On Two-Point Boundary Value Problems for Higher Order Singular Advanced Differential Equations

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In the present report, on the finite interval $]a,b[$ we consider the n-th order advanced ordinary differential equation

$$u^{(n)}(t)=f\big(t,u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\big) \tag{1}$$

under the two-point nonlinear boundary conditions

$$\varphi\big(u(a),\dots,u^{(m-1)}(a)\big)=c_0,\quad u^{(i-1)}(b)=\varphi_i(u^{(n-1)}(b))\ \ (i=1,\dots,n-1). \tag{2}$$

Here $n\geq 2$, $m\in\{1,\dots,n\}$, c_0 is a positive constant, and $f:]a,b]\times\mathbb{R}^n\to\mathbb{R}$, $\tau_i:[a,b]\to[a,b]$ $(i=1,\dots,n)$, $\varphi:\mathbb{R}^m\to\mathbb{R}$, $\varphi_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n-1)$ are continuous functions, $\mathbb{R}=]-\infty,+\infty[$. Moreover,

$$a\leq t<\tau_i(t)\leq b\ \text{ for }\ a\leq t<b\ \ (i=1,\dots,n), \tag{3}$$

$$\varphi(0,\dots,0)=0,\ \ \varphi(x_1,\dots,x_m)\to+\infty\ \text{ as }\ (-1)^{i-1}x_i\to+\infty\ \ (i=1,\dots,m), \tag{4}$$

$$(-1)^{n-i}\varphi_i(x)x\geq 0\ \text{ for }\ x\in\mathbb{R}\ \ (i=1,\dots,n-1). \tag{5}$$

A solution of equation (1) is sought in the space of n-times continuously differentiable functions defined in the interval $]a,b[$.

By $u(a)$ and $u(b)$ (by $u^{(i)}(a)$ and $u^{(i)}(b)$) we denote, respectively, the right and the left limits of the solution u (of the i-th derivative of u) at the points a and b.

A solution u of equation (1) is said to be **a solution of problem** (1), (2) if there exist one-sided limits $u^{(i-1)}(a)$ $(i=1,\dots,m)$, $u^{(k-1)}(b)$ $(k=1,\dots,n)$, and equalities (2) are satisfied.

A solution u is said to be a Kneser solution if

$$(-1)^iu^{(i)}(t)u(t)\geq 0\ \text{ for }\ a<t<b\ \ (i=1,\dots,n-1).$$

In the case, where $\tau(t)\equiv t$, two-point boundary value problems for equation (1) have long attracted the attention of specialists, and most of them, namely, some problems with boundary conditions of type (2), have been studied in sufficient detail (see [1–7] and the references therein). As for the case of advance, i.e. when inequalities (3) hold, two-point boundary value problems for equation (1), as far as we know, remains still unstudied.

The results below fill the above mentioned gap to some extent. They contain unimprovable in a certain sense conditions guaranteeing, respectively, the solvability and unique solvability of problem (1), (2) in the space of Kneser type functions. It should be noted that these conditions do not restrict the growth order of the function f in the phase variables at infinity, and contain the case where the function f has a nonintegrable singularity in the time variable at the point $t=0$, more precisely, the case, where

$$\int\limits_a^b|f(t,x_1,\dots,x_n)|\,dt=+\infty\ \text{ for }\ x_i\neq 0\ \ (i=1,\dots,n).$$

To formulate the main results, we need to introduce the following notations.

$$f^*(t;r)=\max\Big\{|f(t,x_1,\dots,x_n)|:\ \ 0\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}\ \text{ for }\ a<t\le b,\ \ r>0,$$

$$f_*(t;\delta,r)=\min\Big\{|f(t,x_1,\dots,x_n)|:\ \ \delta\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}\ \text{ for }\ a<t\le b,\ \ r>\delta>0,$$

$$\varphi_r(x)=\min\Big\{\varphi(x_1,\dots,x_{m-1},x):\ \ 0\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,m-1)\Big\}$$
$$\text{for }\ m>1,\ \ r>0,\ \ x\in\mathbb{R},$$

$$\varphi_r(x)=\varphi(x)\ \text{ for }\ m=1,\ \ r>0,\ \ x\in\mathbb{R}.$$

Theorem 1. *If along with* (3)–(5) *the conditions*

$$f(t,0,\dots,0)=0,\quad (-1)^n f(t,x_1,\dots,x_n)\ge 0\ \text{ for }\ a<t<b,\ \ (-1)^{i-1}x_i\ge 0\ \ (i=1,\dots,n),\tag{6}$$

$$\int\limits_a^b (t-a)^{n-m} f^*(t;r)\,dt<+\infty\ \text{ for }\ r>0\tag{7}$$

hold, then problem (1), (2) *has at least one nonnegative Kneser solution.*

Theorem 2. *If along with* (3), (5), (6) *the conditions*

$$\varphi(0,\dots,0)=0,\quad \varphi_r(x)\to+\infty\ \text{ for }\ r>0,\ \ (-1)^{n-1}x\to+\infty,\tag{4'}$$

