Application of the Retract Principle to Find Solutions of Discrete Nonlinear Equations

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Let us show using the example of several nonlinear difference equations the possibility of estimating the properties of the solutions using a discrete analogue of the retract principle. To describe this principle, we need to consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \quad k \in \mathbb{N}(k_0), \tag{0.1}$$

where $Y = (Y_0, \dots, Y_{n-1})^T$ and

$$F(k, Y) = \big(F_1(k, Y), \dots, F_n(k, Y)\big)^T : \mathbb{N}(k_0) \times \mathbb{R}^n \to \mathbb{R}^n. \tag{0.2}$$

A solution $Y = Y(k)$ of system (0.1) is defined as a function $Y : \mathbb{N}(k_0) \to \mathbb{R}^n$ satisfying (0.1) for each $k \in \mathbb{N}(k_0)$. The initial problem

$$Y(k_0) = Y^0 = (Y_0^0, \dots, Y_{n-1}^0)^T \in \mathbb{R}^n$$

defines a unique solution to (0.1). Obviously, if $F(k, Y)$ is continuous with respect to Y, then the initial problem (0.1), (0.2) defines a unique solution $Y = Y(k_0, Y^0)(k)$, where $Y(k_0, Y^0)$ indicates a dependence of the solution on the initial point (k_0, Y^0), which depends continuously on the value Y^0. Let $b_i, c_i : \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, \dots, n$ be given functions, satisfying

$$b_i(k) < c_i(k), \quad k \in \mathbb{N}(k_0), \quad i = 1, \dots, n.$$

Define auxiliary functions $B_i, C_i : \mathbb{N}(k_0) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$ as

$$B_i(k, Y) := -Y_{i-1} + b_i(k), \quad C_i(k, Y) := Y_{i-1} - c_i(k),$$

and auxiliary sets

$$\Omega_B^i := \Big\{(k, Y) : \ k \in \mathbb{N}(k_0), \ B_i(k, Y) = 0, \ B_j(k, Y) \le 0, \ C_p(k, Y) \le 0, \\ \forall\, j, p = 1, \dots, n, \ j \ne i\Big\},$$

$$\Omega_C^i := \Big\{(k, Y) : \ k \in \mathbb{N}(k_0), \ C_i(k, Y) = 0, \ B_j(k, Y) \le 0, C_p(k, Y) \le 0, \\ \forall\, j, p = 1, \dots, n, \ p \ne i\Big\},$$

where $i = 1, \dots, n$.

Playing a crucial role in the proofs and being suitable for applications, the following lemma is a slight modification of [3, Theorem 1] (see [5, Theorem 2] also).

Definition 0.1. The set Ω is called the *regular polyfacial set* with respect to the discrete system (0.1) if

$$b_i(k+1)-b_i(k) < F_i(k,Y) < c_i(k+1)-b_i(k),$$

for every $i=1,\dots,n$ and every $(k,Y)\in\Omega^i_B$ and if

$$b_i(k+1)-c_i(k) < F_i(k,Y) < c_i(k+1)-c_i(k),$$

for every $i=1,\dots,n$ and every $(k,Y)\in\Omega^i_C$.

To formulate the following theorem, we need to define sets

$$\Omega(k)=\Big\{(k,Y):\ \ Y=(Y_1,\dots,Y_n)\in\mathbb{R}^n,\ \ b_i(k)<Y_i<c_i(k),\ \ i=1,\dots,n\Big\},$$
$$\Omega_i(k)=\Big\{(Y):\ \ Y\in\mathbb{R},\ \ b_i(k)<Y_i<c_i(k),\ \ i=1,\dots,n\Big\}.$$

Theorem 0.1 ([4, Theorem 4])**.** *Let $F:\mathbb{N}(k_0)\times\overline{\Omega}\to\mathbb{R}^n$. Let, moreover, Ω be regular with respect to the discrete system* (0.1)*, and let the function*

$$G_i(w):=w+F_i(k,Y_1,\dots,Y_{i-1},w,Y_{i+1},\dots,Y_n)$$

be monotone on $\overline{\Omega}_i(k)$ for every fixed $k\in\mathbb{N}(k_0)$, each fixed $i\in\{1,\dots,n\}$, and every fixed

$$(Y_1,\dots,Y_{i-1},Y_{i+1},\dots,Y_n)$$

such that $(k,Y_1,\dots,Y_{i-1},w,Y_{i+1},\dots,Y_n)\in\Omega$. Then, every initial problem $Y(k_0)=Y^*$ with $Y^*\in\Omega(k_0)$ defines a solution $Y=Y^*(k)$ of the discrete system (0.1)*, satisfying the relation*

$$Y^*(k)\in\Omega(k)$$

for every $k\in\mathbb{N}(k_0)$.

