Finding of Periodic Points of Stokes Flow with a Complex Distribution of Motion Velocities of Rectangular Cavity Walls

Oleksandr Kurylko, Oleh Perehuda, Oleksandr Boryseiko

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

E-mails: alexandr.kurylko@knu.ua; perehudao@knu.ua; boryseiko@knu.ua

1 Introduction

In recent years, there has been a significant development of interest in the implementation of a qualitative process of mixing in flows in two-dimensional rectangular cavities without the participation of physical mixers in the process itself. This becomes possible when the flow of an incompressible viscous liquid is periodically excited in a rectangular cavity with the help of tangential velocities applied to its walls. The results obtained in this direction relate to problems in which the side walls in a rectangular cavity are free from loads, which is physically impossible to implement. The purpose of the research is to build a similar model, which is proposed in [1, 4], considering the case of fixed side walls in a rectangular cavity. Moreover, the goal is to find periodic points of the third order and establish their type.

2 Setting of the problem and the main results

The movement of individual flow particles is considered in a known velocity field and is reduced to solving the advection equations, which are a system of first-order ordinary differential equations with a complex functional dependence in the right-hand parts:

$$\frac{dx_i(t)}{dt} = f(x,y), \quad \frac{dy_i(t)}{dt} = g(x,y), \quad i = \overline{1,n} \tag{2.1}$$

with the initial conditions

$$x_i(t) = x_{i0}, \quad y_i(t) = y_{i0}, \quad i = \overline{1,n}.$$

Two-dimensional slow flow of an incompressible viscous fluid can be represented in terms of a biharmonic problem. If such a motion is so slow that the inertial forces containing the squares of the velocities can be neglected compared to the viscous terms, then the stream function ψ satisfies the biharmonic equation

$$\Delta^2\psi = 0. \tag{2.2}$$

In rectangular coordinates, the Euler components of the velocity vector u and v are defined as

$$u = \frac{\partial\psi}{\partial y}\,, \quad v = -\frac{\partial\psi}{\partial x}\,.$$

Flow in a rectangular cavity $|x| \le a$, $|y| \le b$ is caused by the given tangential velocities $U_{top}(x)$ and $U_{bot}(x)$ on the upper ($y = b$) and lower ($y = -b$) walls, respectively, and the side walls $x = a$ are stationary (Figure 1).

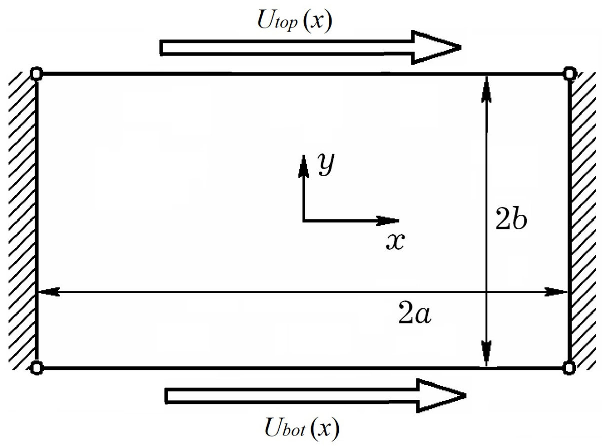

Figure 1. Geometry of a rectangular cavity.

Boundary conditions for equation (2.2) have the form

$$\psi = 0, \quad \frac{\partial \psi}{\partial x} = 0, \quad x = \pm a, \quad |y| \le b, \tag{2.3}$$

$$\psi = 0, \quad \frac{\partial \psi}{\partial y} = \pm U(x), \quad y = \pm b, \quad |x| \le a, \tag{2.4}$$

where

$$U(x) = U_{top}(x) = -U_{bot}(x) = U_1^{(1)} \cos \frac{\pi x}{2a} - U_1^{(2)} \sin \frac{\pi x}{a} \,. \tag{2.5}$$

A detailed description of the construction of the solution to problem (2.2), (2.3), and (2.4) is considered in [2,3]. The resulting solution defines the velocity field, that is, the right-hand sides of the advection equations (2.1).

Important for studying the advection of a passive non-inertial particle is the knowledge of the periodic points of the process of order p, that is, such initial conditions in the advection equation (2.1), when the point accurately returns to its initial position in p periods. A fundamental element of the analysis of the advection process is the classification of periodic points into elliptical and hyperbolic.

We will classify the type of periodic point analytically by determining the eigenvalue λ_1 and λ_2 of the Jacobian matrix of the linearized system (2.1) in the vicinity of the considered point. If λ_1 and λ_2 are complex conjugate, the point is of elliptic type. If λ_1 and $\lambda_2 = \frac{1}{\lambda_1}$ are valid, the time point is of hyperbolic type.

There can also be a situation of $\lambda_1 = \lambda_2 = \pm 1$, which corresponds to the degenerate case where the periodic point is parabolic: in this case, any small change in the velocity field causes the periodic point to become elliptical or hyperbolic.

