Median and p-Median

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1 Introduction

In this paper, we introduce the definition of p-median of a measurable function which is the key concept in the definition of p-oscillation and generalized Kurzweil integral. This new integral is based on minimization of sums of p-oscillations instead of ordinary oscillations which leads to a wider class of integrable functions. We introduce its definition in Section 4. However, it is not obvious how to compute p-oscillation of a given function. It leads us to question if it is possible to classify p-medians of a given function for any p. We can answer this question for $p = 1$, $p = 2$ and $p = \infty$.

2 Preliminaries

Let M be an arbitrary subset of $\mathbb{R}$, then $\mu(M)$ is the Lebesgue measure of M and, for $p \in [1, \infty]$, $L^p(M)$ is, as usual, the space of real valued functions measurable on M and such that $\|f\|_p < \infty$, where

$$\|f\|_p = \bigg(\int\limits_M |f(x)|^p \,\mathrm{d}x \bigg)^{\frac{1}{p}} \text{ if } p \in [1, \infty) \text{ and } \|f\|_\infty = \operatorname*{ess\,sup}_{x \in M} |f(x)|$$

is the usual norm on $L^p(M)$.

3 Median and p-median

Next definition was used in [5] (c.f. Definition 2.5 therein) and it is an analogue of median of random variable in probability and statistics, cf. e.g. [6, Section 1.4].

Definition 3.1 (Median)**.** Let $f : [a, b] \to \mathbb{R}$ be a measurable function. We say that the number $\lambda \in \mathbb{R}$ is the *median* of the function f on $[a, b]$ if there exists a measurable set $M \subset [a, b]$ such that $\mu(M) = \frac{1}{2}(b - a)$, $f \le \lambda$ on M and $f \ge \lambda$ on $[a, b] \setminus M$.

Definition 3.2 (p-median)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in [1, \infty]$. We say that the number $c(p) \in \mathbb{R}$ is the *p-median* of the function f on I if

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \|f - c(p)\|_p.$$

Remark 3.1. The existence of p-median is obvious. Indeed, since the function $g(c) := \|f - c\|_p$ is non-negative, continuous and its limits at $\pm\infty$ are $+\infty$, it follows that it has a minimum.

As we will prove later, median coincides with p-median for $p = 1$. We will also show that for $p = 2$ the p-median coincides with the integral mean value of the given function, while for $p = \infty$ it is simply the arithmetic mean of essential supremum and infimum of the given function. Furthermore, if $f \in L^\infty(I)$, then the relations $\operatorname{ess\,inf} f \leq c(p) \leq \operatorname{ess\,sup} f$ hold for all $p \in [1, \infty]$.

First, we will show that median of a measurable function always exist.

Proposition 3.1. *Every measurable function $f : [a, b] \to \mathbb{R}$ has a median.*

Proof. Let a measurable function $f : [a, b] \to \mathbb{R}$ be given and

$$S_1 := \Big\{\lambda \in \mathbb{R};\ \mu\big(f^{-1}((-\infty, \lambda))\big) \leq \frac{b-a}{2}\Big\},$$
$$S_2 := \Big\{\lambda \in \mathbb{R};\ \mu\big(f^{-1}((\lambda, +\infty))\big) \leq \frac{b-a}{2}\Big\}.$$

The monotonicity of measure implies that $h_1(\lambda) := \mu(f^{-1}((-\infty, \lambda)))$ is non-decreasing on $\mathbb{R}$ and $h_2(\lambda) := \mu(f^{-1}((\lambda, +\infty)))$ is non-increasing on $\mathbb{R}$. Moreover,

$$0 \leq h_i(\lambda) \leq b - a \quad \text{for all} \quad \lambda \in \mathbb{R} \quad \text{and} \quad i \in \{1, 2\}.$$

Denote $A_k := f^{-1}((-\infty, k))$ for $k \in \mathbb{N}$. Then $A_k \subset A_{k+1}$ for each k and in view of the continuity of measure we get

$$\lim_{k\to\infty} \mu(A_k) = \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = b - a.$$

