On Instability of Linear Differential Systems with Smooth Dependence on a Parameter

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Let us consider a one-parameter family of linear differential systems

$$\dot{x} = A_\mu(t)x, \quad x \in \mathbb{R}^2, \quad t \geq 0, \tag{1_μ}$$

whose coefficient matrix is of the form

$$A_\mu(t) := \begin{cases} d_k \operatorname{diag}[1, -1], & 2k-2 \leq t < 2k-1, \\ (\mu + b_k)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 2k-1 \leq t < 2k, \quad k \in \mathbb{N}. \end{cases}$$

Here $\mu k \in \mathbb{R}$ is considered as a parameter; b_k, d_k are arbitrary real numbers.

E. Sorets and T. Spenser have shown in the paper [8] that major characteristic exponent of differential equation

$$\ddot{x} = -\big(K^2\big(\cos t + \cos(\omega t + \theta)\big) + E\big)x, \quad x \in \mathbb{R}^2, \quad t \geq 0$$

is positive for all irrational $\omega \in \mathbb{R}$ and for almost all $\theta \in \mathbb{R}$ on the set of energy values $E \geq 0$, such that it's relative Lebesque measure tends to 1 under increasing to infinity K.

L.-S. Young in the article [9], as a part, have established for all sufficiently big values of $d_k \equiv d > 0$ and $b_k = k\omega$, $k \in \mathbb{N}$, where $\omega \in \mathbb{R} \setminus \mathbb{Q}$ satisfies some diophantine condition holding almost everywhere, that the major characteristic exponent of system (1_μ), which coincides for almost all values of $\mu \in \mathbb{R}$, approximately equal to d.

In the papers [2, 3, 6] we considered the case when the inequality $d_k \geq d > 0$, $k \in \mathbb{N}$, holds. Particularly, in [2], we have proved under condition $d_k \equiv d > 4 \ln 2$ that major characteristic exponent of system (1_μ), is positive for the set of parameter μ with a positive Lebesque measure.

The theorem of the article [3] implies an absence of uniform on $\mu \in \mathbb{R}$ and $t \geq 0$ upper estimations for a solution norms of system (1_μ). Where as, the method developed in the paper [6] essentially uses Parseval's identity for trygonometric sums. It allows to prove an absence of analogous estimations, which are uniform on μ and subexponential on t. Given there the proof of system (1_μ) major characteristic exponent positiveness unfortunately contains invalid statements. The theorem of article [4], that implies the same conclusion, is wrong as well.

In this report we offer the way sufficient to complete the correct proof of specified result.

For all $n \in \mathbb{N}$, an arbitrary $\alpha \in \mathbb{R}$ and set $\chi = \{x_1, \dots, x_n\}$, $x_i \in \mathbb{R}$, $i = \overline{1, n}$, let us denote

$$f_i(x) = f_i(x, x_i) := \ln|x - x_i|, \quad x \neq x_i,$$

and

$$f(x) = f(x, \alpha, \chi) := \alpha + n^{-1} \sum_{i=1}^{n} f_i(x).$$

Lemma ([7]). *For all $a, k, l, \widehat{l} \in \mathbb{R}$ such that $l \geq 1$, $\widehat{l} > 0$, $k > 3 + 2\widehat{l}^{-1}$, and for every set $\chi = \{x_1, \ldots, x_n\}$ and number $\alpha \in \mathbb{R}$, that satisfy the conditions $f(a) > -l$, $\sup\{f(x) : |x - a| \leq 1/2\} < l$, for Lebesque measure of the set*

$$M = M(\alpha, \chi, a, k, l, \widehat{l}) := \Big\{x \in K : \sup_{y \in K} f(y) > f(x) + \widehat{l}\Big\},$$

where $\widetilde{d} := e^{-lk}$, $K := [a - \widetilde{d}/k, a + \widetilde{d}/k]$, the estimation holds $\operatorname{mes} M \leq 48k^{-2}\widetilde{d}/\widehat{l}$.

Let us denote by $X_{A_\mu}(t, s)$, $t, s \geq 0$, Cauchy matrix of system (1_μ).

Theorem. *The major characteristic exponent of system (1_μ), considered as a function of parameter μ, is positive on the set of positive Lebesgue measure in the case when the condition $d_k \geq d > 0$, $k \in \mathbb{N}$, holds.*

Proof. Under

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

we denote the rotation matrix on the angle $\varphi \in \mathbb{R}$ counterclockwise.