Now we formulate a result which is proved in [3] by a retract method sometimes called an Anti-Liapunov method due to the assumptions used being often an opposite to those used when Liapunov method is applied (such an approach goes back to Ważewski, who formulated his topological method formulated for ordinary differential equations). The following theorem is a slight modification of [3, Theorem 1] (see [5, Theorem 2] also).

Theorem 0.2. *Assume that the function $F(k,Y)$ satisfies* (0.1) *and is continuous with respect to Y. Let the inequality*

$$F_i(k,Y)<b_i(k+1)-b_i(k)$$

hold for every $i=1,\dots,n$ and every $(k,Y)\in\Omega^i_B$. Let, moreover, the inequality

$$F_i(k,Y)>c_i(k+1)-c_i(k)$$

hold for every $i=1,\dots,n$ and every $(k,Y)\in\Omega^i_C$. Then, there exists a solution $Y=Y(k)$, $k\in\mathbb{N}(k_0)$ of system (0.1)*, satisfying the inequalities*

$$b_i(k)<Y_{i-1}(k)<c_i(k)$$

for every $k\in\mathbb{N}(k_0)$ and $i=1,\dots,n$.

Definition 0.2. A function $u_{upp}:\mathbb{B}\to\mathbb{R}$ is said to be an *approximate solution* to equation (0.1) of an order g, where $g:\mathbb{N}(k_0)\to\mathbb{R}$, if

$$\lim_{k\to\infty}\big[\Delta^3u_{upp}(k)\pm k^\alpha u^n_{upp}(k)\big]g(k)=0.$$

If the main term (i.e. the term being asymptotically leading) in $u_{upp}(k)$ is a power-type function, we say that it is a *power-type* approximate solution.

1 Discrete analogue of Emden–Fowler second-order non-linear equation

Now let us consider the following second-order non-linear equation

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0, \tag{1.1}$$

where $u : \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution, $\Delta u(k)$ is its first-order forward difference, i.e.,

$$\Delta u(k) = u(k+1) - u(k),$$

$\Delta^2(k)$ is its second-order forward difference, i.e.,

$$\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k),$$

and α, m are real numbers. A function $u = u^* : \mathbb{N}(k_0) \to \mathbb{R}$ is called a solution of equation (1.1) if the equality

$$\Delta^2 u^*(k) \pm k^\alpha (u^*(k))^m = 0$$

holds for every $k \in \mathbb{N}(k_0)$.

Equation (1.1) is a discretization of the classical Emden–Fowler second-order differential equation (we refer, e.g., to [2])

$$y'' \pm x^\alpha y^m = 0,$$

where the second-order derivative is replaced by a second-order forward difference and the continuous independent variable is replaced by a discrete one.

Remark 1.1. We need to assume $m \neq 0$, $m \neq 1$, $s + 2 \neq 0$, and $s + 2 - ms \neq 0$, that is, $m \neq 0$, $m \neq 1$, $\alpha \neq -2$, and $\alpha \neq -2m$.

Let us define

$$s = \frac{\alpha + 2}{m - 1}, \qquad a = \big[\mp s(s+1)\big]^{1/(m-1)}, \tag{1.2}$$

and

$$b = \frac{as(s+1)}{s + 2 - ms}.$$

Remark 1.2. If, in formula (1.2), either the upper variant of sign is in force (i.e. $-$) and $s(s+1) > 0$ or in (1.2) lower variant of sign in force (i.e. $+$) and $s(s+1) < 0$, then the constant m has the form of a ratio m_1/m_2 of relatively prime integers m_1, m_2, and m_2 is odd, the difference $m_1 - m_2$ is odd as well. If this convention holds, formula (1.2) defines two or at least one value. As equation (1.1) splits into two equations, when formulating the results, we assume that a concrete variant is fixed (either with the sign $+$ or with the sign $-$).

Previously in [1, 7, 8] the conditions on the existence of a power-type solution of equation (1.1) were discussed.

Theorem 1.1. *If there exist $\gamma \in (0,1)$, s and $\varepsilon_i > 0$, $i = 1,2,3,4$, such that $P \equiv \frac{\gamma+s+1}{s+1}$ and $Q \equiv \frac{\gamma+s+2}{ms}$ and at least one of the following four conditions is true*

(1) *$ms > 0$, $s > -1$, $\varepsilon_3 < \varepsilon_1 P$, $\varepsilon_4 < \varepsilon_2 P$, $\varepsilon_1 < \varepsilon_3 Q$, $\varepsilon_2 < \varepsilon_4 Q$;*

(2) *$ms < 0$, $s > -1$, $\varepsilon_3 < \varepsilon_1 P$, $\varepsilon_4 < \varepsilon_2 P$, $\varepsilon_2 < -\varepsilon_3 Q$, $\varepsilon_1 < -\varepsilon_4 Q$;*

