The Averaging Method for Optimal Control Problems of Systems of Integro-Differential Equations

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Abstract

This work is devoted to the optimal control of systems of integro-differential equations with rapidly oscillating coefficients and a small parameter. Using the averaging method, it has been proven that the optimal control of the averaged problem, which is a system of ordinary differential equations, is nearly optimal for the original problem. That is, it minimizes the quality criterion with an accuracy up to ε.

1 Problem statement

We consider the nonlinear optimal control problem of integro-differential system with rapidly oscillating coefficients:

$$\begin{cases} \dot{x} = X\Big(\dfrac{t}{\varepsilon}, x, \displaystyle\int\limits_0^t \varphi(t,s,x(s))\,ds, u(t)\Big), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

and a cost function:

$$J_\varepsilon[u] = \int\limits_0^T L(t, x_\varepsilon(t), u(t))\,dt + \phi(x_\varepsilon(T)) \longrightarrow \inf. \tag{1.2}$$

Here, $\varepsilon > 0$ is a small parameter, $T > 0$ is a constant, x is the phase vector in the domain $D \subset \mathbb{R}^d$, $u(t)$ – m-dimensional control vector from a certain functional set.

Furthermore, $x(t,u)$ is the solution to the Cauchy problem $(1.1), (1.2)$ corresponding to the control $u(t)$. Disregarding the dependence on u, we denote it simply as $x(t)$.

We assume that there exists a function $X_0(x,u)$ such that, for uniformly $x \in \mathbb{R}^d$ and $u \in U$, the following limit exists:

$$\lim_{\varepsilon\to 0} \int\limits_0^t \Big[X\Big(\frac{t}{\varepsilon}, x, \varphi_1(t,x), u\Big) - X_0(x,u)\Big]\,d\tau, \tag{1.3}$$

where

$$\varphi_1(t,x) = \int_0^t \varphi(t,s,x)\,ds,$$

$t \in [0,T]$, $s \in [0,T]$.

Note that condition (1.3) means the integral continuity of the function $X(\frac{\tau}{\varepsilon}, x, \varphi(\tau,x), u)$ at the point $\varepsilon = 0$ on the $[0,T]$, $x \in D$, $u \in U$.

The optimal control problems (1.1), (1.2) with rapidly oscillating coefficients correspond to a simpler optimal control problem

$$\begin{cases} \dot{\xi} = X_0(\xi, u(t)), \\ \xi(0) = x_0, \end{cases} \tag{1.4}$$

with a cost function:

$$J_0[u] = \int_0^T L(t, \xi(t), u(t))\,dt + \phi(\xi(T)) \longrightarrow \inf. \tag{1.5}$$

For problems (1.1), (1.2), we assume that the following conditions hold.

Condition 1.1. *The admissible controls are m-dimensional vector functions $u(\cdot)$ such that $u(\cdot) \in U$ – a compact set in $L^2((0,T))$.*

Condition 1.2. *The function $X(t,x,y,u)$ is defined and continuous with respect to the collection of variables in the domain*

$$Q_0 = \Big\{ t \in [0,T], \ \ x \in D \subset \mathbb{R}^d, \ \ y \in \mathbb{R}^n, \ \ u \in U \in \mathbb{R}^m \Big\}.$$

(1) *$X(t,x,y,u)$ satisfies the linear growth condition with respect to x, y in Q_0, i.e. there exists a constant $M > 0$ such that*

$$|X(t,x,y,u)| \le M\big(1 + |x| + |y|\big)$$

for any $(t,x,y,u) \in Q_0$.

(2) *$X(t,x,y,u)$ satisfies the Lipschitz condition with respect to $x \in D \subset \mathbb{R}^d$ and $u \in \mathbb{R}^m$ in Q_0, with constant λ:*

$$\big|X(t,x,y,u) - X(t,x_1,y_1,u_1)\big| \le \lambda\big(|x-x_1| + |y-y_1| + |u-u_1|\big)$$

for any $(t,x,y,u), (t,x_1,y_1,u_1) \in Q_0$.