$$\int\limits_a^b (t-a)^{n-m} f_*(t;\delta,r)\,dt=+\infty\ \text{ for }\ r>\delta>0$$

hold, then problem (1), (2) *has no nonnegative Kneser solution.*

From the above formulated theorems it follows

Corollary 1. *Let conditions* (3), (4′), (5), (6) *hold and let for every constants* $r>0$ *and* $\delta\in\,]0,r[$ *there exist a positive number* $\rho(r,\delta)$ *such that*

$$f^*(t;r)\le\rho(r,\delta)f_*(t;\delta,r)\ \text{ for }\ a<t<b.\tag{8}$$

Then for problem (1), (2) *to have at least one nonnegative Kneser solution, it is necessary and sufficient that condition* (7) *to satisfied.*

Remark 1. Conditions (6) and (8) are satisfied, for example, in the case, where

$$f(t,x_1,\dots,x_n)=\sum_{j=1}^k p_j(t)f_j(x_1,\dots,x_n),$$

where k is any natural number, $p_j:\,]a,b]\to\mathbb{R}$, $f_j:\mathbb{R}^n\to\mathbb{R}$ $(j=1,\dots,k)$ are continuous functions such that

$$(-1)^n p_j(t)\ge 0\ \text{ for }\ a<t<b\ \ (j=1,\dots,k),$$
$$f_j(0,\dots,0)=0\ \ (j=1,\dots,k),$$
$$\min\Big\{f_j(x_1,\dots,x_n):\ \ \delta\le(-1)^{i-1}x_i\le r\ \ (i=1,\dots,n)\Big\}>0\ \text{ for }\ r>\delta>0\ \ (j=1,\dots,k).$$

Example 1. Consider the equation

$$u^{(n)}(t)=p(t)h\big(|u^{(n-1)}(\tau(t))|\big) \tag{9}$$

with the boundary conditions (2), where $m\le n-1$, while $p:]a,b]\to\mathbb{R}$, $h:[0,+\infty[\to\mathbb{R}$, and $\tau:[a,b]\to[a,b]$ are continuous functions, and

$$(-1)^n p(t)\ge 0 \ \text{ for } \ a<t<b, \quad \int\limits_a^b (t-a)^{n-m}|p(t)|\,dt<+\infty,$$

$$\int\limits_a^b |p(t)|\,dt=+\infty, \tag{10}$$

$$h(0)=0, \ \ h(x)>0 \ \text{ for } \ x>0,$$

$$\int\limits_\delta^{+\infty}\frac{ds}{h(s)}<+\infty \ \text{ for } \ \delta>0. \tag{11}$$

If along with (4), (5) the condition

$$a\le t<\tau(t)\le b \ \text{ for } \ a\le t<b \tag{12}$$

is satisfied, then according to Theorem 1 problem (9), (2) has at least one nonnegative Kneser solution. Assume now that all the above conditions are satisfied except of (12) instead of which we have

$$\tau(t)=t \ \text{ for } \ a\le t\le a_0, \quad t<\tau(t)\le b \ \text{ for } \ a_0<t<b,$$

where $a_0\in\,]a,b[$. Show that in this case problem (9), (2) has no nonnegative Kneser solution. Assume the contrary that there exists such a solution u. Then there are $\delta>0$ and $t_0\in\,]a,a_0]$ such that

$$0<\delta\le(-1)^{n-1}u^{(n-1)}(t)<+\infty \ \text{ for } \ a<t\le t_0.$$

On the other hand,

$$|u^{(n-1)}(t)|'=-|p(t)|h\big(|u^{(n-1)}(t)|\big) \ \text{ for } \ a<t\le t_0.$$

Therefore,

$$\int\limits_\delta^{|u^{(n-1)}(t)|}\frac{dx}{h(x)}=\int\limits_t^{t_0}|p(t)|\,dt \ \text{ for } \ a<t\le t_0,$$

which contradicts conditions (10) and (11).

The above constructed example shows that if instead of (3) for some $a_0\in\,]a,b[$ the conditions

$$a\le t<\tau_i(t)\le b \ \text{ for } \ a\le t<b \ \ (i=1,\dots,n-1), \quad \tau_n(t)=t \ \text{ for } \ a\le t\le a_0,$$
$$t<\tau_n(t)\le b \ \text{ for } \ a_0<t<b$$

hold, then conditions (4)–(7) do not guarantee the existence of a nonnegative Kneser solution of problem (1), (2).