Therefore, there is a $k_1 \in \mathbb{N}$ such that $\mu(f^{-1}((-\infty, k_1))) > \frac{b-a}{2}$. Hence, S_1 is bounded from above. Next, we will show that it is non-empty. To this aim, put $B_k := f^{-1}((-\infty, -k))$ for $k \in \mathbb{N}$. We have $B_{k+1} \subset B_k$ for each k and all these sets have finite measures. Thus, using the continuity of measure again, we obtain

$$\lim_{k\to\infty} \mu(B_k) = \mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) = 0.$$

Therefore, there is a $k_2 \in \mathbb{N}$ such that

$$\mu\big(f^{-1}((-\infty, -k_2))\big) < \frac{b-a}{2}\,.$$

In other words, $S_1 \neq \varnothing$. Analogously, we can prove that S_2 is nonempty and bounded from below.

Obviously, $\lambda_1 = \sup S_1 < \infty$ and $-\infty < \lambda_2 = \inf S_2$. Moreover, it is easy to see that $\lambda_2 \leq \lambda_1$. Indeed, if the opposite was true, we could find numbers c_1, c_2 such that $\lambda_1 < c_1 < c_2 < \lambda_2$. In such a case we would have

$$\mu\big(f^{-1}((-\infty, c_1))\big) > \frac{b-a}{2} \quad \text{and} \quad \mu\big(f^{-1}((c_2, +\infty))\big) > \frac{b-a}{2}$$

a contradiction, since

$$f^{-1}((-\infty, c_1)) \cap f^{-1}((c_2, \infty)) = \varnothing,$$

while

$$f^{-1}((-\infty, c_1)) \cup f^{-1}((c_2, \infty)) = [a, b].$$

Let $\lambda_2 < \lambda_1$ and let an arbitrary $\xi \in (\lambda_2, \lambda_1)$ be given. Then we can choose $\xi_1 \in S_1$ and $\xi_2 \in S_2$ in such a way that $\lambda_2 < \xi_2 < \xi < \xi_1 < \lambda_1$. By the definitions of the sets S_1, S_2, we have

$$\mu\big(f^{-1}((-\infty, \xi])\big) \leq \mu\big(f^{-1}((-\infty, \xi_1))\big) \leq \frac{b-a}{2} \quad \text{and} \quad \mu\big(f^{-1}((\xi, \infty))\big) \leq \mu\big(f^{-1}((\xi_2, \infty))\big) \leq \frac{b-a}{2}\,.$$

Thus,

$$\mu\big(f^{-1}((-\infty,\xi])\big)+\mu\big(f^{-1}((\xi,\infty))\big)\leq\mu\big(f^{-1}((-\infty,\xi_1))\big)+\mu\big(f^{-1}((\xi_2,\infty))\big)\leq b-a. \tag{3.1}$$

On the other hand, $f^{-1}((-\infty,\xi])\cup f^{-1}((\xi,\infty))=[a,b]$ and this together with (3.1) yields

$$\mu\big(f^{-1}((-\infty,\xi])\big)+\mu\big(f^{-1}((\xi,\infty))\big)=b-a,$$

i.e. any $\xi\in(\lambda_1,\lambda_2)$ is the median of f.

It remains to consider the case $\lambda_1=\lambda_2$. Thus, let $\lambda^*:=\lambda_1=\lambda_2$. Then, since $(-\infty,\lambda^*)\subset S_1$, we have $\mu(f^{-1}((-\infty,\lambda)))\leq\frac{1}{2}(b-a)$ for all $\lambda<\lambda^*$ and, thanks to the continuity of measure,

$$\mu\big(f^{-1}((-\infty,\lambda^*))\big)=\lim_{\lambda\to\lambda^*}\mu\big(f^{-1}((-\infty,\lambda))\big)\leq\frac{b-a}{2}\,. \tag{3.2}$$

Similarly,

$$\mu(f^{-1}(\lambda^*,\infty))\leq\frac{b-a}{2}\,. \tag{3.3}$$

If one of the relations (3.2), (3.3) reduces to the equality, then λ^* will be the median of f. Indeed, if $\mu(f^{-1}(\lambda^*,\infty))=\frac{b-a}{2}$, then for $M=f^{-1}(\lambda^*,\infty))$, we have

$$f(M)=(\lambda^*,\infty),\ \ \mu(M)=\frac{b-a}{2}\ \text{ and }\ f([a,b]\setminus M)\subset(-\infty,\lambda^*].$$

