On Bounded and Periodic Solutions of Planar Systems of ODE

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We consider the system

$$u' = f_1(u,v), \quad v' = f_2(u,v) \tag{S}$$

subject to the conditions

$$u(0) = u(\omega), \quad v(0) = v(\omega), \tag{P}$$

and

$$u(0) = c, \quad \sup\big\{|u(t)| + |v(t)| : \ t \geq 0\big\} < +\infty. \tag{B}$$

Here, f_1 and f_2 are Carathéodory functions on $[0,\omega] \times \mathbb{R}^2$, and ω-periodic with respect to the independent variable.

Definition 1. Solutions (u_1,v_1) and (u_2,v_2) of (S), (P) are said to be consecutive if $u_1(t) \leq u_2(t)$ for $t \in [0,\omega]$, $u_1 \not\equiv u_2$, and for every solution (u,v) of (S), (P) satisfying $u_1(t) \leq u(t) \leq u_2(t)$ for $t \in [0,\omega]$, either $u_1 \equiv u$ or $u_2 \equiv u$ holds.

Property O. We will say that (S), (P) possesses Property O if there exists $\varepsilon > 0$ such that every solution (u,v) of (S), (P) satisfies

$$\min\big\{|v(t)| : \ t \in [0,\omega]\big\} \leq \varepsilon.$$

Remark 1. Consider the problem

$$u' = \lambda \cos^2(3u)\psi(v), \quad v' = \cos^2 t \sin u - \frac{1}{4}\,, \quad u(0) = u(\omega), \quad v(0) = v(\omega). \tag{$*$}$$

It is clear that for every c, function $(u,v) := (\frac{\pi}{6}\,, c + \frac{1}{8}\,\sin(2t))$ is a solution of $(*)$, and consequently, $(*)$ does not have Property O.

Hypothesis B. We will say that $f_1 : [0,\omega] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Hypothesis B if

$$f_1(t,x,\,\cdot\,) : \mathbb{R} \to \mathbb{R} \ \text{ is non-decreasing for a.e. } \ t \in [0,\omega], \ \ x \in \mathbb{R}, \tag{1}$$

and

$$f_1(t,x,y)\operatorname{sgn} y \geq 0 \ \text{ for } \ t \in [0,\omega], \ \ x,y \in \mathbb{R}.$$

Proposition 1. *Let Hypothesis B hold, and*

$$\operatorname{meas}\big\{t \in [0,\omega] : \ f_1(t,x,y) \neq 0\big\} > 0 \ \textit{ for } \ x \in \mathbb{R}, \ \ y \in \mathbb{R} \setminus \{0\}.$$

Then problem (S), (P) *has Property O.*

Property V. We will say that (S), (P) possesses Property V if for every pair (u_1,v_1) and (u_2,v_2) of solutions of (S), (P), satisfying $u_1 \equiv u_2$, the identity $v_1 \equiv v_2$ is fulfilled.

Example presented in Remark 1 shows that $(*)$ does not have Property V.

Proposition 2. *Let* $f_1(t,x,y) := f_0(t,x)\psi(y)$, *where* $f_0(t,x) \geq h_0(t) \geq 0$ *for* $t \in [0,\omega]$, $x \in \mathbb{R}$, $h_0 \not\equiv 0$, *and* ψ *is an increasing continuous function with* $\psi(0) = 0$. *Then* (S), (P) *possesses Property V.*

Hypothesis L_x. We will say that f_1 satisfies Hypothesis L_x if (1) holds and for every $r > 0$ and $\varepsilon > 0$, there exist $p_{r\varepsilon} \in L([0,w])$ such that

$$\big|f_1(t,x,y) - f_1(t,x',y)\big| \leq p_{r\varepsilon}(t)|x - x'| \ \text{ for } \ t \in [0,w], \ \ |x - x'| \leq \varepsilon, \ \ |y| \leq r.$$

Definition 2. Function $(\alpha,\beta) : [0,w] \to \mathbb{R}^2$ is said to be a lower function of (S), (P) if $\beta = \beta_0 + \beta_1$, $\alpha, \beta_0 \in AC([0,w])$, β_1 is non-decreasing, $\beta_1'(t) = 0$ for a.e. $t \in [0,\omega]$, $\alpha(0) = \alpha(\omega)$, $\beta(0+) \geq \beta(\omega-)$, and

$$\alpha'(t) = f_1(t,\alpha(t),\beta(t)), \quad \beta'(t) \geq f_2(t,\alpha(t)) \ \text{ for a.e. } \ t \in [0,\omega].$$

