On Control Problems for Systems with Fractional Derivative and Aftereffect

Vladimir P. Maksimov

Perm State University, Perm, Russia
E-mail: maksimov@econ.psu.ru

1 Introduction

We consider a controlled system of linear functional differential equations with a fractional derivative and aftereffect. The well-known definition of the fractional Caputo derivative of the order $\alpha \in (0,\,1)$ is used. The system under study includes, in addition to the Caputo derivative, a linear Volterra operator of a general form defined on the trajectories and a linear Volterra operator defined on the controls. For the system, an initial state is fixed. The aim of controlling is given by a prescribed value of a linear target vector-functional. The question of the solvability to the control problem is studied for two cases: the case without constraints regarding control and the case of linear constraints with respect to control. The approach used is based on the theory of abstract functional differential equations (AFDE) developed by the heads of the Perm Seminar, professors N. V. Azbelev and L. F. Rakhmatullina, and systematically presented in [2].

2 Preliminaries

Let $L_\infty = L^n_\infty[0,T]$ be the space of measurable and bounded in essence functions $z : [0,T] \to \mathbb{R}^n$ with the norm $\|z\|_{L_\infty} = \operatorname{vraisup}(|z(t)|,\ t \in [0,T])$ (here and below $|\cdot|$ stands for a norm in $\mathbb{R}^n$). $AC_\infty = AC^n_\infty[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ with the derivative $\dot{x} \in L_\infty$ and the norm $\|x\|_{AC_\infty} = |x(0)| + \|\dot{x}\|_{L_\infty}$, $L_2 = L^r_2[0,T]$ is the space of square summable functions $u : [0,T] \to \mathbb{R}^r$ with the inner product $\langle u,\, v\rangle = \int\limits_0^T u'(t)\cdot v(t)\,dt$ (the symbol $(\,\cdot\,)'$ stands for transposition).

Consider the linear fractional functional differential system

$$\mathcal{D}^\alpha x = \mathcal{T}x + f, \tag{2.1}$$

where $\mathcal{D}^\alpha$ is the Caputo derivative of the order $\alpha \in (0,1)$ (see, for instance, [4]),

$$(\mathcal{D}^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\dot{x}(s)}{(t-s)^\alpha}\,ds$$

($\Gamma(\,\cdot\,)$ is the Euler gamma-function), $\mathcal{T} : AC^n_\infty[0,T] \to L^n_\infty[0,T]$ is linear bounded Volterra operator with the property: there exists $p > 0$ such that the inequality

$$|(\mathcal{T}x)(t)| \le p \max_{s\in[0,t]} |x(s)|, \ \ t \in [0,T] \tag{2.2}$$

holds for any $x \in AC^n_\infty[0,T]$.

All our constructions below are based on the representation of solutions to (2.1) with the initial condition $x(0) = 0$.

Let us denote

$$(Kz)(t) = (\mathcal{T} J^\alpha z)(t),$$

where the fractional integration operator $J^\alpha : L^n_\infty[0,T] \to AC^n_\infty[0,T]$ is defined by the equality

$$(J^\alpha z)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, z(s)\, ds$$

(see, for instance, [4]).

Remark. Equations (2.1) and $\dot{x} = \mathcal{T}x + f$ are two different representatives of the AFDE [2] $\delta x = \mathcal{T}x + f$. Therewith the theory of the latter uses the representation $x = V\frac{d}{dt}x + x(0)$, where $(Vz)(t) = \int_0^t z(s)\, ds$, while the theory of (2.1) is based on the representation $x = J^\alpha \mathcal{D}^\alpha x + x(0)$. The space AC_∞ is isomorphic to the direct product $L_\infty \times \mathbb{R}^n$ with two possible isomorphisms: $x = Vz + \beta$ and $x = J^\alpha z + \beta$, correspondingly.