Now, assume that both inequalities (3.2) and (3.3) are strict. Then, as obviously

$$[a,b]=f^{-1}\big(((-\infty,\lambda^*))\big)\cup f^{-1}(\{\lambda^*\})\cup f^{-1}(((\lambda^*,\infty))),$$

the set $f^{-1}(\{\lambda^*\})$ is nonempty and $\mu(f^{-1}(\{\lambda^*\}))>0$. We can define

$$h(t)=\mu\big([a,t]\cap f^{-1}(\{\lambda^*\})\big)\ \text{ for }\ t\in[a,b].$$

As h is continuous on $[a,b]$, $h(a)=0$ and $h(b)=b-a$, we can find a $t_0\in[a,b]$ such that

$$h(t_0)=\mu\big([a,t_0]\cap f^{-1}(\{\lambda^*\})\big)=\frac{b-a}{2}-\mu(f^{-1}(-\infty,\lambda^*))>0.$$

Furthermore,

$$f^{-1}(\{\lambda^*\})=A\cup B,$$

where

$$A:=[a,t_0]\cap f^{-1}\big(\{\lambda^*\}\big)\ \text{ and }\ B:=(t_0,b]\cap f^{-1}\big(\{\lambda^*\}\big)$$

are disjoint. Simultaneously,

$$\mu\big(A\cup f^{-1}(-\infty,\lambda^*)\big)=\frac{b-a}{2}\,,\quad \mu\big(B\cup f^{-1}((\lambda^*,\infty))\big)=\frac{b-a}{2}\,,$$
$$f(x)\leq\lambda^*\ \text{ for }\ x\in A\cup f^{-1}((-\infty,\lambda^*))\ \text{ and }\ f(x)\geq\lambda^*\ \text{ for }\ x\in B\cup f^{-1}((\lambda^*,\infty)).$$

It follows easily that λ^* is the median of the function f. □

Example 3.1. Median doesn't have to be uniquely determined, as shown by the following example. The median of the function $f:[0,2]\to\mathbb{R}$ given by the formula

$$f(x)=\begin{cases}0 & \text{if } x\in[0,1),\\ 1 & \text{if } x\in[1,2]\end{cases}$$

can be any number from the interval $[0,1]$. Indeed, let $M=[0,1)$. Then given an arbitrary $\lambda\in[0,1]$, we have $f\leq\lambda$ on M and $f\geq\lambda$ on $[0,2]\setminus M$.

On the other hand, it is easy to verify that if $p>1$, then all the p-medians of the function f on $[0,2]$ are equal to $\frac{1}{2}$.

Example 3.2. The median of the function $\sin x$ on $[-\pi, \pi]$ is zero, as well as all its p-medians with $p > 1$ (shown in [3]).

Example 3.3. The median of the function $\sin x$ on the interval $[0, \pi]$ equals $\frac{1}{2}\sqrt{2}$ because $\sin x \geq \frac{\sqrt{2}}{2}$ for all $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ and $\sin x \leq \frac{\sqrt{2}}{2}$ for all $x \in [0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$. On the other hand, its p-median $c(\infty)$ for $p = \infty$ equals $\frac{1}{2}$, while

$$c(2) = \frac{1}{\pi} \int_0^\pi \sin x \, \mathrm{d}x = \frac{2}{\pi} \, .$$

Proposition 3.2. *Median of the* ***continuous*** *function $f : [a, b] \to \mathbb{R}$ is uniquely determined.*

Proof. For a contradiction, let us suppose that f has two medians λ_1, λ_2 such that $\lambda_1 < \lambda_2$. Then, by the definition of the median, there are measurable sets $M_1, M_2 \subset [a, b]$, of measure $\frac{b-a}{2}$ and such that

$$f(x) \leq \lambda_1 \text{ for all|; } x \in M_1 \text{ and } f(x) \geq \lambda_1 \text{ for all|; } x \in [a, b] \setminus M_1$$

and

$$f(x) \leq \lambda_2 \text{ for all|; } x \in M_2 \text{ and } f(x) \geq \lambda_2 \text{ for all|; } x \in [a, b] \setminus M_2.$$