Condition 1.3. *The function $\varphi(t,s,x)$ is defined and continuous in the domain $Q_1 = \{t \in [0,T],\ s \in [0,T],\ x \in D\}$ and satisfies the linear growth and the Lipschitz conditions with respect to x, i.e., $\exists\, L_\varphi$ such that*

$$\begin{aligned} \big|\varphi(t,s,x) - \varphi(t,s,x_1)\big| &\le L_\varphi |x - x_1|, \\ |\varphi(t,s,x)| &\le L_\varphi\big(1 + |x|\big). \end{aligned}$$

Condition 1.4. *Uniformly with respect to $x \in D$, $u \in \mathbb{R}^m$, the limit (1.3) exists.*

Condition 1.5. *The function $L(t,x,u)$ is defined and continuous with respect to the collection of arguments in the domain $Q_1 = \{t \in [0,T],\ x \in \mathbb{R}^d,\ u \in \mathbb{R}^m\}$, where:*

(1) *$L(t,x,u)$ is uniformly bounded on $[0,T]$ with $u \in \mathbb{R}^m$ and continuous with respect to $x \in \mathbb{R}^d$.*

(2) *$L(t,x,u)$ satisfies the Lipschitz condition with respect to u in Q_1 with constant $\lambda > 0$.*

(3) *The function $\phi : \mathbb{R}^d \to \mathbb{R}$ is continuous with respect to x.*

According to Conditions 1.1, 1.2 and Theorem 3.1 from [1], it follows that for any continuous admissible control $u(t)$, the solution of the Cauchy problem $X(t,u)$ exists and is unique on the entire interval $[0,T]$. The problems (1.1), (1.4) make sense for all admissible controls.

2 Main results

The following theorem guarantees the closeness of solutions of the corresponding Cauchy problems (1.1), (1.4) for small ε on a finite time interval.

Theorem 2.1. *Let Conditions 1.1–1.3 hold. Then for any $\eta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\eta)$ such that $0 < \varepsilon \le \varepsilon_0$, for the solutions $x(t,u)$, $\xi(t,u)$ of the Cauchy problems* (1.1) *and* (1.4) *satisfy the following estimate*

$$|x(t,u) - \xi(t,u)| \le \eta,$$

for all $t \in [0,T]$ and all admissible controls $u(t)$.

Proof. We will choose the fixed $\eta > 0$. For any $\varepsilon > 0$ and any admissible control $u(t)$, we estimate the difference between $x(t,u)$ and $\xi(t,u)$. For simplicity, let's denote $x(t,u) = x(t)$ and $\xi(t,u) = \xi(t)$. We will also omit the dependence of $x(t)$ on ε.

Since U is compact in $L^2((0,T))$, for the given η, there exists a finite grid. Thus, for the chosen control $u(t)$ from the grid such that $\frac{\eta e^{-\lambda}}{4\lambda}$: $u_1(t), \dots, u_n(t)$, where $N = N(\eta)$. Then, for the chosen control $u(t)$, there exists a subsequence $u_j(t)$ from the grid such that

$$\|u(\,\cdot\,) - u_j(\,\cdot\,)\|_{L_2} \le \frac{\eta}{4\lambda}\, e^{-\lambda}.$$

Thus, since $u(t)$ is compact in $L^2((0,T))$, all $u(t)$ satisfy the inequality, where there exists $K > 0$ such that

$$\int\limits_0^T |u(t)|\, dt \le K.$$

Then

$$|x(t)| \le |x_0| + MT + M \int\limits_0^T \Big(|x(s)| + L_\varphi \int\limits_0^s \big(1 + |x(\tau)|\big)\, d\tau \Big)\, ds.$$

Since, by the Bellman–Gronwall inequality, we get

$$|x(t)| \le C, \quad |\xi(t)| \le C, \tag{2.1}$$

where C is a constant. The estimate for $|\xi(t)|$ was obtained in the same way.

