# Deep Neural Network for Approximate Solution of One System of Nonlinear Integro-differential Equations

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The analysis of electromagnetic field penetration into materials, its mathematical representation, and subsequent computational solutions represent a crucial aspect of applied mathematics. This phenomenon often involves the production of heat energy, which modifies the medium's permeability and affects diffusion processes. Such effects arise from the strong dependence of the material's conductivity on temperature. The mathematical description of these processes, like many other real-world problems, results in systems of nonlinear partial differential equations and integro-differential equations. In the quasistatic approximation, the system of Maxwell's equations can be expressed in a following form [17]

$$\frac{\partial H}{\partial t} = -\nabla \times (\nu_m \nabla \times H), \quad c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\nabla \times H)^2. \tag{1}$$

The equations (1) describe the evolution of magnetic fields and temperature within the medium, considering Joule heating effects. The coefficients for thermal capacity and electrical conductivity are assumed to depend on temperature. Under specific assumptions, as shown in [5], the system of Maxwell's equations can be reduced to a nonlinear parabolic-type integro-differential equation [5]

$$\frac{\partial H}{\partial t} = -\nabla \times \left[ a\left( \int_0^t |\nabla \times H|^2 \, d\tau \right) \nabla \times H \right], \tag{2}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

The integro-differential equation (2) derived in this context is complex, and only specific cases of this model have been studied in depth (see references such as [5–20, 24] and references therein). By assuming a specific structure for the magnetic field, in particular, if $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, the vector equation (2) becomes as the following system of nonlinear integro-differential equations:

$$\begin{aligned} &\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[ a\left( \int_0^t \left[ \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau \right) \frac{\partial U}{\partial x} \right] = 0, \\ &\frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left[ a\left( \int_0^t \left[ \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau \right) \frac{\partial V}{\partial x} \right] = 0. \end{aligned} \tag{3}$$

This note aims to apply Deep Neural Network (DNN) approach implemented in [15] to solve the Dirichlet initial-boundary value problem for system (3). The focus is on approximate solution to the nonlinear equations using neural network architectures, where the diffusion coefficient has the following form $a(S) = (1 + S)^p$, $0 < p \le 1$.

Thus, our goal is to apply DNN for the approximate solution of the the following nonlinear initial-boundary value problem with nonhomogeneous right-hand sides

$$\begin{gathered}
\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial U}{\partial x}\bigg]=f_1(x,t),\\
\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\bigg[\bigg(1+\int_0^t\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial V}{\partial x}\bigg]=f_2(x,t),\\
U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\ \ t\in[0,T],\\
U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1],
\end{gathered}\tag{4}$$

where $f_1$, $f_2$, $U_0$, and $V_0$ are the given functions.

Descriptive and measurable features, along with the numerical solutions for problem (4) and one-dimensional analogue of (2) type models have been thoroughly studied in the literature (see, for example, [1, 2, 4–16, 18–20, 24], and the references therein). As mentioned earlier, our goal is to explore an alternative method for solving partial differential equations (PDEs) using Machine Learning techniques. Machine learning, specifically neural networks, is utilized to create surrogate models that predict PDE solutions at any point within the domain. Neural networks, which consist of input, hidden, and output layers, offer flexibility in terms of architecture and the number of neurons per layer (see, for example, [16]). In this approach, the solution to the problem is approximated by a neural network output, and the network parameters are optimized during training. A significant advantage of using DNNs in solving PDEs is their ability to incorporate physical laws into the learning process, reducing the volume of required training data (as discussed in [3, 21–23]).

The methodology allows to define a residual for the nonlinear problem (4), involving the approximate solution $(u(x,t,\rho),\,v(x,t,\rho))$ which is evaluated at specific training points

$$\begin{aligned}
R(x,t,\rho)=\;&\frac{\partial u(x,t,\rho)}{\partial t}+\frac{\partial v(x,t,\rho)}{\partial t}-f_1(x,t)-f_2(x,t)\\
&-\frac{\partial}{\partial x}\bigg\{\bigg(1+\int_0^t\Big[\Big(\frac{\partial u(x,t,\rho)}{\partial x}\Big)^2+\Big(\frac{\partial v(x,t,\rho)}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial u(x,t,\rho)}{\partial x}\bigg\}\\
&-\frac{\partial}{\partial x}\bigg\{\bigg(1+\int_0^t\Big[\Big(\frac{\partial u(x,t,\rho)}{\partial x}\Big)^2+\Big(\frac{\partial v(x,t,\rho)}{\partial x}\Big)^2\Big]\,d\tau\bigg)^p\,\frac{\partial v(x,t,\rho)}{\partial x}\bigg\}.
\end{aligned}\tag{5}$$

The neural network is trained by minimizing a cost function that combines residual (5) with boundary and initial condition constraints [3, 15, 21–23].