Using these properties, we get

$$\frac{b-a}{2} = \mu(M_2) \geq \mu\big(f^{-1}((-\infty, \lambda_2))\big) \geq \mu(M_1) + \mu\big(f^{-1}((\lambda_1, \lambda_2))\big) = \frac{b-a}{2} + \mu\big(f^{-1}((\lambda_1, \lambda_2))\big).$$

It follows that $\mu(f^{-1}((\lambda_1, \lambda_2))) = 0$. However, the preimage of an open interval (λ_1, λ_2) under a continuous mapping f must be an open set. Therefore, $f^{-1}((\lambda_1, \lambda_2))$ is an open set of measure zero, so it must be empty. Thus, the range H_f of the function f must be a subset of the set $[0, \lambda_1] \cup [\lambda_2, \infty]$. Since the continuous image of the interval $[a, b]$ is again an interval, it must be either $H_f \subset [0, \lambda_1]$ or $H_f \subset [\lambda_2, \infty]$. But, in the former case it is $f(x) \leq \lambda_1$ for all $x \in [a, b]$ which implies that λ_2 can not be the median of f. Similarly, in the latter case we have $f(x) \geq \lambda_2$ for all $x \in [a, b]$ which means that λ_1 can not be the median of f. These conclusions contradicts our assumption, of course. □

Proposition 3.3. *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^\infty(I)$. Put*

$$A := \operatorname*{ess\,inf}_{x \in I} f(x) \quad \textit{and} \quad B := \operatorname*{ess\,sup}_{x \in I} f(x).$$

Then

$$\inf_{c \in \mathbb{R}} \|f - c\|_p = \inf_{c \in [A,B]} \|f - c\|_p \quad \textit{for all} \quad p \in [1, \infty].$$

Furthermore,

$$\inf_{c \in \mathbb{R}} \|f - c\|_\infty = \left\| f - \frac{A+B}{2} \right\|_\infty = \frac{B-A}{2} \, .$$

Proof.
(i) First, let us prove the first part of the statement, i.e. that the sought number c will always lie in the interval $[A, B]$. In other words, we want to show that

$$\inf_{z \in [A,B]} \|f - z\|_p \leq \|f - c\|_p \text{ for all } c \in (-\infty, A) \cup (B, \infty).$$

If $c \in (B, \infty)$, then for almost all $x \in [a, b]$ we have

$$|f(x) - c| = c - f(x) > B - f(x) = |f(x) - B|.$$

Consequently, $|f(x)-c|^p > |f(x)-B|^p$ for a.e. $x\in I$ and, thus, $\|f-c\|_p \ge \|f-A\|_p$.

In case $p=\infty$ we have

$$\|f-c\|_\infty \ge \|f-B\|_\infty \ \text{ if } \ c>B \ \text{ and } \ \|f-c\|_\infty \ge \|f-A\|_\infty \ \text{ if } \ c<A.$$

To summarize,

$$\inf_{c\in\mathbb{R}} \|f-c\|_p = \inf_{c\in[A,B]} \|f-c\|_p \ \text{ for all } \ p\in[1,\infty].$$

(ii) Let us prove the remaining part of the statement. If $c\in\mathbb{R}$ is an arbitrary constant, then

$$\operatorname{ess\,inf}(f(x)-c) = A-c \ \text{ and } \ \operatorname{ess\,sup}(f(x)-c) = B-c.$$

Thus

$$\|f-c\|_\infty = \max\big\{|A-c|,|B-c|\big\}.$$

Function $y(c)=\max\{|A-c|,|B-c|\}$ has a minimum for $c=\frac{A+B}{2}$ and, therefore,

$$\inf_{c\in\mathbb{R}} \|f-c\|_\infty = \max\bigg\{\Big|A-\frac{A+B}{2}\Big|,\Big|B-\frac{A+B}{2}\Big|\bigg\} = \frac{B-A}{2}\,. \qquad \square$$