According to estimations (40) from paper [6], the inequality holds

$$\int_0^{2\pi} X_{A_\mu}(2k, 0\, d\mu \geq 2\pi \prod_{j=1}^{k} \operatorname{ch} d_j \geq 2\pi(1 + 2^{-1}d^2)^k.$$

Hence, and because of the equality $X_{A_\mu}(2k, 2k-1) = U(\mu + b_k)$, we have the relation

$$\int_0^{2\pi} X_{A_\mu}(2k-1, 0)\, d\mu \geq 2\pi(1 + 2^{-1}d^2)^{k-1}. \tag{2}$$

Remark. In cited article F_k should been defined by the formula $F_k = \kappa_k E + \kappa_{k-1} \operatorname{sh} d_k I$. Followed by estimations (40) an equality in (41) is in general incorrect. Really, for every continuous function $f(\cdot) : \mathbb{R} \to (0, +\infty)$ and numbers $p > q$ the next formula holds [1, p. 167]

$$\exp\left\{\frac{1}{p-q} \int_q^p \ln f(t)\, dt\right\} \leq \int_q^p \frac{1}{p-q} \ln f(t)\, dt. \tag{3}$$

Whereas estimation (41) from paper [6] demands the opposite to (3) inequality. So all subsequent statements of this article are not justified. Hence the conclusion of Theorem 2 in [6] cannot be thought as sufficiently proved.

Here we give another way that allows to avoid the indicated failures.

From estimation (2) it follows the existence of $\gamma_k \in [0, 2\pi]$ such that the inequality holds

$$\|X_{A_{\gamma_k}}(2k, 0)\| \geq (1 + 2^{-1}d^2)^k. \tag{4}$$

Denote by $x_{ij}(t, \mu)$, $i, j = \overline{1, 2}$, the elements of matrix $X_{A_\mu}(t, 0)$.

In the papre [7] after the formula (36) we have proved that $x_{ij}(2n-1, \mu)$, $i, j = \overline{1, 2}$, is a uniform polynome $P_{n,i,j}(\sin\mu, \cos\mu)$ degree $n-1$ on $\sin\mu$ and $\cos\mu$.

For every real $\mu \neq \pi(2^{-1} + m)$, $m \in \mathbb{Z}$, the equality holds

$$P_{n,i,j}(\sin\mu, \cos\mu) = \cos^n \mu P_{n,i,j}(\operatorname{tg}\mu, 1).$$

In the opposite case when $\mu \neq \pi m$, $m \in \mathbb{Z}$, we have the formula

$$P_{n,i,j}(\sin\mu,\cos\mu) = \sin^n \mu P_{n,i,j}(1,\operatorname{ctg}\mu).$$

Denote

$$\delta_n = \delta_n(\mu) := \begin{cases} 0, & \text{if } |\cos\mu| \geq \dfrac{1}{\sqrt{2}}\,, \\ 1, & \text{if } |\cos\mu| < \dfrac{1}{\sqrt{2}}\,. \end{cases}$$

The next relation is correct

$$P_{n,i,j}(\sin\mu,\cos\mu) = \cos^n\big(\mu + 2^{-1}\pi\delta_n(\mu)\big)\,P_{n,i,j}\big(\operatorname{tg}^{1-\delta_n}\mu,\operatorname{ctg}^{\delta_n}\mu\big). \tag{5}$$

The equality

$$\widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\mu)}\mu) = \sum_{i=1}^{2}\sum_{j=1}^{2} P^2_{n,i,j}\big(\operatorname{tg}^{1-\delta_n}\mu,\operatorname{tg}^{-\delta_n}\mu\big)$$

defines a polynome $\widehat{P}_n(\,\cdot\,):\mathbb{R}\to\mathbb{R}$.

Next formulas hold

$$\begin{aligned}\|X_{A_\mu}(2n-1,0)\|^2 = \max_{y\in\mathbb{R}^2}\frac{\|X_{A_\mu}(2n-1,0)y\|^2}{\|y\|^2} &= \max_{\zeta\in\mathbb{R}}\left\|(x^2_{ij}(2n-1,\mu))^2_{i,j=1}\begin{pmatrix}\cos\zeta\\ \sin\zeta\end{pmatrix}\right\|^2 \\ &= \max_{\zeta\in\mathbb{R}}\sum_{i=1}^{2}\Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2. \end{aligned} \tag{6}$$

They imply the inequalities

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}x^2_{ij}(2n-1,\mu) &\leq \sum_{i=1}^{2}\max_{j\in\{1,2\}}x^2_{ij}(2n-1,\mu) \\ &= \sum_{i=1}^{2}\max_{\zeta\in\{0,2^{-1}\pi\}}\Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2 \\ &\leq \sum_{i=1}^{2}\max_{\zeta\in\mathbb{R}}\Big(x_{i1}(2n-1,\mu)\cos\zeta + x_{i2}(2n-1,\mu)\sin\zeta\Big)^2 \overset{(5)}{=} \|X_{A_\mu}(2n-1,0)\|^2 \\ &\overset{(6)}{\leq} \max_{\zeta\in\mathbb{R}}\sum_{i=1}^{2}\big(x_{i1}(2n-1,\mu)\cos\zeta\big)^2 + \big(x_{i2}(2n-1,\mu)\sin\zeta\big)^2 \leq \sum_{i=1}^{2}\sum_{j=1}^{2}x^2_{ij}(2n-1,\mu). \end{aligned} \tag{7}$$