Analogously, $(\gamma,\delta) : [0,w] \to \mathbb{R}^2$ is said to be an upper function of (S), (P) if $\delta = \delta_0 + \delta_1$, $\gamma, \delta_0 \in AC([0,w])$, δ_1 is non-increasing, $\delta_1'(t) = 0$ for a.e. $t \in [0,\omega]$, $\gamma(0) = \gamma(\omega)$, $\delta(0+) \leq \delta(\omega-)$, and

$$\gamma'(t) = f_1(t,\gamma(t),\delta(t)), \quad \delta'(t) \leq f_2(t,\gamma(t)) \ \text{ for a.e. } \ t \in [0,\omega].$$

Definition 3. Solution (u,v) of (S), (P) is said to be upper weakly stable (lower weakly stable) if for every $\varepsilon > 0$, there exist a lower function (α,β) (resp. an upper function (γ,δ)) of (S), (P) such that

$$u(t) \leq \alpha(t) \leq u(t) + \varepsilon \ \text{ for } \ t \in [0,\omega], \ \ \alpha \not\equiv u$$
$$\Big(\text{resp. } \ u(t) - \varepsilon \leq \gamma(t) \leq u(t) \ \text{ for } \ t \in [0,\omega], \ \ \gamma \not\equiv u\Big).$$

Remark 2. Let $f_1(t,x,0) \equiv 0$, $f_2(t,0) \equiv 0$, $\varepsilon_0 > 0$ and $f_2(t,\cdot)$ is non-increasing on $[-\varepsilon_0,\varepsilon_0]$. It is not difficult to verify that solution $(u,v) := (0,0)$ is both u.w.s and l.w.s.

The next proposition (partially) justifies introduced terminology.

Proposition 3. *Let Hypothesis L_x be fulfilled and* (S), (P) *possess Property V. Let, moreover,* (u,v) *be a Lyapunov stable solution of* (S), (P). *Then* (u,v) *is both u.w.s and l.w.s.*

Definition 4. Let $\alpha, \gamma \in C([0,\omega])$, $\alpha(t) \leq \gamma(t)$ for $t \in [0,\omega]$, $a \in [0,\omega[$, $\alpha(a) < \gamma(a)$ and $c \in]\alpha(0), \gamma(0)[$. We say that (S), (P) possesses property $Z_{\alpha\gamma}(a,c)$ if for every solution (u,v) of (S), (P) satisfying $\alpha(t) \leq u(t) \leq \gamma(t)$ for $t \in [0,\omega]$, the inequality $u(a) \neq c$ holds.

Remark 3. It is clear that if (u_1,v_1) and (u_2,v_2) are consecutive solutions of (S), (P), then there exist $a \in [0,\omega[$ and $c \in]u_1(a), u_2(a)[$ such that (S), (P) possesses Property $Z_{u_1u_2}(a,c)$.

Now we are able to formulate results.

Consecutive solutions

Theorem 1. *Suppose that* (S), (P) *possesses Property O and* (u_1,v_1) *and* (u_2,v_2) *are solutions of* (S), (P) *satisfying* $u_1(t) \leq u_2(t)$ *for* $t \in [0,\omega]$. *Let, moreover,* $a \in [0,\omega[$, $u_1(a) < u_2(a)$, $c \in]u_1(a), u_2(a)[$ *and* (S), (P) *possesses Property* $Z_{u_1u_2}(a,c)$. *Then there exist consecutive solutions* (u_*,v_*) *and* (u^*,v^*) *of* (S), (P) *such that*

$$u_1(t) \leq u_*(t) \leq u^*(t) \leq u_2(t) \ \text{ for } \ t \in [0,\omega], \ \ u_*(a) < c < u^*(a).$$

Proposition 4. *Let Hypothesis B hold,* (u_*,v_*) *and* (u^*,v^*) *are consecutive solutions of* (S), (P) *and* $u_*(t) < u^*(t)$ *for* $t \in [0,\omega]$. *Then, if* (u_*,v_*) *is u.w.s, then* (u^*,v^*) *is not l.w.s and vice versa, if* (u^*,v^*) *is l.w.s, then* (u_*,v_*) *is not u.w.s.*

