Some Sufficient Conditions for Almost Reducibility of Millionshchikov Systems

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Consider a linear differential system

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \tag{1}$$

with continuous and bounded coefficient matrix A.

System (1) is said to be almost reducible [2] (or approximately similar, see [6]) if for any $\delta > 0$ there exists a Lyapunov transformation reducing system (1) to the form

$$\dot{x} = Bx + Q_\delta(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geq 0,$$

where B is a constant matrix and Q_δ satisfy the condition $\|Q_\delta(t)\| \leq \delta$. Note that the matrix B is the same for all $\delta > 0$.

The property of almost reducibility plays a crucial role in many issues related to Erugin's problem on Lyapunov regularity of linear systems with almost periodic coefficients. This problem was posed by N. P. Erugin at a mathematical seminar at the Institute of Physics and Mathematics of Byelorussian Academy of Sciences in 1956. The original formulation of Erugin's problem involved proving the hypothesis of Lyapunov regularity of all systems with almost periodic coefficients, see [4, pp. 121, 137] and also [5].

Erugin's problem has been solved by V. M. Millionshchikov who has proved the following two statements.

(i) *Let $\mathcal{H}(A)$ be the hull of A, i.e. the uniform closure of all shifts $A_\tau(t) := A(t+\tau)$. If A is almost periodic, then almost all systems with coefficient matrices from $\mathcal{H}(A)$ are Lyapunov regular* [17].

(ii) *There exists some Lyapunov irregular system* (1) *with almost periodic coefficients* [19].

It should be noted that the proof in [19] is not completely constructive and use the following result from [18].

(iii) *If there exists a non-almost reducible system with coefficient matrix from $\mathcal{H}(A)$, then there exists an irregular system with coefficient matrix from $\mathcal{H}(A)$* [18].

By virtue of (iii), to prove statement (ii) it is sufficient to construct some non-almost reducible system with almost periodic coefficients. To this end V. M. Millionshchikov introduced a special class of limit periodic linear systems and constructed the required system within that class. Now such systems are usually called Millionshchikov systems. A comprehensive study of such systems was made by A. V. Lipntskii in [8–15]. In particular, an explicit example of Lyapunov-irregular Millionshchikov system is given in [8] (see also [21]). However, no effective tools are known for recognising almost reducibility for these systems.

A number of almost reducibility criteria are known for general systems and systems with almost periodic coefficients, see e.g. [3, 7, 20]. However, most of these criteria are based on properties of some solution sets for such systems. In [16] we propose a sufficient condition for almost reducibility of Millionshchikov systems based on properties of periodic approximations to the system under consideration. Our goal here is to give some corollaries of this result.

Let coefficient matrix A has the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t + \tau_k), \tag{2}$$

where A_k, $k = 0, \ldots, +\infty$, are periodic matrices with the periods T_k and τ_k are arbitrary real numbers. If each matrix A_k is everywhere continuous and series (2) converges uniformly on the entire time axis $\mathbb{R}$, then the matrix A is limit-periodic [1, p. 32] and, therefore, almost periodic.

In what follows we suppose that $T_0 = 2$, $T_k \in \mathbb{N}$, and $T_{k+1}/T_k = m_i \in \mathbb{N}$ for all $k = 0, \ldots, +\infty$. We also suppose that $m_k > 1$, $k = 0, \ldots, +\infty$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take some continuous function $\omega : [0, 1] \to \mathbb{R}$ such that $\omega(0) = \omega(1) = 0$ and $\int\limits_0^1 \omega(t)\, dt = 1$. Take also a sequence $\varphi : \mathbb{N} \to [0, \pi/2[$. As usually, the values of the sequence φ we denote by φ_k, $k \in \mathbb{N}$.

Now let us define the matrices A_k by the following equalities:

$$A_0(t) = \begin{cases} \omega(t)D, & \text{for } \ t \in [0, 1[\,, \\ 0, & \text{for } \ t \in [1, 2[\end{cases} \tag{3}$$

for $k = 0$ and

$$A_k(t) = \begin{cases} -\varphi_k \omega(t) J, & \text{for } \ t \in [0, 1[\,, \\ 0, & \text{for } \ t \in [1, T_i[\end{cases} \tag{4}$$

for all $k = 1, \ldots, +\infty$.

It can be easily shown that if

$$\sum_{k=1}^{\infty} \varphi_k < +\infty,$$

then system (1) with the coefficient matrix A defined by (3) and (4) is limit periodic.

Definition 1. We say that system (1) with the coefficient matrix A defined by (3), (4), and (2) with $\tau_k = 0$ is a gathered Millionshchikov system.

Definition 2. System (1) with the coefficient matrix A defined by (3), (4), and (2) with $\tau_k \in 2\mathbb{Z}$, is said to be a Millionshchikov system. We say that this system corresponds to a gathered Millionshchikov system with the same matrices A_k, $k = 0, \ldots, +\infty$.

Note that any gathered Millionshchikov system has the coefficient matrix of the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t).$$

Let

$$S_m(t) = \sum_{k=0}^{m} A_k(t), \quad m = 1, \ldots, +\infty,$$

where A_k are defined by (3) and (4). It can be easily seen that each matrix S_m is T_m-periodic. Now for arbitrary $m \in \mathbb{N}$ consider a periodic linear system

$$\dot{z} = S_m(t)z, \quad z \in \mathbb{R}^2, \quad t \in \mathbb{R}. \tag{5}$$

Denote the Cauchy matrix of system (5) by Z_m. Then the monodromy matrix of system (5) can be written as $Z_m(T_m, 0)$ and the eigenvalues of $Z_m(T_m, 0)$ are the Floquet multipliers of system (5). If these numbers are real, then we can find some real eigenvectors of $Z_m(T_m, 0)$ and the angle β_m between them.

Definition 3 ([16])**.** We say that gathered Millionshchikov system (1) is a real-type system if all Floquet multipliers of each corresponding system (5) with $m \in \mathbb{N}$ are real.

Theorem 1 ([16])**.** *Suppose that system* (1) *is a real-type gathered Millionshchikov system. If the angle β_m is separated from zero for all $m \in \mathbb{N}$, then system* (1) *is almost reducible.*

Definition 4. We say that a gathered Millionshchikov system (1) minorises another gathered Millionshchikov system if the angles φ_k of the first system are not greater than the corresponding angles of the second system.

Theorem 2. *If system* (1) *satisfies conditions of Theorem 1, then all minorizing it gathered Millionshchikov systems are almost reducible.*

Theorem 3. *If system* (1) *satisfies conditions of Theorem 1, then all corresponding to it Millionshchikov systems are almost reducible.*

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