(3) $ms < 0$, $-2 \neq s < -1$, $\varepsilon_4 < -\varepsilon_1 P$, $\varepsilon_3 < -\varepsilon_2 P$, $\varepsilon_2 < -\varepsilon_3 Q$, $\varepsilon_1 < -\varepsilon_4 Q$;

(4) $ms > 0$, $-2 \neq s < -1$, $\varepsilon_4 < -\varepsilon_1 P$, $\varepsilon_3 < -\varepsilon_2 P$, $\varepsilon_1 < \varepsilon_3 Q$, $\varepsilon_2 < \varepsilon_4 Q$,

then there exists K such that for all $k_0 > K$ there exists a solution $u(k)$ to equation (1.1) *such that for all $k \in \mathbb{N}(k_0)$ the following inequalities*

$$-\frac{\varepsilon_1}{k^\gamma} < \Big(u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\Big)\Big(\frac{b}{k^{s+1}}\Big)^{-1} < \frac{\varepsilon_2}{k^\gamma}\,, \tag{1.3}$$

$$-\frac{\varepsilon_3}{k^\gamma} < \Big(\Delta u(k) - \Delta\Big(\frac{a}{k^s}\Big) - \Delta\Big(\frac{b}{k^{s+1}}\Big)\Big)\Big(\Delta\Big(\frac{b}{k^{s+1}}\Big)\Big)^{-1} < \frac{\varepsilon_4}{k^\gamma}\,, \tag{1.4}$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\Big(\frac{1}{k}\Big) < \Big(\Delta^2 u(k) - \Delta^2\Big(\frac{a}{k^s}\Big) - \Delta^2\Big(\frac{b}{k^{s+1}}\Big)\Big)\Big(\Delta^2\Big(\frac{b}{k^{s+1}}\Big)\frac{ms}{s+2}\Big)^{-1} < \frac{\varepsilon_2}{k^\gamma} + O\Big(\frac{1}{k}\Big) \tag{1.5}$$

hold.

Theorem 1.2. *If there exist $s > -1$ and $\varepsilon_i > 0$, $i = 1, 2, 3, 4$, such that one of the following conditions hold*

(1) $ms > 0$, $\varepsilon_3 < \varepsilon_1$, $\varepsilon_2 > \varepsilon_4$, $\varepsilon_3 > \dfrac{ms}{s+2}\,\varepsilon_1$, $\varepsilon_4 > \dfrac{ms}{s+2}\,\varepsilon_2$;

(2) $ms < 0$, $\varepsilon_3 < \varepsilon_1$, $\varepsilon_2 > \varepsilon_4$, $\varepsilon_3 > -\dfrac{ms}{s+2}\,\varepsilon_2$, $\varepsilon_4 > -\dfrac{ms}{s+2}\,\varepsilon_1$,

then for some K for all $k_0 > K$ there exist a solution $u(k)$ to equation (1.1) *such that for all $k \in \mathbb{N}(k_0)$ and $\gamma = 0$* (1.3)–(1.5) *hold.*

To prove these theorems we had to transform the discrete second-order non-linear equation to the system of two discrete equations, and applying theorems in preliminaries we get the above theorems. For more details to the proof we refer to [1, 7].

2 Another second-order non-linear difference equation

Let us consider the problem of the existence of a nontrivial solution to the equation

$$\Delta^2 v(k) = -k^s (\Delta v(k))^3 \tag{2.1}$$

such that the limit $\lim_{k\to\infty} v(k)$ exists and is finite. More exactly, under the condition $s > 1$, we prove the existence of a solution to equation (2.1) such that the limit

$$\lim_{k\to\infty} v(k) = 0.$$

Theorem 2.1. *Let $s > 1$. Let ε_i, γ_i, $i = 1, 2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then there exists a solution $v = v(k)$ to equation* (2.1) *such that*

$$-\varepsilon_1 |c| k^{-\alpha} < v(k) - ck^{-\alpha} < \gamma_1 |c| k^{-\alpha},$$
$$-\varepsilon_2 \gamma_2 \Delta\big(|c| k^{-\alpha}\big) < \Delta v(k) - (\Delta(ck^{-\alpha})) < \gamma_2 \Delta\big(|c| k^{-\alpha}\big),$$

and

$$\Delta^2 v(k) = O(1)$$

for all $k \in \mathcal{Z}_{k_0}^{\infty}$ provided that k_0 is sufficiently large.

Opposite to the equation in the previous chapter where Theorem 0.1 was used, in this case Theorem 0.2 is applied (for details see [6]).

3 Conclusion

In this article we discussed two different non-linear discrete equations. To prove some properties to its solutions, we used the retract principle described in this article. It can be concluded that other nonlinear discrete differential equations can be investigated in a similar way.

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