Unstable solution

Theorem 2. *Let Hypothesis B and Hypothesis L_x hold and* (S), (P) *possess Property O. Let, moreover,* (α,β) *and* (γ,δ) *be lower and upper functions of* (S), (P), $\alpha(t)\le\gamma(t)$ *for* $t\in[0,\omega]$, $a\in[0,\omega[\,$, $c\in\,]\alpha(a),\gamma(a)[$ *and* (S), (P) *possess property* $Z_{\alpha\gamma}(a,c)$. *Then, there exist unstable solution* (u,v) *of* (S), (P) *such that*

$$\alpha(t)\le u(t)\le\gamma(t)\ \ for\ \ t\in[0,w].$$

Corollary. *Let Hypothesis B and Hypothesis L_x hold, problem* (S), (P) *possess Property O, and*

$$f_1(t,x+\omega_1,y)=f_1(t,x,y),\quad f_2(t,x+\omega_1)=f_2(t,x)\ \ for\ \ t\in[0,\omega],\ \ x,y\in\mathbb{R},$$

where $\omega_1>0$. *Let, moreover,* (S), (P) *be solvable and possess no more than countable many solutions. Then* (S), (P) *has countably many unstable solutions.*

Bounded solutions

Theorem 3. *Let Hypothesis B and Hypothesis L_x hold,* $r_0>0$, $h_0\in L([0,\omega])$ *be nontrivial non-negative, and*

$$|f_1(t,x,\sigma r_0)|\ge h_0(t)\ \ for\ \ t\in[0,\omega],\ \ x\in\mathbb{R},\ \ \sigma\in\{-1,1\}.$$

Let, moreover, (u_*,v_*) *and* (u^*,v^*) *be consecutive solutions of* (S), (P), $u_*(t)<u^*(t)$ *for* $t\in[0,\omega]$, *and* (u_*,v_*) *is u.w.s* ((u^*,v^*) *is l.w.s*)*. The, for every* $c\in\,]u_*(0),u^*(0)[\,$, *problem* (S), (B) *has a solution* (u,v) *such that*

$$\begin{gathered}u_*(t)\le u(t)\le u^*(t),\quad u(t)\le u(t+\omega)\ \ for\ \ t\ge 0\\ \Big(u_*(t)\le u(t)\le u^*(t),\quad u(t)\ge u(t+\omega)\ \ for\ \ t\ge 0\Big)\end{gathered}\tag{2}$$

and

$$\begin{gathered}\lim_{n\to+\infty}\max\big\{|u^*(t)-u(t)|:\ t\in[n\omega,(n+1)\omega]\big\}=0\\ \Big(\lim_{n\to+\infty}\max\big\{|u_*(t)-u(t)|:\ t\in[n\omega,(n+1)\omega]\big\}=0\Big).\end{gathered}$$

If, moreover, the Cauchy problem for (S) *is uniquely solvable, then all inequalities in* (2) *hold in the strong sense.*

As an example, we consider the system

$$u'=f_0(t,u)\psi(v),\quad v'=p_0(t,u)\sin u+q(t).\tag{S$'$}$$

Here, we suppose that

$$\begin{gathered}p_0(t,x)\le p(t)\ \text{ for }\ t\in[0,\omega],\ \ x\in\mathbb{R},\\ 0\le h_0(t)\le f_0(t,x)\le h(t)\ \text{ for }\ t\in[0,\omega]\ \ x\in\mathbb{R},\ \ h_0\not\equiv 0,\end{gathered}$$

and $\psi\in C(\mathbb{R})$,

$$\psi(y)\operatorname{sgn}y\ge 0,\quad |\psi(y)|\le 1\ \text{ for }\ y\in\mathbb{R}.$$

Solvability of (S′), (P)

Theorem 4. *Let* $\|h\|_L < 2\pi$ *and*

$$\big\|[p]_+\big\|_L + \bigg|\int\limits_0^\omega g(s)\,\mathrm{d}s\bigg| \le \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{4}\,.$$

Then, for every $k \in \mathbb{Z}$, *there exists a solution* (u_k, v_k) *of* (S′), (P) *such that*

$$\operatorname{Range}(u_k - 2k\pi) \subseteq \Big[\frac{\pi}{2} - \frac{1}{4}\,\|h\|_L, \frac{3\pi}{2} + \frac{1}{4}\,\|h\|_L\Big]$$

and

$$\Big[\frac{\pi}{2} + \frac{1}{4}\,\|h\|_L, \frac{3\pi}{2} - \frac{1}{4}\,\|h\|_L\Big] \cap \operatorname{Range}(u_k - 2k\pi) \ne \varnothing.$$

In the next theorem, another localization of solutions is stated.