Remark 3.2. Analogously, if instead of p-norm we consider the supremum norm $\|f\| = \sup_{x\in I}|f(x)|$, we get

$$\inf_{c\in\mathbb{R}} \|f-c\| = \Big\|f-\frac{1}{2}\big(\sup f(x)+\inf f(x)\big)\Big\|.$$

Proposition 3.4. *Let $I\subset\mathbb{R}$ be a bounded interval and $f\in L^2(I)$. Then*

$$\inf_{c\in\mathbb{R}} \|f-c\|_2 = \bigg\|f-\frac{1}{\mu(I)}\int_I f(t)\,\mathrm{d}t\bigg\|_2.$$

In other words, for $p=2$ the p-median of f equals to the integral mean value of f.

Proof. Let $c\in\mathbb{R}$. Since $L^2(I)\subset L^1(I)$ for I bounded, both integrals $\int_I f(x)\,\mathrm{d}x$ and $\int_I f^2(x)\,\mathrm{d}x$ exist and are finite. Therefore,

$$g(c) := \|f-c\|_2^2 = \int_I (f(x)-c)^2\,\mathrm{d}x = \int_I f^2(x)\,\mathrm{d}x - 2c\int_I f(x)\,\mathrm{d}x + c^2\mu(I).$$

This is a quadratic function of c with a positive leading coefficient and thus it must have a minimum. Its derivative is

$$g'(c) = -2\int_I f(x)\,\mathrm{d}x + 2c\,\mu(I).$$

Hence

$$g'(c)=0 \ \text{ if and only if } \ c=\frac{1}{\mu(I)}\int_I f(x)\,\mathrm{d}x.$$

This is its stationary point, and the function g takes a minimum there. Therefore, it is also a minimum of the function $\|f-c\|_2$. $\square$

4 Oscillations and HK^p integral

In this section we introduce the notions of oscillation which is the key concept in the definition of HKS^p integral. Next definition is taken from [5, Definition 2.3].

Definition 4.1 (Oscillations)**.** Let $I \subset \mathbb{R}$ be a bounded interval and $p \in [1, \infty]$. We define the p-oscillation of a measurable function $f : I \to \mathbb{R}$ as

$$\operatorname{osc}_p(f, I) := (\mu(I))^{-\frac{1}{p}} \inf_{c \in \mathbb{R}} \|f - c\|_p.$$

Here and in what follows we set $\frac{1}{p} = 0$ if $p = \infty$.

The following proposition is taken from [5, Proposition 2.6].

Proposition 4.1 (Oscillation and median relation)**.** *Let $\lambda \in \mathbb{R}$ be the median of the function f on the bounded interval $I \subset \mathbb{R}$ and $p \in [1, \infty]$. Then*

$$\operatorname{osc}_p(f, I) \leq (\mu(I))^{-\frac{1}{p}} \|f - \lambda\|_p \leq 2^{1-\frac{1}{p}} \operatorname{osc}_p(f, I).$$

In particular, for $p = 1$ we get

$$\operatorname{osc}_1(f, I) = (\mu(I))^{-1} \|f - \lambda\|_1. \tag{4.1}$$

It implies that

$$\inf_{c \in \mathbb{R}} \|f - c\|_1 = \|f - \lambda\|_1.$$

In other words, median coincides with p-median for $p = 1$.

Next, we will introduce the new definition of generalized Kurzweil integral based on minimization of sum of p-oscillations instead of ordinary oscillations which leads to a wider class of integrable functions.

Definition 4.2. We say that $\{[a_i, b_i], x_i\}_{i=1}^n$ ($n \in \mathbb{N}$) is a **tagged partition** of the interval $I \subset \mathbb{R}$ if the intervals $[a_i, b_i]$ are non-overlapping, their union is I and $x_i \in [a_i, b_i]$ for every $i \in \{1, \ldots, n\}$.