Since Assumption 1.2, we get

$$|x(t) - \xi(t)| \le \int\limits_0^t \bigg| X\Big(\frac{s}{\varepsilon}, x(s), \int\limits_0^s \varphi(s,\tau,x(\tau))\, d\tau, u_j(s)\Big) - X_0(\xi(s), u_j(s)) \bigg|\, ds$$
$$+ 2\lambda \Big(\int\limits_0^T |u(s) - u_j(s)|^2\, ds \Big)^{\frac{1}{2}} \le I_1 + \frac{\eta}{2}\, e^{-\lambda T}.$$

Then we will evaluate I_1 using Conditions 1.2, 1.3, we have

$$I_1 \le \int_0^t \left(\lambda|x(s) - \xi(s)| + \int_0^s |x(t) - \xi(t)| L_\varphi \, d\tau \right) ds$$
$$+ \int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_j(s) \right) - X_0(\xi(s), u_j(s)) \right) ds. \quad (2.2)$$

Since any function from $L^2((0,T))$ can be approximated in the L^2 – norm by a continuous function, and any continuous function on a closed interval can be approximated by a piecewise constant function, for $u_j(t)$ we take a continuous function $u_c(t)$ and a piecewise constant function $u_c(t)$ such that the inequalities hold:

$$\|u_j - u_{c_j}\|_{L^2} < \frac{\eta}{16\lambda} e^{-\lambda T}, \quad (2.3)$$
$$\|u_{c_j}(t) - u_{p_j}(t)\|_{L_2} < \frac{\eta}{16\lambda} e^{-\lambda T} \quad (2.4)$$

for all $t \in [0,T]$.

Using estimates (2.3) and (2.4), we evaluate the last integral from (2.2):

$$\int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_j(s) \right) - X_0(\xi(s), u_j(s)) \right) ds$$
$$\le \int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right) ds + \frac{\eta}{4} e^{-\lambda T}.$$

We split the integral from the last inequality into two integrals, and I_2 and I_3

$$\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right] ds$$
$$= \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau)) \, d\tau, u_p(s) \right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s)) \, d\tau, u_p(s) \right) \right] ds$$
$$+ \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s)) \, d\tau, u_p(s) \right) - X_0(\xi(s), u_p(s)) \right] ds = I_2 + I_3.$$

If necessary, by dividing the segment $[0,T]$ with points $\{t_k\}_0^R$ ($t_0 = 0$, $t_R = T$), it can be assumed that on each interval $[t_k, t_{k+1})$, all components of the vector function $u_p(t)$ take constant values, i.e., $u_p(t_k) = u_p(t_k)$ for $t \in [t_k, t_{k+1})$. Here, the natural $R = R(\eta)$ is fixed for a fixed choice of η.

Now, let us choose a natural n and divide the segment $[0,T]$ into equal n parts using the points $t_i = i \cdot n^{-1}$ ($i = \overline{0,n}$). We assume n is large enough such that each interval $[t_k, t_{k+1})$ contains points t_i. As a result, we obtain n intervals of the form $[t_i, t_{i+1})$. If, for some k and i, $t_i < t_k < t_{i+1}$, the interval $[t_i, t_{i+1})$ is divided into two intervals, $[t_i, t_k)$ and $[t_k, t_{i+1})$. Consequently, the segment $[0,T]$ is divided into no more than $n + R$ intervals, each with a length not exceeding $\frac{1}{n}$. The division points are again denoted as t_i, and the total number of intervals $[t_i, t_{i+1})$ is denoted by $K = K(\eta)$. Clearly, $K \le n + R$, and $u_p(t) = u_p(t_i)$ for $t \in [t_i, t_{i+1})$. Let us denote $\xi_i = \xi(t_i)$, and $u_p(t_i) = u_{pi}$. Then