Throughout the following, we assume that $K : L^n_\infty[0,T] \to L^n_\infty[0,T]$ is a regular integral Volterra operator:

$$(Kz)(t) = \int_0^t K(t,s)\, z(s)\, ds.$$

This condition is fulfilled for wide classes of operators $\mathcal{T}$ including the operators of inner superposition with delay [1] and Volterra integral ones. In such cases, the representation of the kernel $K(t,s)$ can be obtained in an explicit form.

Under above condition (2.2), the operator K has the resolvent operator $R : (I-K)^{-1} = I + R$, where I is the identity operator, see [9], and R is an integral Volterra operator too:

$$(Rf)(t) = \int_0^t R(t,s) f(s)\, ds$$

with the resolvent kernel $R(t,s)$ [11, Theorem 2.2, p. 119].

As is shown in [9], the Cauchy problem for (2.1) with the initial condition $x(0) = 0$ is uniquely solvable, and its solution has the representation

$$x(t) = (Cf)(t) = \int_0^t C(t,s) f(s)\, ds, \tag{2.3}$$

where $C(t,s)$ is the Cauchy matrix that is defined by the equality

$$C(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, E + \int_s^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\, R(\tau,s)\, d\tau \tag{2.4}$$

(here and below E is the identity $(n\times n)$-matrix). Note that, for $\alpha = 1$, (2.4) takes the form [5]

$$C(t,s) = E + \int_s^t R(\tau,s)\, d\tau.$$

Operator $C : L_\infty \to AC_\infty$ is called the Cauchy operator.

3 The problem formulation

The development of the theory of fractional dynamics led to the works on the dynamics of fractional systems with control. A detailed review of the fundamental works on the control theory of systems with fractional derivatives is given in [3].

Consider the fractional functional differential system under control

$$\mathcal{D}^{\alpha}x = \mathcal{T}x + Fu + f, \tag{3.1}$$

where $F : L_2^r[0,T] \to L_\infty^n[0,T]$ is a linear Volterra operator that is responsible for the implementation of control actions $u \in L_2^r[0,T]$.

Without loss of generality we assume that the initial state of the system is zero:

$$x(0) = 0. \tag{3.2}$$

The goal of control we set by the equality

$$\ell x \equiv \sum_{i=1}^{m} A_i x(t_i) + \int_0^T B(\tau)x(\tau)\,d\tau = \beta \in \mathbb{R}^N, \tag{3.3}$$

where t_i, $i = 1,\dots,m$ are fixed points from $[0,T]$, A_i are constant $(N \times n)$-matrices, $B(\,\cdot\,)$ is $(N \times n)$-matrix with summable elements, β is prescribed constant vector. We study the solvability of the control problem (3.1)–(3.3) for two cases: the case without constraints regarding control and the case of linear constraints with respect to control. Some results are presented in the next section.

4 Main results

Let's denote

$$\Theta_i(s) = A_i\big\{F^*\big[\chi_i(\,\cdot\,)C(t_i,\,\cdot\,)\big]\big\}(s), \ \ i = 1,\dots,m; \quad \Phi(s) = \int_s^T B(\tau)C(\tau,s)\,d\tau; \tag{4.1}$$

$$\Theta_{m+1}(s) = \big\{F^*\Phi(\,\cdot\,)\big\}(s); \quad M(s) = \sum_{i=1}^{m+1}\Theta_i(s); \quad W = \int_0^T M(s)M'(s)\,ds. \tag{4.2}$$

Here $\chi_i(\,\cdot\,)$ is the characteristic function of the segment $[0,t_i]$, F^* is the adjoint operator to F, M is the so called moment matrix.

Theorem 4.1 ([10])**.** *Control Problem* (3.1)–(3.3) *is solvable for any $f \in L_\infty^n[0,T]$ and $\beta \in \mathbb{R}^N$ if and only if the $(N \times N)$-matrix W defined by the equalities* (4.1), (4.2) *is invertible. The control $u_0(t) = M'(t)\,d$ with $d = W^{-1}[\beta - \ell Cf]$, solves the problem.*

In the case of polyhedral constraints with respect to the control u:

$$\Lambda \cdot u(t) \le \gamma, \ \ \gamma \in \mathbb{R}^{N_1}, \ \ t \in [0,T], \tag{4.3}$$

with constant $(N_1 \times r)$-matrix (it is assumed that the set of solutions to the system $\Lambda \cdot v \le \gamma$ is nonempty), there arises the problem of the description to the set of all target values β such that the problem (3.1)–(3.3), (4.3) is solvable. Such a set is called the attainability set to the problem.