Theorem 5. *Let* $\|h\|_L < \pi$ *and*

$$\big\|[p]_+\big\|_L + \bigg|\int\limits_0^\omega g(s)\,\mathrm{d}s\bigg| < \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2}\,. \tag{3}$$

Then, for every $k \in \mathbb{Z}$, *there exists solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *of* (S′),(P) *such that*

$$\operatorname{Range}(u_{1k} - 2k\pi) \subset \Big]-\frac{\pi}{2}, \frac{\pi}{2}\Big[, \quad \operatorname{Range}(u_{2k} - 2k\pi) \subset \Big]\frac{\pi}{2}, \frac{3\pi}{2}\Big[.$$

It is not difficult to verify the validity of

Proposition 5. *Let* $\|h\|_L < \pi$, $i \in \{0, 1\}$ *and*

$$(-1)^{i+1}\int\limits_0^\omega q(s)\,\mathrm{d}s > \big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2}\,.$$

Then, every solution (u, v) *of* (S′), (P) *satisfies*

$$\Big\{(-1)^i\,\frac{\pi}{2} + 2\pi n : \ n \in \mathbb{Z}\Big\} \cap \operatorname{Range} u = \varnothing.$$

Conservative solutions of (S′), (P)

Suppose in addition that

$$\psi \quad \text{is increasing on } \mathbb{R}. \tag{4}$$

Then, by virtue of Proposition 1 and 2, (S′), (P) possesses Property O and Property V. Taking into account Theorem 1, 4, 5 and Proposition 5, we get

Theorem 6. *Let* (4) *hold,* $\|h\|_L < \pi$ *and*

$$\big\|[p]_+\big\|_L - \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{2} < \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \le \big\|[p]_-\big\|_L \cos\frac{\|h\|_L}{4} - \big\|[p]_+\big\|_L\,.$$

Then, for every $k \in \mathbb{Z}$, *there exist a pair of consecutive solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *of* (S′), (P) *such that:*

(1) *If* $\int\limits_0^\omega q(s)\,\mathrm{d}s \geq 0$, *then*

$$\mathrm{Range}(u_{1k}-2k\pi) \subseteq \Big[\frac{\pi}{2}-\frac{1}{4}\,\|h\|_L\,,\frac{3\pi}{2}\Big[\,, \quad \mathrm{Range}(u_{2k}-2k\pi) \subset \Big]\frac{3\pi}{2}\,,\frac{7\pi}{2}\Big[\,;$$

(2) *If* $\int\limits_0^\omega q(s)\,\mathrm{d}s \leq 0$, *then*

$$\mathrm{Range}(u_{1k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{5\pi}{2}\Big[\,, \quad \mathrm{Range}(u_{2k}-2k\pi) \subseteq \Big[\frac{5\pi}{2}\,,\frac{7\pi}{2}+\frac{1}{4}\,\|h\|_L\Big[\,.$$

Theorem 7. *Let* (4) *hold,* $\|h\|_L < \pi$ *and* (3) *be fulfilled. Then, for every* $k \in \mathbb{Z}$, *there exist two pairs of consecutive solutions* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *and* (u_{3k}, v_{3k}) *and* (u_{4k}, v_{4k}) *of* $(\mathrm{S}'), (\mathrm{P})$ *such that* $u_{2k}(t) \leq u_{3k}(t)$ *for* $t \in [0,\omega]$,

$$\mathrm{Range}(u_{1k}-2k\pi) \subseteq \Big]-\frac{\pi}{2}\,,\frac{\pi}{2}\Big[\,, \quad \mathrm{Range}(u_{2k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[$$

and

$$\mathrm{Range}(u_{3k}-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[\,, \quad \mathrm{Range}(u_{4k}-2k\pi) \subseteq \Big]\frac{3\pi}{2}\,,\frac{5\pi}{2}\Big[\,.$$