Definition 4.3. Let an arbitrary positive function $\delta : [a, b] \to \mathbb{R}^+$ be given. We say that the tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ is **δ-fine** if

$$[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i)) \text{ for all } i \in \{1, \ldots, n\}.$$

Definition 4.4 (Generalized Kurzweil integral)**.** Let $I \subset \mathbb{R}$ be an interval, f, F be functions measurable on I. We say that F is an indefinite **HK^p integral** of a function f if for all $\varepsilon > 0$ there exists $\delta_\varepsilon : I \to \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} \operatorname{osc}_p \big(F - f(x_i)\, x, [a_i, b_i]\big) < \varepsilon$$

holds for each **δ_ε-fine** tagged partition $\{[a_i, b_i], x_i\}_{i=1}^n$ of the interval I.

5 Examples

Example 5.1. Next example shows that even if the p-median $c(p)$ is determined uniquely for all $p\in[1,\infty]$, the function $p\mapsto c(p)$ **need not be monotone**, in general. Indeed, for the function

$$f(x)=\begin{cases}\sin^2 x & \text{if } x\in[0,\pi],\\ \sin x & \text{if } x\in(\pi,2\pi]\end{cases}$$

we have $c(1)=c(\infty)=0$. On the other hand, $c(2)$ is negative, as by Proposition 3.4 we have

$$c(2)=\frac{1}{2\pi}\int\limits_0^{2\pi} f(x)\,\mathrm{d}x=\frac{1}{2\pi}\Big(\frac{\pi}{2}-2\Big)<0.$$

Example 5.2. Let $f(x)=\begin{cases}1 & \text{if } x\in J,\\ 0 & \text{if } x\in I\setminus J,\end{cases}$ where $I\subset\mathbb{R}$ is bounded interval and $J\subset I$ its subinterval.

It was shown in [3, Example 2.2.6] that the p-medians $c(p)$ of this function are uniquely determined and they are explicitly given by the formula

$$c(p)=\bigg(\Big(\frac{\mu(I\setminus J)}{\mu(J)}\Big)^{\frac{1}{p-1}}+1\bigg)^{-1}\ \text{ for }\ p\in(1,\infty).$$

Notice that the limit of p-medians as $p\to+\infty$ is indeed the arithmetic mean of essential suprema and infima, i.e $\lim\limits_{p\to+\infty}c(p)=\frac{1}{2}=c(\infty)$. Further, notice also that

$$\lim_{p\to 1+}c(p)=1=c(1)\ \text{ if }\ \mu(J)>\mu(I\setminus J)\ \text{ and }\ \lim_{p\to 1+}c(p)=0=c(1)\ \text{ if }\ \mu(J)<\mu(I\setminus J),$$

i.e. the limit of p-medians $c(p)$ as $p\to 1+$ is indeed the median of f.

If $\mu(I\setminus J)=\mu(J)$, then $c(p)=\frac{1}{2}$ for all $p\in(1,\infty]$, while the median of f is not unique as it can be any number from the interval $[0,1]$.

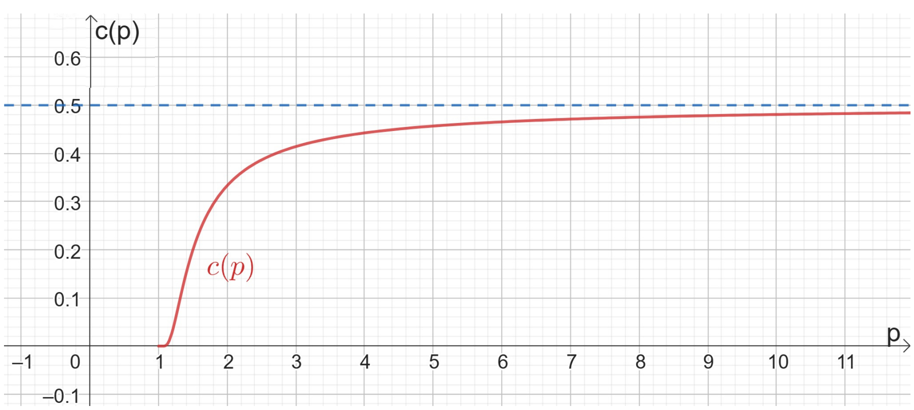

Figure 1. Graph of function $c(p)$ in case $\frac{\mu(I\setminus J)}{\mu(J)}=2$

Example 5.3. Let

$$f(x)=\begin{cases}8 & \text{if } x\in[0,1],\\ 0 & \text{if } x\in(1,6],\\ -4 & \text{if } x\in(6,10].\end{cases}$$