It should be noted that, in most works, the attainability is understood in relation to the terminal target vector-functional $\ell x = x(T)$. In contrary to this, we consider the essentially more general case of the target vector-functional, and the term ℓ-attainability seems to be more proper. Here we consider the question on the inner (lower by inclusion) estimates to the ℓ-attainability set. In the case of the systems with the derivative of the integer order, this question is studied in [6–8]. All constructions used to obtain the inner estimates are based on the representation (2.3) of solutions to the system (3.1).

Using the moment matrix M, the equality (3.3) that defines the aim of control is reduced to the integral form with respect to control u:

$$\int_0^T M(t) \cdot u(t)\, dt = \beta.$$

Thus the control problem (3.1)–(3.3), (4.3) is reduced to the system

$$\int_0^T M(t) \cdot u(t)\, dt = \beta \in \mathbb{R}^N, \quad \Lambda \cdot u(t) \leq \gamma, \quad t \in [0, T].$$

The inner estimate of the ℓ-attainability set is based on the following constructions. Let us split the segment $[0, T]$ onto partial ones by the points $\vartheta_1, \ldots, \vartheta_{\mathcal{K}-1} : 0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{\mathcal{K}-1} < T = \vartheta_{\mathcal{K}}$, and denote by $\chi_i(t)$ the characteristic function of the interval $(\vartheta_{i-1}, \vartheta_i]$. We restrict the class of controls by piecewise constant ones of the form

$$u(t) = \sum_{i=1}^{\mathcal{K}} d_i \chi_i(t), \tag{4.4}$$

where $d_i \in \mathbb{R}^m$ are constant vectors. Next we define constant $(N \times r)$-matrices M_i by the equalities

$$M_i = \int_{\vartheta_{i-1}}^{\vartheta_i} M(t)\, dt, \quad i = 1, \ldots, \mathcal{K}.$$

Let us fix a collection of vectors $\lambda_1, \ldots, \lambda_j, \ldots, \lambda_{\mathcal{N}} \in R^N$, and, for every j, set the linear programming problem

$$\sum_{i=1}^{\mathcal{K}} \lambda_j' \cdot M_i d_i \to \max, \quad \Lambda \cdot d_i \leq \gamma, \quad i = 1, \ldots, \mathcal{K}. \tag{4.5}$$

Let $\lambda_{j_1}, \ldots, \lambda_{j_{\mathcal{N}_1}}$ be a subset of the collection $\{\lambda_j\}$, $j = 1, \ldots, \mathcal{N}$ such that, for any its element the problem (4.5) has a solution $D^{j_k} = (d_1^{j_k}, \ldots, d_{\mathcal{K}}^{j_k})$, $k = j_1, \ldots, j_{\mathcal{N}_1}$. Every such solution, after substitution of it into (4.4), defines a program control $u^{j_k}(t)$ that gives an attainable value of the target vector-functional ℓ:

$$\ell x = \int_0^T M(t) \cdot u^{j_k}(t)\, dt = \rho^{j_k}.$$

The collection of such values (points in $\mathbb{R}^N$) allows one to obtain an inner estimate to the ℓ-attainability set.

Theorem 4.2. *Let $\mathcal{P} \subset \mathbb{R}^N$ be the set of all linear convex combinations of the points ρ^{j_k}, $k = j_1, \ldots, j_{\mathcal{N}_1}$. Then any value $\beta \in \mathcal{P}$ is an ℓ-attainable value in the problem* (3.1)–(3.3), (4.3).

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