If, moreover, $p(t) \leq 0$ *for* $t \in [0,\omega]$, *then* (u_{1k}, v_{1k}) *is u.w.s and* (u_{4k}, v_{4k}) *is l.w.s.*

Unstable solutions of $(\mathrm{S}'), (\mathrm{P})$

First note that Hypothesis L_x now reads as follows: for every $\varepsilon > 0$, there exists $p_\varepsilon \in L([0,\omega])$ such that

$$\big|f_0(t,x)-f_0(t,x')\big| \leq p_\varepsilon(t)|x-x'| \ \text{ for } \ t \in [0,\omega], \ \ |x-x'| \leq \varepsilon. \tag{5}$$

Theorem 8. *Let* (5) *be fulfilled, and the conditions of Theorem 6 (resp. Theorem 7) hold. Then from every pair of consecutive solutions of* $(\mathrm{S}'), (\mathrm{P})$, *at least one of them is unstable. In particular,* $(\mathrm{S}'), (\mathrm{P})$ *possesses at least countably many unstable solutions.*

Theorem 9. *Let* (4) *and* (5) *hold,* $p(t) \leq 0$ *for* $t \in [0,\omega]$, $\|h\|_L < \pi$, *and*

$$\Big|\int\limits_0^\omega g(s)\,\mathrm{d}s\Big| < \|p\|_L \cos\frac{\|h\|_L}{2}\,. \tag{6}$$

Then, for every $k \in \mathbb{Z}$, *the problem* $(\mathrm{S}'), (\mathrm{P})$ *has an unstable solution* (u_k, v_k) *such that*

$$\mathrm{Range}(u_k-2k\pi) \subset \Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[\,.$$

Bounded solution of (S') and its asymptotics

First mention that under the assumptions of Theorem 4, one can show that for every $c \in \mathbb{R}$, the problem $(\mathrm{S}'), (\mathrm{B})$ is solvable. However, we are interested in the existence of non-periodic solutions of $(\mathrm{S}'), (\mathrm{B})$.

Theorem 10. *Let* (5) *hold, and the conditions of Theorem 6 be fulfilled. Let, moreover,* (u_{1k}, v_{1k}) *and* (u_{2k}, v_{2k}) *be solutions of* $(\mathrm{S}'), (\mathrm{P})$ *appearing in the conclusion of Theorem 6. Then, for every* $k \in \mathbb{Z}$*, there exists a non-periodic solution* (u_k, v_k) *of* $(\mathrm{S}'), (\mathrm{B})$ *such that*

$$u_{1k}(t) \leq u_k(t) \leq u_{2k}(t) \ \textit{ for } \ t \geq 0.$$

Theorem 11. *Let* (4) *and* (5) *hold,* $\|h\|_L < \pi$*, and* (6) *be fulfilled (clearly, conditions of Theorem 7 hold). Let, moreover,* $k \in \mathbb{Z}$ *and* (u_{ik}, v_{ik})*,* $i = 1, 2, 3, 4$*, be solutions of* $(\mathrm{S}'), (\mathrm{P})$*; their existence is stated in Theorem 7.*

Then, for every $c \in \,]u_{1k}(0), u_{2k}(0)[\,$*, the problem* $(\mathrm{S}'), (\mathrm{B})$ *has a solution* (u_k, v_k) *such that*

$$u_{1k}(t) \leq u_k(t) \leq u_{2k}(t), \quad u_k(t) \leq u_k(t+\omega) \ \textit{ for } \ t \geq 0, \tag{7}$$

and

$$\lim_{n \to +\infty} \max \big\{ |u_k(t) - u_{2k}(t)| : \ t \in [n\omega, (n+1)\omega] \big\} = 0,$$

while, for every $c \in \,]u_{3k}(0), u_{4k}(0)[\,$*, the problem* $(\mathrm{S}'), (\mathrm{B})$ *possesses a solution* (u_k, v_k) *such that*

$$u_{3k}(t) \leq u_k(t) \leq u_{4k}(t), \quad u_k(t) \geq u_k(t+\omega) \ \textit{ for } \ t \geq 0, \tag{8}$$

and

$$\lim_{n \to +\infty} \max \big\{ |u_k(t) - u_{3k}(t)| : \ t \in [n\omega, (n+1)\omega] \big\} = 0,$$

If, moreover, ψ *is a Lipschitz function, then all inequalities in* (7) *and* (8) *hold in the strict sense.*

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