The Jacobian elements of the matrix M are calculated by solving system (2.1) for four initial conditions $(\overline{x} + \epsilon, \overline{y})$, $(\overline{x} - \epsilon, \overline{y})$, $(\overline{x}, \overline{y} + \epsilon)$, $(\overline{x}, \overline{y} - \epsilon)$, where $(\overline{x}, \overline{y})$ are the rectangular coordinates of a periodic point, and ϵ is an arbitrarily small value,

$$\begin{cases} M_{xx} = \dfrac{x_{(0,pT)}(\overline{x} + \epsilon, \overline{y}) - x_{(0,pT)}(\overline{x} - \epsilon, \overline{y})}{2\epsilon}\,, \quad M_{xy} = \dfrac{x_{(0,pT)}(\overline{x}, \overline{y} + \epsilon) - x_{(0,pT)}(\overline{x}, \overline{y} - \epsilon)}{2\epsilon}\,, \\[2ex] M_{yx} = \dfrac{y_{(0,pT)}(\overline{x} + \epsilon, \overline{y}) - y_{(0,pT)}(\overline{x} - \epsilon, \overline{y})}{2\epsilon}\,, \quad M_{xy} = \dfrac{y_{(0,pT)}(\overline{x}, \overline{y} + \epsilon) - y_{(0,pT)}(\overline{x}, \overline{y} - \epsilon)}{2\epsilon}\,, \end{cases} \tag{2.6}$$

where p is the order of the periodic point.

The condition that the determinant of the matrix M must be equal to one is used when checking the accuracy of calculations.

Figure 2 shows periodic points (in red), which have the following coordinates: $A_L = (-2.01, 0)$, $A_C = (0, 0)$, $A_R = (2.01, 0)$. Coordinates of periodic points and parameters $U_1^{(1)}$ and $U_1^{(2)}$ in (2.5) were selected according to the following algorithm:

(1) periodic points A_L and A_C must belong to the same flow line;

(2) points A_L and A_R are equidistant from the central point A_C.

With such values of $U_1^{(1)}$ and $U_1^{(2)}$, the periodic points A_L and A_C pass into each other in a half-period $\tau = \frac{1}{2}T$ ($T = 2$, τ varies from 0 to $\frac{1}{2}T$), exchange positions (the transition occurs clockwise), and the right A_R remains stationary.

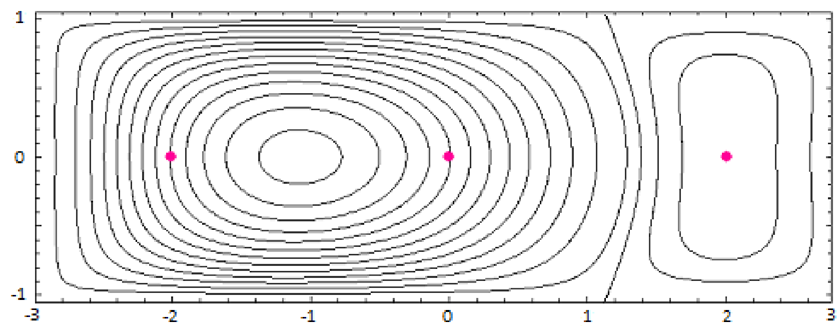

Figure 2. Picture of streamlines in a rectangular cavity $a = 3$ and $b = 1$ with unmovable side walls over the half period $0 < t < \frac{1}{2}T$.

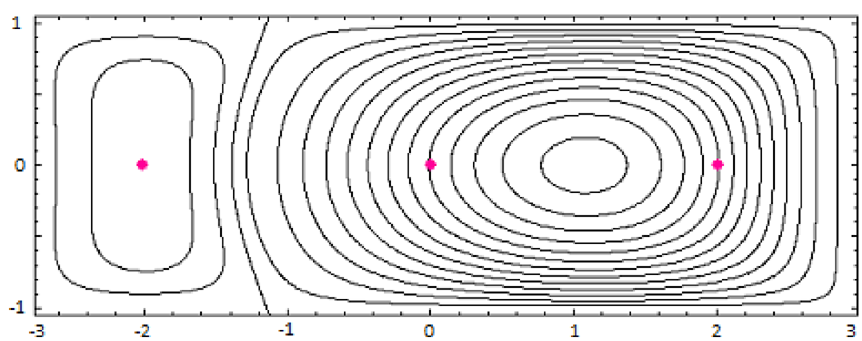

Figure 3. Picture of streamlines in a rectangular cavity $a = 3$ and $b = 1$ with unmovable side walls over the half period $\frac{1}{2}T < t < T$.

In the time period from $\frac{1}{2}T$ to T, the found velocity at the boundaries changes its value to the opposite, begins to act in the opposite direction. In this case, the left periodic point remains stationary, and the central and right point move into each other, changing their positions (the transition occurs counter-clockwise). The corresponding picture of streamlines along with the observed points is shown in Figure 3. In three full periods T, the points will return to their original positions.

Thus, the found points are periodic points of the third order of the elliptic type. The type of these points was determined numerically and analytically according to the methodology proposed in this work. These points play an important role in the theory of mixing liquids and are called “ghost rods”.

References

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