Test experiments, adopting the setup described in [16], were conducted to validate this approach. The results demonstrate the potential of neural networks to approximate solutions effectively, even for complex nonlinear PDEs. The experiments used TensorFlow for training and incorporated various parameter settings to replicate and extend prior findings.

The source terms $f_1$ and $f_2$ were selected such a way that the exact solutions are given as follows:

$$U(x,t)=x(1-x)\sin(2\pi x-t),\quad V(x,t)=x(1-x)\cos(2\pi x-t).$$

In Figures 1 and 2 the results of the exact and approximate solutions are given for $U$ and $V$ respectively. The plots are for the case $p=0.5$ in the problem (4).

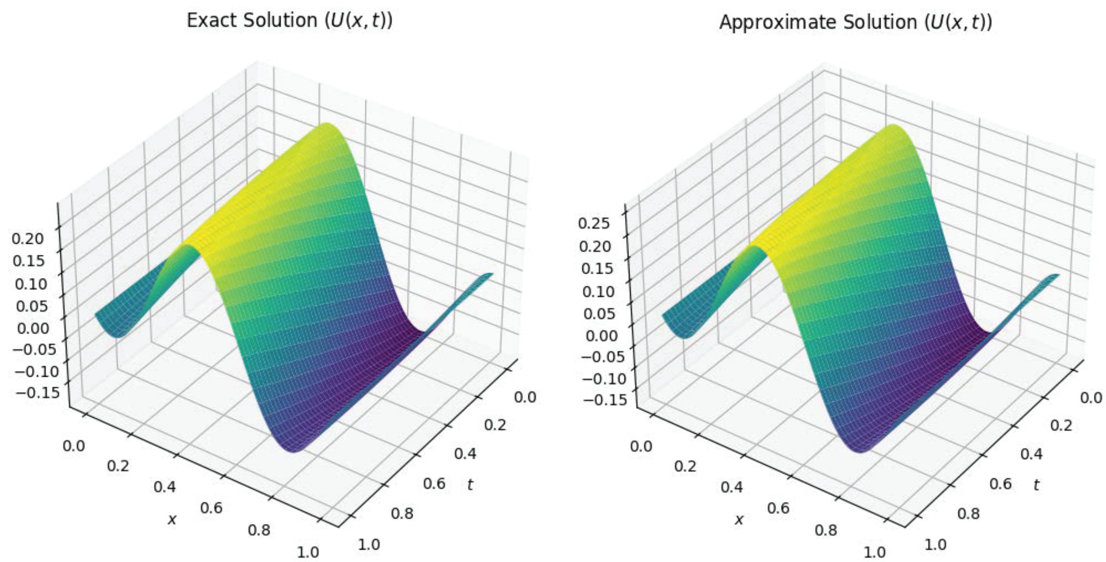


**Figure 1.** Exact and numerical solutions for $U(x,t)$.

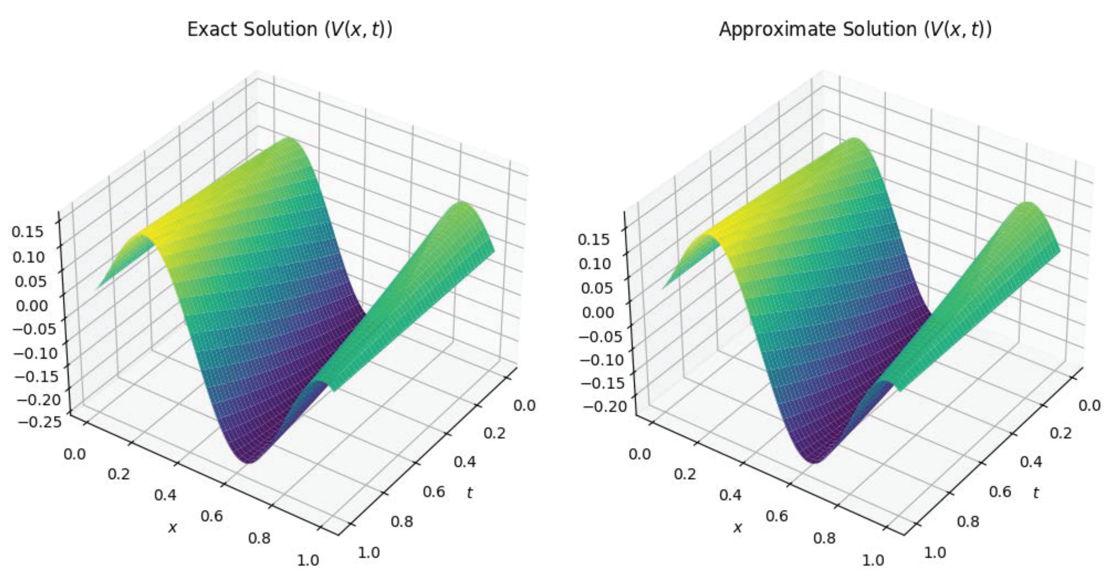


**Figure 2.** Exact and numerical solutions for $V(x,t)$.

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