To simplify the presentation, we will consider the question on the uniqueness of a solution of problem (1), (2) in the case where the boundary conditions (2) have the form

$$\begin{gathered}\sum_{i=1}^m \alpha_i|u^{(i-1)}(a)|^{\mu_i}\operatorname{sgn}(u^{(i-1)}(a))=c_0,\\ u^{(j-1)}(b)=\beta_j|u^{(n-1)}(b)|^{\nu_j}\operatorname{sgn}(u^{(n-1)}(b)) \ \ (j=1,\dots,n-1),\end{gathered} \tag{2'}$$

where

$$(-1)^{i-1}\alpha_i \geq 0, \quad \mu_i > 0 \ \ (i=1,\dots,m), \quad \alpha_m \neq 0,$$
$$(-1)^{n-j}\beta_j \geq 0, \quad \nu_j > 0 \ \ (j=1,\dots,n-1).$$

Evidently, in this case the function

$$\varphi(x_1,\dots,x_m) \equiv \sum_{i=1}^{m} \alpha_i |x_i|^{\mu_i} \operatorname{sgn}(x_i)$$

satisfies conditions ($4'$), and the functions $\varphi_j(x) = \beta_j |x|^{\nu_j} \operatorname{sgn}(x)$ $(j=1,\dots,n-1)$ – conditions (5).

We will say that the function f is **locally Lipschitzian in the phase variables on the set** $[a_0,b]\times\mathbb{R}^n$ if for every $r>0$ there exists $\ell(r)>0$ such that

$$\big|f(t,x_1,\dots,x_n) - f(t,y_1,\dots,y_n)\big| \leq \ell(r)\sum_{i=1}^{n}|x_i-y_i| \ \text{ for } \ a_0 \leq t \leq b, \ \ \sum_{i=1}^{n}\big(|x_i|+|y_i|\big) \leq r.$$

Theorem 3. *Let along with* (3), (7) *the condition*

$$f(t,0,\dots,0)=0, \quad (-1)^n f(t,x_1,\dots,x_n) \geq (-1)^n f(t,y_1,\dots,y_n) \geq 0$$
$$\text{for } \ a<t<b, \ \ (-1)^{i-1}x_i \geq (-1)^{i-1}y_i \geq 0 \ \ (i=1,\dots,n)$$

hold. Let, moreover, there exist $a_0 \in]a,b[$ such that the function f is locally Lipschitzian in the phase variables on the set $[a_0,b]\times\mathbb{R}^n$. Then problem (1), ($2'$) *has a unique nonnegative Kneser solution.*

Finally, we consider two nontrivial particular cases of equation (1):

$$u^{(n)}(t) = \sum_{i=1}^{n} p_i(t) f_i\big(u^{(i-1)}(\tau_i(t))\big), \tag{13}$$

$$u^{(n)}(t) = \sum_{i=1}^{n} p_i(t)\big|u^{(i-1)}(\tau_i(t))\big|^{\lambda_i} \operatorname{sgn}\big(u^{(i-1)}(\tau_i(t))\big), \tag{14}$$

where $p_i :]a,b] \to \mathbb{R}$, $f_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ are continuous functions, while λ_i $(i=1,\dots,n)$ are constants.

Corollary 1 and Theorem 3 yield the following results.

Corollary 2. *Let the functions τ_i $(i=1,\dots,n)$ satisfy inequalities* (3), *and let the functions p_i and f_i $(i=1,\dots,n)$ be such that*

$$(-1)^{n+i-1}p_i(t) \geq 0 \ \text{ for } \ a<t<b \ \ (i=1,\dots,n), \tag{15}$$
$$f_i(0)=0, \ \ (-1)^{i-1}f_i(x)>0 \ \text{ for } \ (-1)^{i-1}x>0 \ \ (i=1,\dots,n).$$

Then for problem (13), ($2'$) *to have at least one nonnegative Kneser solution, it is necessary and sufficient that the conditions*

$$\int_a^b (t-a)^{n-m}|p_i(t)|\,dt < +\infty \ \ (i=1,\dots,n) \tag{16}$$

to satisfied.

Corollary 3. *Let the functions* τ_i *and* p_i $(i = 1, \dots, n)$ *satisfy inequalities* (3) *and* (15)*, and let* f_i $(i = 1, \dots, n)$ *be locally Lipschitzian functions such that*

$$f_i(0) = 0, \ \ (-1)^{i-1} f_i(x) \geq (-1)^{i-1} f_i(y) > 0 \ \ \textit{for} \ \ (-1)^{i-1} x \geq (-1)^{i-1} y > 0 \ \ (i = 1, \dots, n).$$

Then for problem (13)*,* (2′) *to have a unique nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

From Corollaries 2 and 3 it follow Corollaries 4 and 5, respectively.

Corollary 4. *Let*

$$\lambda_i > 0 \ \ (i = 1, \dots, n),$$

and let the functions τ_i*,* p_i $(i = 1, \dots, n)$ *satisfy inequalities* (3) *and* (15)*. Then for problem* (14)*,* (2′) *to have at least one nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

Corollary 5. *Let*

$$\lambda_i \geq 1 \ \ (i = 1, \dots, n),$$

and let the functions τ_i*,* p_i $(i = 1, \dots, n)$ *satisfy inequalities* (3) *and* (15)*. Then for problem* (14)*,* (2′) *to have a unique nonnegative Kneser solution, it is necessary and sufficient that conditions* (16) *be satisfied.*

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