$$I_2 \le \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \Big[X\Big(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\,d\tau, u_{pi}\Big) - X\Big(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Big)\Big]\,ds$$
$$+ \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \Big[X\Big(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Big) - X\Big(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\,d\tau, u_{pi}\Big)\Big]\,ds$$
$$\le \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi(\tau) - \xi_i|\,d\tau\,ds + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi_i - \xi(\tau)|\,ds + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi_i - \xi(s)|\,d\tau\,ds$$
$$\le 2 \sum_{i=0}^{K-1} \lambda\, \frac{MT(1+C)}{n^2} \Big(1 + \int_{t_i}^{t_{i+1}} ds \int_0^s L_\varphi\,d\tau\Big) \le \lambda MT(1+C)\,\frac{n+R}{n^2} \Big(1 + L_\varphi \frac{T}{n}\Big).$$

Then, for a chosen $\eta > 0$, there exists a number n such that for all $\varepsilon > 0$, the following holds:

$$I_2 \le \frac{\eta}{8}\, e^{-\lambda T}.$$

For estimating the integral I_3, we split it over the interval $[0, T]$ into a sum of integrals

$$\Big| \int_0^t \Big[X\Big(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\,d\tau, u_p(s)\Big) - X_0(\xi(s), u_p(s))\Big]\,ds \Big|$$
$$\le \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + \int_{t_i}^{t_{i+1}} ds \int_0^s L_\varphi |\xi(s) - \xi_i|\,d\tau + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\,ds + I_4,$$

where

$$I_4 = \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \Big[X\Big(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi_i)\,d\tau, u_{pi}\Big) - X_0(\xi_i, u_{pi})\Big]\,ds.$$

In terms of $\varphi_1(t,x)$, we have

$$\int_{t_i}^{t_{i+1}} \Big[X\Big(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Big) - X_0(\xi_i, u_{pi})\Big]\,ds$$
$$= \int_0^{t_{i+1}} \Big[X\Big(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Big) - X_0(\xi_i, u_{pi})\Big]\,ds$$
$$+ \int_0^{t_i} \Big[X\Big(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s,\xi_i), u_{pi}\Big) - X_0(\xi_i, u_{pi})\Big]\,ds. \quad (2.5)$$

To estimate (2.5), it is necessary to use the lemma.

Lemma 2.1. *The convergence in (2.5) is uniform with respect to ξ_i, u_{pi}, and $t_i \in [0,T]$, by subsequence $\varepsilon_n \to 0$.*

Since K is fixed, then, due to the proven lemma, Condition 1.3 holds for small ε_n (depending on K), but independent of ξ_i, u_{pi} and t_i, we have

$$I_6 \leq \frac{\eta}{8}\, e^{-\lambda T}.$$

So we have established that for small enough ε_n

$$|x_{\varepsilon_n}(t) - \xi(t)| < \eta, \quad t \in [0,T].$$

We get

$$\sum_{i=0}^{K-1} \int\limits_{t_i}^{t_{i+1}} \left(X\left(\frac{s}{\varepsilon}, \xi_i, \int\limits_0^s \varphi(s,\tau,\xi_i)\, d\tau, u_{pi}\right) - X_0(\xi_i, u_{pi}) \right) ds < \frac{\eta}{16}\, e^{-\lambda T}.$$

So,

$$I_3 \leq \frac{\eta}{8}\, e^{-\lambda T}.$$

Hence, the following can be obtained from the proof of I_2,

$$I_1 \leq \lambda\biggl(\int\limits_0^t |x(s) - \xi(s)|\, ds + \int\limits_0^s L_\varphi |x(\tau) - \xi(\tau)|\, d\tau \biggr) + \frac{\eta}{4}\, e^{-\lambda T} \leq \frac{\eta}{2}\, e^{-\lambda T}.$$

The reasoning outlined above can be applied to each function $u_1(t), u_2(t), \ldots, u_n(t)$ from the constructed grid. Due to its finiteness, there exists a unique choice i for each function in the system.