Hence, for some $\varkappa \in [1,2]$ we have the equalities

$$\begin{aligned}\widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\mu)}\mu) &\overset{(5)}{=} \sum_{i=1}^{2}\sum_{j=1}^{2}\cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big)P^2_{n,i,j}(\sin\mu,\cos\mu) \\ &= \cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big)\sum_{i=1}^{2}\sum_{j=1}^{2}x^2_{ij}(2n-1,\mu) \\ &\overset{(7)}{=} \varkappa\cos^{-n}\big(\mu + 2^{-1}\pi\delta_n(\mu)\big)\|X_{A_\mu}(2n-1,0)\|^2. \end{aligned} \tag{8}$$

For all $\mu \in \mathbb{R}$, such that $\delta_n(\mu) = 0$, the next estimation is correct

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| = |\cos\mu| \geq \frac{1}{\sqrt{2}}\,.$$

In the opposite case the formulas hold

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| = \big|\cos(\mu + 2^{-1}\pi)\big| = |\sin\mu| = \sqrt{1-\cos^2\mu} \geq \frac{1}{\sqrt{2}}\,.$$

The both cases united imply the inequality

$$\big|\cos(\mu + 2^{-1}\pi\delta_n(\mu))\big| \geq \frac{1}{\sqrt{2}}\,, \quad \mu \in \mathbb{R}. \tag{9}$$

According to relation (10) from the paper [5], we have the estimation

$$\|X_{A_\mu}(t,0)\| \leq e^{th}, \ \text{ where } \ h := \sup_{k\in\mathbb{N}} d_k. \tag{10}$$

From formulas (8)–(10) the next estimations follow

$$\widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\mu)}\mu) \overset{(8),\,(9)}{\leq} 2^{n/2}\|X_{A_\mu}(2n-1,0)\|^2 \overset{(10)}{\leq} 2^{n/2}e^{h(2n-1)}. \tag{11}$$

The relations (4) and (8) imply the inequalities

$$\widehat{P}_n(\operatorname{tg}^{1-2\delta_n(\gamma_n)}\gamma_n) \overset{(8)}{\geq} \|X_{A_{\gamma_n}}(2n-1,0)\| \overset{(4)}{\geq} (1+2^{-1}d^2)^{n-1}. \tag{12}$$

Due to main algebra theorem, there exist $\alpha \in \mathbb{R}$ and $\beta_j \in \mathbb{C}$, $j = \overline{1, 2n-2}$, such that

$$\widehat{P}_n(\nu) = \alpha \prod_{j=1}^{2n-2}(\nu - \beta_j). \tag{13}$$

Let us put in lemmas conditions (here $[\,\cdot\,]$ denotes a whole part of the number)

$$l := 1+h, \quad \widehat{l} := \frac{\widehat{d}}{4}, \quad k := \max\big\{2^{11}\widehat{d}^{-1}, 4+2[\widehat{l}^{-1}]\big\}, \quad \widetilde{d} := e^{-lk}, \quad f(\,\cdot\,) := \frac{1}{2n-2}\ln\widehat{P}_n(\,\cdot\,).$$

Denote $\widetilde{\gamma}_n = \operatorname{tg}^{1-2\delta_n(\gamma_n)}\gamma_n$.

For all $\nu \in [\widetilde{\gamma}_n - \widetilde{d}/k, \widetilde{\gamma}_n + \widetilde{d}/k]$ there exists $\mu = \mu(\nu) \in [\gamma_n - \widetilde{d}/k, \gamma_n + \widetilde{d}/k]$ such that $\nu = \operatorname{tg}^{1-2\delta_n(\mu)}\mu$.

Hence, as a consequence of formula (11), for such ν the estimation holds

$$f(\nu) \overset{(11)}{\leq} \frac{1}{2n-2}\ln(2^{n/2}e^{h(2n-1)}) = \frac{n\ln 2 + h(2n-1)}{2n-2} \leq 1+h. \tag{14}$$

Denote $\widehat{d} := \frac{1}{2}\ln(1+2^{-1}d^2)$.

