2.2 it easily follows

Corollary 3.1. *Let there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a.e. on I inequalities (3.1) and*

$$[p(t)]_+ \le \frac{500}{(b-a)^4} \ \ \Big([p(t)]_- \le \frac{949}{(b-a)^4} \Big),$$

hold. Then problem $(1.2), (1.3_1)$ $((1.2), (1.3_2))$ has at least one solution.

The following theorems of the uniqueness of the solution for our nonlinear problem follows from Theorems 1.2, 1.3 and 2.1.

Theorem 3.2 ([3])**.** *Let there exist $p^* \in D_+(I)$ such that a. e. on I the inequality*

$$\big[f(t, x_1) - f(t, x_2) \big] \operatorname{sgn}(x_1 - x_2) < \big[p^*(t) - p(t) \big] |x_1 - x_2|$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem $(1.2), (1.3_1)$ has at most one solution.

Theorem 3.3 ([3])**.** *Let there exist $p_* \in D_-(I)$ such that a.e. on I the inequality*

$$\big[f(t, x_1) - f(t, x_2) \big] \operatorname{sgn}(x_1 - x_2) > \big[p_*(t) - p(t) \big] |x_1 - x_2|$$

holds for $x_1, x_2 \in R$, $x_1 \ne x_2$. Then problem $(1.2), (1.3_2)$ has at most one solution.

References

[1] E. Bravyi and S. Mukhigulashvili, On solvability of two-point boundary value problems with separating boundary conditions for linear ordinary differential equations and totally positive kernels. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2020*, Tbilisi, Georgia, December 19-21, pp. 42–46; http://www.rmi.ge/eng/QUALITDE-2021/Bravyi_Mukhigulashvili_workshop_2020.pdf.

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[3] M. Manjikashvili and S. Mukhigulashvili, Two-point boundary value problems for 4th order ordinary differential equations. *Miskolc Math. Notes* **25** (2024), no. 1, 399–409.