Thus, from an arbitrary sequence of solutions $x_{\varepsilon_n}(t)$ of problem (1.1), one can select a subsequence of solutions $x_{\varepsilon_n}(t)$, which converges uniformly for $t \in [0,T]$ to the same limiting function $\xi(t)$. Therefore, the entire family x_ε converges uniformly in $t \in [0,T]$, $u \in U$ as $\varepsilon \to 0$ to $\xi(t)$.

The theorem is proved. □

Theorem 2.2. *Let*

$$J_\varepsilon^* = \inf_{u(\,\cdot\,) \in U} J_\varepsilon[u],$$
$$J_0^* = \inf J_0[u].$$

Let Conditions 1.1–1.5 hold. Then problems (1.1), (1.2) *and* (1.4), (1.5) *have solutions* $(x_\varepsilon^*(t), u_\varepsilon^*(t))$, $(\xi^*(t), u^*(t))$, *respectively. Moreover,*

(1)
$$J_\varepsilon^* \to J_0^* \ \ as \ \ \varepsilon \to 0.$$

(2) *For any $\eta > 0$, there exists ε_0 such that for $\varepsilon < \varepsilon_0$,*
$$|J_\varepsilon^* - J_\varepsilon(u^*)| < \eta,$$
i.e., the optimal control of the averaging problem is nearly optimal for the original problem.

(3) *There exists a sequence $\varepsilon_n \to 0$, $n \to \infty$, such that*
$$x_{\varepsilon_n}^*(t) \to \xi^*(t) \ \ uniformly \ on \ [0,T], \tag{2.6}$$
and
$$u_{\varepsilon_n}^*(t) \to u^*(t) \ \ in \ L^2((0,T)). \tag{2.7}$$

If the averaging problem (1.4), (1.5) *has a unique solution, then the convergence results* (2.6) *and* (2.7) *hold for all* $\varepsilon \to 0$.

Proof.

(1) First, let us prove the continuity of $J_\varepsilon(u)$ with respect to $u \in L^2((0,1))$ for each $\varepsilon > 0$.

Let $u_1(t), u_2(t)$ be arbitrary admissible controls for problem (1.1), (1.2), and let $x(t,u_1), x(t,u_2)$ be the corresponding trajectories.

Using Condition 1.2 and Gronwall's inequality, we have

$$\sup_{t\in[0,1]} |x(t,u_1) - x(t,u_2)| \le \lambda \|u_1 - u_2\|_{L^2} e^{\lambda}. \tag{2.8}$$

Thus,

$$\begin{aligned} |J_\varepsilon(u_1) - J_\varepsilon(u_2)| &\le \lambda \|u_1 - u_2\|_{L^2} \\ &+ \int_0^T \Big[L\big(t, x(t,u_2), u_1(t)\big) - L\big(t, x(t,u_2), u_2(t)\big)\Big]\, dt + \big|\Phi(x(T,u_1)) - \Phi(x(T,u_2))\big|. \end{aligned} \tag{2.9}$$

Estimate (2.1) is uniform for any admissible control $u(t)$.

Thus, from (2.1), we have that $x(t,u)$ does not go beyond the boundaries of the area B_c-sphere of radius C with center at for $t \in [0,T]$.

Due to (1) from Condition 1.5 and Cantor's theorem, the function $L(t,x,u)$ will be uniformly continuous with respect to $x \in B_c$, uniformly relative to $t \in [0,T]$ and $u \in \mathbb{R}^m$. Therefore, from (2.8) and (2.9), the continuity of $J_\varepsilon(u)$ with respect to the L^2-norm follows.

By similar considerations, we establish the continuity of the functional $J_0(u)$ with respect to u.

Now, considering the compactness of the set of admissible controls, we establish the existence of $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ and $(\xi^*(t), u^*(t))$ – optimal solutions of (1.1), (1.2) and (1.4), (1.5), respectively.