Inequalities (12) imply the relation

$$f(\widetilde{\gamma}_n) \overset{(12)}{\geq} \frac{1}{2n-2}\ln(1+2^{-1}d^2)^{n-1} \geq \frac{1}{2}\,. \tag{15}$$

Then, considering (13) and (14), due to lemma we have the inequality

$$\overline{\operatorname{mes}}\Big\{\mu \in \Big[\gamma_n - \frac{\widetilde{d}}{k}, \gamma_n + \frac{\widetilde{d}}{k}\Big]:\ \ \frac{1}{2n-2}\ln\widehat{P}_n(\operatorname{tg}^{\delta_n}\gamma_n) > \frac{1}{2n-2}\ln\widehat{P}_n(\operatorname{tg}^{\delta_n}\mu) + \frac{\widehat{d}}{4}\Big\} \leq 48k^{-2}\widetilde{d}\,\frac{4}{\widehat{d}}. \tag{16}$$

For all $\mu=\mu(\nu)\in[\gamma_n-\widetilde{d}/k,\gamma_n+\widetilde{d}/k]$ the next formulas are correct

$$\begin{aligned}\Big|\cos\big(\mu+2^{-1}\pi\delta_n(\mu)\big)\Big|-\Big|\cos\big(\gamma_n+2^{-1}\pi\delta_n(\mu)\big)\Big|\\ \geq -\Big|\cos\big(\mu+2^{-1}\pi\delta_n(\mu)\big)-\cos\big(\gamma_n+2^{-1}\pi\delta_n(\mu)\big)\Big|\geq -\frac{\widetilde{d}}{k}\,. \quad (17)\end{aligned}$$

Thus, denote $\varepsilon:=\widetilde{d}/k$, for all $\mu\in[\gamma_n-\varepsilon,\gamma_n+\varepsilon]$ with exception of the set W_n which Lebesgue measure $\operatorname{mes}W_n\leq\dfrac{\varepsilon}{4}$ by the cause of (16) we have the estimations

$$\begin{aligned}\frac{1}{2n-1}\,\ln\|X_{A_\mu}(2n-1,0)\| &\overset{(8),\,(16)}{\geq}\frac{1}{2n-1}\,\ln\widehat{P}_n(\operatorname{tg}^{\delta_n}\gamma_n)\\ &+\frac{1}{2n-1}\,\ln\big|\cos^n(\mu-2^{-1}\pi\delta_n(\mu))\big|-\big|\cos(\gamma_n+2^{-1}\pi\delta_n(\mu))\big|-\frac{\widehat{d}}{4}\\ &\overset{(8),\,(17)}{\geq}\frac{1}{2n-1}\,\ln\|X_{A_{\gamma_n}}(2n-1,0)\|-\frac{\widetilde{d}}{k}-\frac{\widehat{d}}{4}\overset{(15)}{\geq}\frac{\widehat{d}}{5}\,. \quad (18)\end{aligned}$$

The set of limit points of sequence $\{\gamma_k\}_{k=1}^{\infty}$ is not empty.

Let us denote by γ_∞ some of them.

For an arbitrary $n\in\mathbb{N}$, there exists $k(n)\geq n$ such that $|\gamma_{k(n)}-\gamma_\infty|<\frac{\varepsilon}{2}$.

Denote also

$$W_\infty:=\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}W_{k(n)}=\operatorname*{Lim}_{m\to+\infty}\bigcap_{n\geq m}W_{k(n)}.$$

The next relations hold

$$\operatorname{mes}W_\infty=\lim_{m\to+\infty}\operatorname{mes}\bigcap_{n\geq m}W_{k(n)}\leq\lim_{m\to+\infty}\sup_{n\geq m}\operatorname{mes}W_{k(n)}\leq\frac{\varepsilon}{4}\,. \quad (19)$$

We have the inclusions

$$\begin{aligned}\widetilde{M}:=\big[\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon\big]\setminus W_\infty\\ =\bigcap_{m\in\mathbb{N}}\bigcup_{n\geq m}\Big([\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon]\setminus W_{k(n)}\Big)\\ \subset\bigcap_{m\in\mathbb{N}}\bigcup_{n\geq m}\Big(\big[\gamma_{k(n)}-\varepsilon,\gamma_{k(n)}+\varepsilon\big]\setminus W_{k(n)}\Big). \quad (20)\end{aligned}$$

Thus for all $\mu\in\widetilde{M}$, as a consequence of formula (18), the next estimations are correct

$$\lambda_{\max}(A_\mu)\geq\varlimsup_{n\to+\infty}\frac{1}{2k(n)-1}\,\ln\big\|X_{A_\mu}(2k(n)-1,0)\big\|\overset{(18),\,(20)}{\geq}\frac{\widehat{d}}{5}>0.$$

As well, relations (19) imply the inequality

$$\operatorname{mes}\widetilde{M}\leq\operatorname{mes}\big[\gamma_\infty-2^{-1}\varepsilon,\gamma_\infty+2^{-1}\varepsilon\big]-\operatorname{mes}W_\infty\geq\frac{\varepsilon}{4}\,.$$

The theorem is proved. □

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