Then the p-median $c(p)$ of f on $[0,10]$ is determined uniquely for any $p \in [1,\infty]$, but no explicit formula for $c(p)$ is available. One can verify that $c(1) = c(3) = 0$, while $c(2) < 0$ and $c(\infty) > 0$. Thus, the function $p \mapsto c(p)$ is not monotone. Notice that the arithmetic mean of suprema and infima is $c(\infty) = 2$, while the integral mean value evaluates $c(2) = -\frac{4}{5}$.

For a given $p > 1$ let us denote again $g(c) := \|f-c\|_p^p$. Obviously, $g(c) = (8-c)^p+5|c|^p+4(4+c)^p$ and g is continuous $[-4,8]$. Furthermore, $g'(c) = -p(8-c)^{p-1} + 5p|c|^{p-1}\operatorname{sgn} c + 4p(4+c)^{p-1}$ for $c \neq 0$. One can verify that g' is continuous and increasing on $[-4,0) \cup (0,8]$, while $g'(-4) < 0$, $g'(8) > 0$ and $g'(0-) = g'(0+) = p(4^p - 8^{p-1}))$. In particular, for a given $p \in (1,\infty)$, there is exactly one point $c(p) \in (-4,8)$ such that $g'(c(p)) = 0$. This defines implicitly the function $p \mapsto c(p)$. In addition, g is decreasing on $[-4,c(p)]$ and increasing on $[c(p),8]$. Finally, notice that $g'(0-) = g'(0+) = 0$ if and only if $8^{p-1} = 4^p$, i.e. if and only if $p = 3$, i.e. $c(3) = 0$.

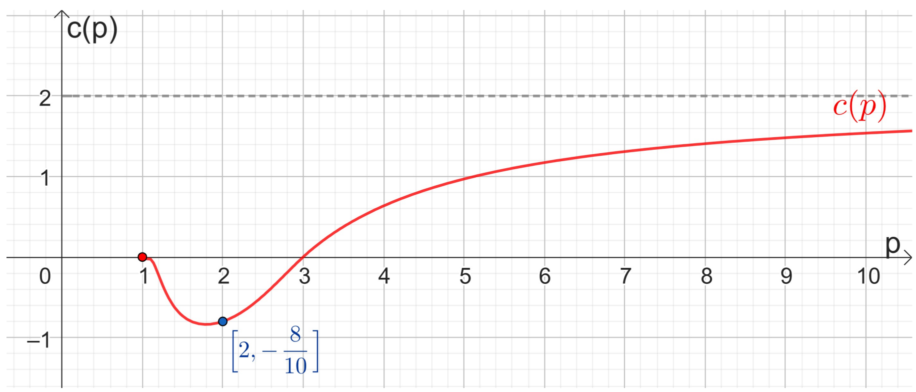

Figure 2. Graph of $c(p)$

6 Open problems

- (Uniqueness of p-median) Let $I \subset \mathbb{R}$ be a bounded interval and $f \in L^p(I)$ for some $p \in (1,\infty]$. Is there a unique number $c(p) \in \mathbb{R}$ such that

$$\inf_{c\in\mathbb{R}} \|f-c\|_p = \|f-c(p)\|_p\,?$$

  We have proved the uniqueness of p-medians if $p > 1$ for step function, analogously as in Example 5.3, but still we don't know if there is uniqueness in general for $f \in L^p(I)$. If yes, it would be interesting to investigate properties of function $p \mapsto c(p)$.

- (Limits of p-medians) In Example 5.2 we have seen that for the function f considered there the limit of p-medians $c(p)$ as $p \to \infty$ is $c(\infty)$ and the limit of $c(p)$ as $p \to 1+$ is the median of f. The question is whether this is true in general.

- (Properties of p-medians) Is function $p \mapsto c(p)$ continuous? Is it differentiable? Is it true that $\lim_{p\to 1_+} c'(p) = 0$?

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