Now, we prove that $J^*_\varepsilon \to J^*_0$ as $\varepsilon \to 0$. Choose an arbitrary $\eta > 0$ and fix it. Then

$$J^*_\varepsilon \le J_\varepsilon(u^*) = J^*_0 + J_\varepsilon(u^*) - J_0(u^*).$$

But

$$|J_\varepsilon(u^*) - J_0(u^*)| \le \int_0^T \Big| L\big(t, x(t,u^*), u^*(t)\big) - L\big(t, \xi(t), u^*(t)\big)\Big|\, dt + \big|\Phi(x(T,u^*)) - \Phi(\xi(T))\big|. \tag{2.10}$$

From Theorem 2.1 we have

$$\max_{t\in[0,1]} |x(t,u^*) - \xi^*(t)| \to 0, \ \ \varepsilon \to 0. \tag{2.11}$$

Now, considering the uniform continuity of the function $L(t,x,u)$ with respect to $x \in B_c$, uniformly for $t \in (0,T]$ and $u \in \mathbb{R}^m$, it follows from (2.10), (2.11) and Condition 1.5 that there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, we have

$$|J_\varepsilon(u^*) - J_0| < \eta,$$

then

$$J^*_\varepsilon < J^*_0 + \eta. \tag{2.12}$$

From other side, as $\varepsilon < \varepsilon_0$, we get

$$J^*_0 \le J_0(u^*_\varepsilon) = J^*_\varepsilon + \big(J_0(u^*_\varepsilon) - J_\varepsilon(u^*_\varepsilon)\big). \tag{2.13}$$

Therefore

$$J_0^* < J_\varepsilon^* + \eta.$$

From (2.12) and (2.13) it follows that

$$J_\varepsilon^* \to J_0^*, \quad \varepsilon \to 0. \tag{2.14}$$

Then, statement (1) is proved.

The proof of statement (2) follows from the following inequality

$$|J_\varepsilon^* - J_\varepsilon(u^*)| \le |J_\varepsilon^* - J_0^*| + \big|J_0(u^*) - J_\varepsilon(u^*)\big|.$$

Let's move on to the proof of the next statement. Since u is compact in $L^2((0,1))$, it follows that from the family u_ε^*, we can extract a subsequence $u_{\varepsilon_n}^*$ that converges in $L^2((0,1))$.

Let

$$\lim_{\varepsilon_n \to 0} u_{\varepsilon_n}^* = u_0. \tag{2.15}$$

Consider the auxiliary systems. Using the auxiliary systems and Theorem 2.1, through simple considerations, we obtain

$$\sup_{t\in[0,T]} |x_{\varepsilon_n}^*(t) - \xi(t)| \to 0, \quad \varepsilon_n \to 0. \tag{2.16}$$

Accordingly,

$$J_{\varepsilon_n}^* = J_{\varepsilon_n}(u_{\varepsilon_n}^*) = \int_0^T L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big)\, dt + \Phi(x_{\varepsilon_n}^*(T))$$

$$= \int_0^T L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big)\, dt + \phi(X_{\varepsilon_n}^*(T)) + \int_0^T \Big[L\big(t, x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t)\big) - L\big(t, x_{\varepsilon_n}^*(t), u_0(t)\big)\Big]\, dt. \tag{2.17}$$

From (2) of Condition 1.5 and (2.15), it follows that the last term in (2.17) tends to zero as $\varepsilon_n \to 0$. Let's consider the limit in equation (2.17) as $\varepsilon_n \to 0$, using (2.14) and (2.16), we have

$$J_0^* = \int_0^T L(t, \xi(t), u_0(t))\, dt + \Phi(\xi(T)).$$

Thus, $(\xi(t), u_0(t))$ is the optimal solution of the averaged problem $(1.4), (1.5)$, proving statement (3).

If the problem $(1.4), (1.5)$ has a unique solution, as shown earlier, it follows that any convergent sequence $(x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t))$ converges to the only uniquely defined solution. This completes the proof of statement (4). □

References

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